Correspondence Rules in $SU(3)$

by

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Abstract

In this thesis, I present a path to the correspondence rules for the generators of the $su(3)$ symmetry and compare my results with the $SU(2)$ correspondence rules. Using these rules, I obtain analytical expressions for the Moyal bracket between the Wigner symbol of a Hamiltonian $\hat{H}$, where this Hamiltonian is written linearly or quadratically in terms of the generators, and the Wigner symbol of a general operator $\hat{B}$. I show that for the semiclassical limit, where the $SU(3)$ representation label λ tends to infinity, this Moyal bracket reduces to a Poisson bracket, which is the leading term of the expansion (in terms of the semiclassical parameter ϵ), plus correction terms. Finally, I present the analytical form of the second order correction term of the expansion of the Moyal bracket.

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The Road goes ever on and on
Down from the door where it began.
Now far ahead the Road has gone,
And I must follow, if I can,
Pursuing it with eager feet,
Until it joins some larger way
Where many paths and errands meet.
And whither then? I cannot say.

Bilbo Baggins,
The Lord of The Rings

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Chapter 1

Introduction

A productive approach to analyze quantum systems is the mapping between quantum states in Hilbert space and c-valued functions defined in the classical phase space. In this approach, one reformulates the standard quantum mechanics with the tools of classical mechanics and obtains insights about the correspondence between quantum and classical mechanics [11]. An operator $\hat{A}$ is associated with functions in phase space (called the Wigner symbol $W_{\hat{A}}(q,p)$ of this operator) , and averages are computed by integrating the corresponding Wigner symbols of the density operator and the operator $\hat{A}$ over phase space.

The concept of the Wigner symbol appeared for the first time in a paper by Wigner in 1932 [31] concerning quantum corrections to classical statistical mechanics. The mapping between an operator $\hat{A}$ and a c-valued function $W(q,p)$ in phase space was introduced by Weyl in 1927 [30] and proved independently by Moyal in 1949 [17]. Because quantum operators do not necessarily commute, whereas the product of ordinary functions $fg = gf$, one must restore the quantum feature by introducing a special type of multiplication called the $\star$-product. This operation is defined as

$$W_{\hat{A}\hat{B}}(\Omega) := W_{\hat{A}} \star W_{\hat{B}} \tag{1.0.1}$$

and it was introduced by Groenewold in 1946 [8] to make it possible to compute the semiclassical dynamics of a system under the action of a Hamiltonian $\hat{H}$.

Some properties of the Wigner symbol of the density matrix are not intuitive. For instance, the symbol of the density matrix can take negative values, something not possible for a classical probability distribution. Experimental confirmation of this feature was given by M. G. Raymer *et al* [25] and D.J. Wineland *et al* in 1996 [15]. In the latter experimental work, the authors reconstructed the density matrices and Wigner functions for quantum states of harmonic oscillator like motion for $^9\mathrm{Be}^+$, and showed characteristics of Wigner functions that are purely quantum. For instance, the distribution function had regions of negative probability, which highlights the nonclassical behavior of the quantum harmonic oscillator states. Therefore, the distribution function that will be presented here will be somewhat different than those encountered in classical statistical mechanics and will be called a quasi-distribution due to the nature of the nonclassical (negative) regions assumed in phase space.

Although the literature describes a vast range of applications of Wigner functions for harmonic oscillator systems and angular momentum (also known as systems of $SU(2)$ symmetry) written in books such as [11, 24, 32] and scientific articles or review papers such as [5, 9, 14, 15, 25], this is not the case for Wigner functions of $SU(3)$ symmetry. A possible reason for this is that the construction of $SU(3)$ Wigner symbols are much more mathematically and computationally challenging than the $SU(2)$ case.

This construction, just like in the $SU(2)$ counterpart, depends on many results such as: finding the quantization kernel $\hat{w}$ that makes the connection between quantum mechanical operators and classical c-valued functions, obtaining tensor operators $\hat{T}^{(\lambda,\mu)}_{\nu I}$ of a given irrep (λ, μ) of $SU(3)$ and Clebsch-Gordan and Racah coefficients for $SU(3)$.

The objective of this thesis is to provide a path to correspondence rules for $SU(3)$ systems. In order to accomplish this I had to obtain analytical expressions for $SU(3)$ Clebsch-Gordan coefficients. These coefficients are essential to replace the action of a generator on the quantization kernel $\hat{w}$ by a differential action on this kernel. I also obtained this differential action. With this it is then possible to obtain a differential realization for the action of generators on the symbols of general operators - i.e. the so-called correspondence rules. These in turn are used to understand the asymptotic (semiclassical) limit where in particular the so-called Moyal bracket, defined in terms of the special $\star$ product mentioned above [33]

$$\{W_{\hat{A}}, W_{\hat{B}}\}_{\mathcal{M}} := W_{\hat{A}} \star W_{\hat{B}} - W_{\hat{B}} \star W_{\hat{A}} \tag{1.0.2}$$

reduces to the classical Poisson bracket

$$\{\mathcal{G}, \mathcal{Z}\}_{\mathcal{P}} = \sum_{i=1}^{N} \frac{\partial \mathcal{G}}{\partial q_i}\frac{\partial \mathcal{Z}}{\partial p_i} - \frac{\partial \mathcal{G}}{\partial p_i}\frac{\partial \mathcal{Z}}{\partial q_i}. \tag{1.0.3}$$

where (q_i, p_i) is a pair of coordinates in the N-dimensional space and $\mathcal{G}$ and $\mathcal{Z}$ are functions of these variables. Examples of $\star$-product for position-momentum, and spin systems, will be given later.

For this introduction, I will discuss the basic ideas of the Wigner function formalism and give some examples to illustrate the power of this semiclassical approach. These basic ideas will be crucial for the understanding of the generalizations in later chapters for the case of $SU(3)$ systems.

1.1 Formulations of Quantum Mechanics and Classical Mechanics

In classical mechanics, the motion of a particle in one dimension is described by a pair of coordinates (q, p) of the phase space, where q represents the position of the particle and p its momentum. The Hamiltonian $H = H(q, p)$ will dictate the allowed trajectories of the particle that evolves according to Hamilton's equations

$$\dot{q} = \{q, H\}_{\mathcal{P}} \quad \dot{p} = \{p, H\}_{\mathcal{P}} \tag{1.1.1}$$

where $\{\alpha, \beta\}_{\mathcal{P}}$ is the Poisson bracket as defined in equation (1.0.3) for the functions $\alpha = \alpha(q, p)$ and $\beta = \beta(q, p)$. If we are working with an ensemble of n particles, where n is large, it is more convenient to represent this system with a classical distribution $\rho = \rho(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ that evolves according to

$$\frac{d\rho}{dt} = \{\rho, H\}_{\mathcal{P}} + \frac{\partial \rho}{\partial t} \quad . \tag{1.1.2}$$

However, if the density distribution ρ is constant in time, equation (1.1.2) becomes [6]

$$\frac{\partial \rho}{\partial t} = -\{\rho, H\}_{\mathcal{P}} \quad . \tag{1.1.3}$$

In the quantum mechanical formalism, it is not possible to identify a particle with an exact position and momentum at the same time. The position and momentum of a particle are represented by operators that do not commute, that is if we represent the position operator as $\hat{q}$ and momentum operator as $\hat{p}$, we will find that $\hat{q}\hat{p} \neq \hat{p}\hat{q}$. Moreover, the commutator of these two operators is

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar\mathbb{1} \quad . \tag{1.1.4}$$

We can represent the state of a system by a wave function $|\psi\rangle$ that lives in a complex Hilbert space $\mathcal{H}$, and its evolution under a Hamiltonian operator $\hat{H}$, acting on this wave function, is given by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \quad . \tag{1.1.5}$$

As mentioned before, a productive approach to make the connections between classical and quantum mechanics is the formalism of quantum mechanics in phase space. In this approach, an operator $\hat{f}$ acting on the Hilbert space $\mathcal{H}$ is mapped into a c-valued function $W_f(q,p)$ (the Wigner symbol of $\hat{f}$) in phase space. Inversely, by using the Weyl quantization we can transform any real function $W_f(q,p)$ in phase space into a corresponding operator $\hat{f}$ of the Hilbert space $\mathcal{H}$. Since the Wigner function is a quasi-distribution, we can use an equation similar to that of equation (1.1.3) to determine the time evolution of the Wigner function distribution. This evolution is described by the so-called quantum Liouville equation [9]

$$i\hbar\frac{\partial W_{\hat{\rho}}(q,p,t)}{\partial t} = -\{W_{\hat{\rho}}(q,p,t), W_{\hat{H}}\}_{\mathcal{M}} = -\{W_\rho, W_{\hat{H}}\}_{\mathcal{P}} - \frac{\hbar^2}{24}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 W_{\hat{\rho}}(q,p,t)}{\partial p^3} + \ldots \quad , \tag{1.1.6}$$

where $\{W_{\hat{f}}, W_{\hat{g}}\}_{\mathcal{M}}$ is the Moyal bracket as defined in equation (1.0.2) for the distributions $W_{\hat{f}}$ and $W_{\hat{g}}$.

The way we evaluate averages in Schrödinger quantum mechanics is different than the way averages are calculated in classical mechanics. Given the operator $\hat{Q}$ and the density operator $\hat{\rho} = |\psi\rangle\langle\psi|$ in a Hilbert space $\mathcal{H}$, the average of this operator is written as

$$\langle\hat{Q}\rangle = \mathrm{Tr}\Big(\hat{Q}\hat{\rho}\Big) = \langle\psi|\hat{Q}|\psi\rangle \quad , \tag{1.1.7}$$

where the state $|\psi\rangle$ is linearly expanded into any desired basis $\{|\phi_i\rangle\,; i = 1,2,3,\ldots\}$.

In contrast, the averages in classical mechanics are calculated by integration of the probability distributions that represent the operators $\hat{Q}$ and $\hat{\rho}$ over entire phase space. If the Wigner symbol of the operator and Wigner function of the density operator are written as $W_{\hat{Q}}(q,p)$ and $W_{\hat{\rho}}(q,p)$, the average of this operator in the classical phase space is given by

$$\langle\hat{Q}\rangle = \frac{1}{2\pi\hbar}\int_{\infty}^{\infty}\int_{\infty}^{\infty} dqdp W_Q(q,p)W_\rho(q,p) \quad . \tag{1.1.8}$$

The advantage of using equation (1.1.8), instead of equation (1.1.7), is that the number of variables in phase space does not increase when the dimensions of the quantum system increase. For instance, in spin particle systems, the dimensions of the matrices that represent the operators increase with respect to the spin number as $(2S+1)\times(2S+1)$. Therefore, using equation (1.1.7) becomes computationally expensive for systems of high dimensions, while equation (1.1.8) can yield an excellent approximation in the limit of large S, always using two angles on the sphere. Moreover, in $SU(3)$ systems, the dimension of the matrices increase like the square of the number of particles, highlighting the increased savings that come from using the phase space formalism in systems with these symmetries.

1.2 Constructing the Wigner Function of a Particle

1.2.1 A first derivation

Following Schleich in [24], the appropriate operator to describe the state of a particle in one dimension is the density operator $\hat{\rho}$[1]. Considering a quantum jump, which is defined as a transition between two states, from position x' to x'', with $y = x'' - x'$ being the distance between the two points, the strength of this transition is given by $\langle x''| \hat{\rho} |x'\rangle$. In addition, let us define the center of the jump by $x \equiv (x' + x'')/2$, which will produce the following two equations:

$$x' = x - \tfrac{1}{2}y, \qquad x'' = x + \tfrac{1}{2}y \quad . \tag{1.2.1}$$

The strength of this quantum jump is

$$\rho\left(x + \tfrac{1}{2}y, x - \tfrac{1}{2}y\right) \equiv \left\langle x + \tfrac{1}{2}y\right| \hat{\rho} \left|x - \tfrac{1}{2}y\right\rangle \quad , \tag{1.2.2}$$

and by performing a Fourier transform with respect to the quantum jump y on equation (1.2.2), it is possible to find a distribution that is dependent on the position and momentum at the same time

$$W(x,p) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \exp\left(-\frac{i}{\hbar}py\right) \left\langle x + \tfrac{1}{2}y\right| \hat{\rho} \left|x - \tfrac{1}{2}y\right\rangle \quad . \tag{1.2.3}$$

This quasi-distribution is normalized, that is

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp W(x,p) = 1 \quad . \tag{1.2.4}$$

The Wigner function is a Fourier transform that depends on the center of the jump $x \equiv (x' + x'')/2$ and the Fourier transform of the jump distance, which was the variable p. Therefore, this quasi-distribution depends only on two classical quantities x and p. Moreover, for a pure state, $\hat{\rho} = |\psi\rangle \langle\psi|$, equation (1.2.3) reduces to

$$W(x,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \exp\Big(-\frac{i}{\hbar}py\Big) \psi^*(x - \tfrac{1}{2}y)\psi(x + \tfrac{1}{2}y) \quad , \tag{1.2.5}$$

where $\psi(x) \equiv \langle x|\psi\rangle$ represents the position of the state $|\psi\rangle$.

1.2.2 Introducing the quantization kernel $\hat{w}$

An alternate derivation, more in the spirit of the approach of this thesis, is obtained by considering the phase space symbol of a density matrix as the trace of the quantization kernel $\hat{w}$ and the operator $\hat{\rho}$

$$W_{\hat{\rho}}(q,p) = 2\,\mathrm{Tr}(\hat{w}(q,p)\hat{\rho}) \quad . \tag{1.2.6}$$

The quantization kernel of equation (1.2.6) is defined as [21, 26]

$$\hat{w}(q,p) = \hat{D}(q,p)\hat{P}\hat{D}^{\dagger}(q,p) \quad , \tag{1.2.7}$$

[1] For more information about the density operator $\hat{\rho}$, refer to appendix A

where

$$\hat{D}(q,p) = \exp\left\{\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right\} \tag{1.2.8}$$

is a displacement operator and $\hat{P}$ is the parity operator

$$\hat{P} = \int dq \, |-q\rangle \langle q| = \int dp \, |-p\rangle \langle p| \quad . \tag{1.2.9}$$

We can invert the expression of equation (1.2.6) to obtain the operator $\hat{\rho}$, that is going from a one-dimensional system in phase space to Hilbert space $\mathcal{H}$

$$\hat{\rho} = \frac{1}{\pi\hbar}\int dqdp\hat{w}(q,p)W_{\hat{\rho}}(q,p) \quad . \tag{1.2.10}$$

To recover the Wigner function of equation (1.2.5), we start with the definition of equation (1.2.6)

$$W_{\rho}(q,p) = 2\,\mathrm{Tr}(\hat{w}(q,p)\hat{\rho}) \tag{1.2.11}$$

where $\hat{\rho} = |\psi\rangle\langle\psi|$. By using equations (1.2.8) and (1.2.9), one can recover the explicit form of the quantization kernel $\hat{w}(q,p)$ to be

$$\hat{w}(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy \exp\Big(-\frac{2iqy}{\hbar}\Big)\,|p+y\rangle\langle p-y| \quad . \tag{1.2.12}$$

Substituting equation (1.2.12) in equation (1.2.6) we obtain

$$\tilde{W}_{\hat{\rho}}(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy \exp\Big(-\frac{2iqy}{\hbar}\Big)\psi^*(p+y)\psi(p-y) \quad , \tag{1.2.13}$$

which is a Fourier transform of equation (1.2.5). Therefore, taking a Fourier transform of equation (1.2.13) gives us

$$W_{\hat{\rho}}(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy \exp\Big(-\frac{ipy}{\hbar}\Big)\psi^*\Big(q-\frac{1}{2}y\Big)\psi\Big(q+\frac{1}{2}y\Big) \quad , \tag{1.2.14}$$

which is the definition of Wigner function of equation (1.2.5).

1.2.3 Quasi-distribution functions

The Wigner function is one of a family of possible quasi-distribution functions, which differ by a choice of ordering of operators. In other words, the formalism does not solve the ordering problem, but we can consider an ordering rule to overcome this issue. A reasonable choice for this thesis is the Weyl ordering, which produces the Wigner function of equation (1.2.5). For instance, in this ordering the polynomial form $q^n p^m$ takes the form [9]

$$q^n p^m \to \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\hat{q}^{n-r}\hat{p}^m\hat{q}^r \tag{1.2.15}$$

and as an example, the polynomial q^2p^2 takes the form

$$q^2p^2 = \frac{1}{4}\left(\hat{q}^2\hat{p}^2 + 2\hat{q}\hat{p}^2\hat{q} + \hat{p}^2\hat{q}^2\right). \tag{1.2.16}$$

Therefore, the Wigner function is commonly associated with the so-called symmetric ordering. Other well-known examples of phase space quasi-distributions are the Husimi Q-function associated with normal ordering, and the Glauber-Sudarshan P function, associated with antinormal ordering.

In order to compute average values of an operator $\hat{f}$ using the Q-function for $\hat{\rho}$, one needs the P-function for $\hat{f}$; contrariwise, using the P-function for $\hat{\rho}$ requires the use of the Q-function for $\hat{f}$. On the other hand, one only has to know the Wigner symbol for $\hat{\rho}$ and $\hat{f}$ to compute an average. In addition to this economy, the first order correction to expansion of the Moyal bracket actually vanishes for the Wigner function, but not for the other orderings. For instance, if we look at equation (1.1.6), the Poisson bracket is the dominant term of this expansion. However, the first order approximation appear with a third order derivative with to respect to the momentum.

1.2.4 The Overlap of quantum states in phase space

Let us present some properties of the Wigner symbols.

An important property when dealing with two pure states, say $\hat{\rho}_1$ and $\hat{\rho}_2$, is the calculation of the overlap of these two states $|\langle\psi_1|\psi_2\rangle|^2$ as

$$\mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2) = |\langle\psi_1|\psi_2\rangle|^2 = 2\pi\hbar\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp W_{\hat{\rho}_1}(x,p)W_{\hat{\rho}_2}(x,p) \tag{1.2.17}$$

where equation (1.2.17) is known as the trace product rule [24].

Therefore, the overlap of the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ is proportional to the integration of the two Wigner functions, $W_{|\psi_1\rangle}$ and $W_{|\psi_2\rangle}$, over the phase space. In addition, the functions $W_{|\psi_1\rangle}$ and $W_{|\psi_2\rangle}$ are real functions [9].

1.2.5 Some conditions on the Wigner Function

Due to the nature of the definition of equation (1.2.5) and the uncertainty principle, one cannot squeeze a state to a region smaller than $2\pi\hbar$ [9, 24]. Consider now two identical density operators $\hat{\rho}_1 = \hat{\rho}_2 = \hat{\rho}$, such that

$$\mathrm{Tr}\left(\hat{\rho}^2\right) \leq 1$$

where the inequality happens when the density operator $\hat{\rho}$ does not represent a pure state. Using equation (1.2.17), it is possible to show that

$$|W(x,p)| \leq \frac{1}{\pi\hbar} \quad . \tag{1.2.18}$$

This shows that the Wigner function has an upper bound of $\frac{1}{\pi\hbar}$ for one dimensional and normalized systems. This is clearly shown in Schleich [24].

1.2.6 Negativity of the Wigner Function

If we go back to equation (1.2.17) and consider the case

$$\langle \psi_1 | \psi_2 \rangle = 0 \tag{1.2.19}$$

we get

$$\mathrm{Tr}(\hat{\rho}_1 \hat{\rho}_2) = 0 \Rightarrow \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp W_{\hat{\rho}_1}(x,p) W_{\hat{\rho}_2}(x,p) = 0 \quad , \tag{1.2.20}$$

which means that at least one of the quasi-distributions $W_{\hat{\rho}_1}(x,p)$ or $W_{\hat{\rho}_2}(x,p)$ must have negative values for some regions of phase space [24, 32]. This negativity is a pure quantum phenomenon that characterizes the Wigner function as a quasi-distribution. Figure (1.1) is a comparison between an experimental reconstruction of the Wigner function for a harmonically bound $^9\mathrm{Be}^+$ ion and the theoretical result of the same system for the case which is described by the following distribution

$$W_{|m=1\rangle}(x,p) = -\frac{1}{\pi\hbar} \exp\left\{ -\left[\left(\frac{p}{\hbar\kappa} \right)^2 + (\kappa x)^2 \right] \right\} L_1 \left\{ 2 \left[\left(\frac{p}{\hbar\kappa} \right)^2 + (\kappa x)^2 \right] \right\} \quad . \tag{1.2.21}$$

where $\kappa \equiv (\frac{m\Omega}{\hbar})^{\frac{1}{2}}$ with m being the mass of the system and Ω the frequency.

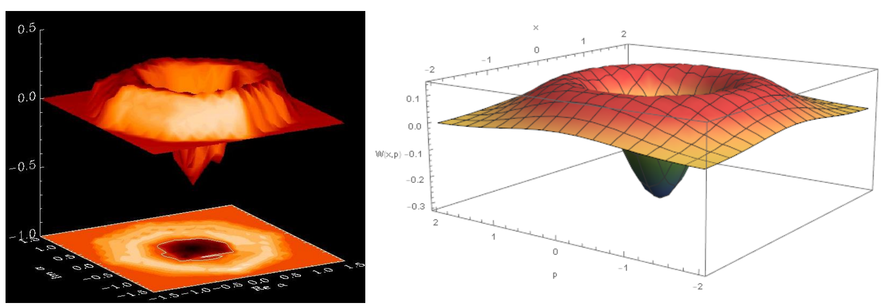

Figure 1.1: On the left: Reconstruction of the Wigner function of the harmonic coherent state $W_{|1\rangle}(x,p)$ for $^9\mathrm{Be}^+$ ions by D.J. Wineland *et al* in [15]. On the right: The constructed Wigner function for the same state using equation (1.2.21).

However, it is not every Wigner function that will exhibit negativity. An experimental result on squeezed coherent states given by M. G. Raymer *et al* [25] is shown in figure (1.2) where the Wigner function is positive in every location of phase space. And although these quasi-distributions may assume negative probabilities as depicted in figure (1.1), we can still obtain the correct marginal distribution if we integrate the Wigner function over the entire range of the complementary variable and multiply this result by $2\pi\hbar$.

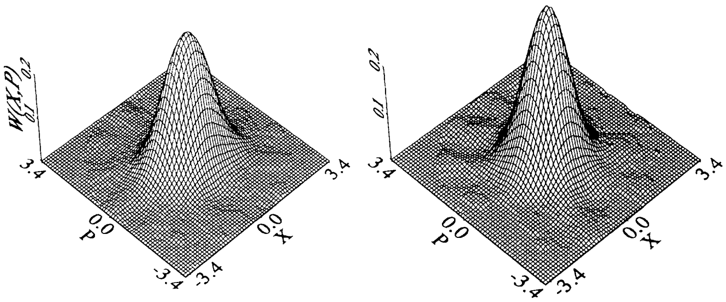

Figure 1.2: Experimental reconstruction of the Wigner functions of the squeezed (left) and vacuum states (right) [25].

1.3 Correspondence Rules for $\hat{q}$ and $\hat{p}$.

A general result from quantum mechanics is that the position $\hat{q}$ and momentum $\hat{p}$ operators do not commute. In fact, the commutator of these two quantities is found to be

$$[\hat{q}, \hat{p}] = i\hbar\mathbb{1} \quad .$$

However, the quantities q and p are ordinary variables in classical mechanics and do commute. When one performs a mapping of operators from a Hilbert space $\mathcal{H}$ into c-valued functions of the phase space of classical mechanics it is necessary to preserve the non-commutativity of the operators of the Hilbert space $\mathcal{H}$. The introduction of an operator that maintains the ordering of the quantum mechanical operators in phase space is fundamental to the quasi-distribution formalism. This operator is the $\star$-product and it is defined as

$$W_{\hat{f}}(q,p) \star W_{\hat{g}}(q,p) := W_{\hat{f}\hat{g}}(q,p) \quad , \tag{1.3.1}$$

and it has the properties of associativity and non-commutativity:

$$W_{\hat{f}} \star (W_{\hat{g}} \star W_{\hat{h}}) = (W_{\hat{f}} \star W_{\hat{g}}) \star W_{\hat{h}} \tag{1.3.2}$$

$$W_{\hat{f}} \star W_{\hat{g}} \neq W_{\hat{g}} \star W_{\hat{f}} \tag{1.3.3}$$

For the case of a particle in the (q,p) phase space, we can write the star product of equation (1.3.1) in a closed form [33]

$$W_{\hat{f}\hat{g}}(q,p) = W_{\hat{f}}(q,p) \exp\left(-\frac{i\hbar}{2}\Gamma\right) W_{\hat{g}}(q,p) \tag{1.3.4}$$

with

$$\Gamma := \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} \quad , \tag{1.3.5}$$

where $\frac{\overleftarrow{\partial}}{\partial s_i}$ represents an operation on the function to the left of Γ. However, an advantageous representation of the $\star$-product was introduced by Bopp in [3]. Given two operators $\hat{f}$ and $\hat{g}$ in a Hilbert space $\mathcal{H}$, the $\star$-product of equation (1.3.1) is written as

$$W_{\hat{f}}(q,p) \star W_{\hat{g}}(q,p) = f(Q,P) W_{\hat{g}}(q,p) \tag{1.3.6}$$

where

$$Q := q + \frac{i\hbar}{2}\frac{\partial}{\partial p}, \quad P := p - \frac{i\hbar}{2}\frac{\partial}{\partial q} \quad . \tag{1.3.7}$$

and $f(Q,P)$ is a differential operator that depends on the quantum mechanical operator $\hat{f}$ only and acts on the Wigner symbol $W_{\hat{g}}(q,p)$. Moreover, equation (1.3.6) is an example of the correspondence rules in the (q,p) phase space.

Once we have defined the $\star$-product, we can introduce the Moyal bracket

$$\{W_{\hat{f}}, W_{\hat{g}}\}_{\mathcal{M}} := W_{\hat{f}} \star W_{\hat{g}} - W_{\hat{g}} \star W_{\hat{f}} \tag{1.3.8}$$

which is defined as the symbol of the commutator

$$W_{[\hat{f},\hat{g}]} := W_{\hat{f}} \star W_{\hat{g}} - W_{\hat{g}} \star W_{\hat{f}} \quad . \tag{1.3.9}$$

The evolution in quantum mechanics is given by the von Neumann equation

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = \left[\hat{H}, \hat{\rho}\right] \quad , \tag{1.3.10}$$

where $\hat{H}$ is the Hamiltonian of the system and $\hat{\rho}$ is the density matrix. Multiplying equation (1.3.10) by $\hat{w}(q,p)$ and tracing it, one finds that the exact evolution of a quantum particle in phase space is

$$i\hbar\frac{\partial}{\partial t}W_{\hat{\rho}} = \{W_{\hat{H}}, W_{\hat{\rho}}\}_{\mathcal{M}} \quad . \tag{1.3.11}$$

The correspondence rules defined in equation (1.3.6) have been extended to the context of angular momentum ($SU(2)$ systems) [10, 33] and I show in this thesis that these rules can be written in the context of $SU(3)$ systems.

1.4 What this Thesis is about

This thesis is about correspondence rules $SU(3)$ Wigner functions. It represents a partial extension of known results for $SU(2)$ Wigner functions.

The quantization kernel for $SU(2)$ and $SU(3)$ makes heavy use of tensor operators and coupling coefficients ($SU(2)$ and $SU(3)$, respectively). The expression for this kernel is found in the literature [12, 14].

To obtain the correspondence rules one must multiply and decompose tensor operators. This is a more technical aspect which involves recoupling coefficients. One must also obtain differential identities for group functions. These technical steps are well-known for $SU(2)$; the bulk of the work of my thesis is to obtain various $SU(3)$ coefficients in analytical form, and find the appropriate differential identities.

There is considerable literature on coupling and recoupling coefficients for $SU(3)$. Unfortunately, none of these results were readily usable because:

- as there is no uniform convention for the sign of $SU(3)$ states and matrix elements between those states, the known analytical expressions for the required $SU(3)$ coupling and recoupling coefficients were tabulated using a phase convention not immediately compatible with the current calculations,

- the form of the appropriate SU(3) group functions are not widely known,

- the form of the differential operators depend on the factorization of $SU(3)$ transformation.

In view of the above, this thesis is organized as follows. I will first review some features of angular momentum systems: the Wigner function, its evolution, the $\star$-product and correspondence rules.

Next, I will discuss the evaluation of $SU(3)$ coupling coefficients, with emphasis on the construction of tensor operators and other coefficients needed in my thesis.

Finally, I will discuss the path to correspondence rules in $SU(3)$, and possible future applications.

There is a considerable amount of useful background material that usually comes with this topic. This includes density matrix theory, tensor operators, *etc.* This important material has been placed in several appendices, along with complementary material, so as to allow easier reading of the thesis.

Chapter 2

Review of SU(2) Clebsch-Gordan Coefficients and Spherical Tensor Operators

In quantum mechanics one must often combine systems, especially when dealing with multiparticle systems. The simplest and best known example is the combination of angular momenta. Suppose we have two particles with angular momentum observables $\hat{L}_i^1$ and $\hat{L}_i^2$. These are the operators corresponding to projections of each angular momentum vector. Since the first and second particles are independent, the states of the systems are simple products $|L_1 m_1\rangle|L_2 m_2\rangle$, where the individual states are eigenstates of $\hat{L}_z^k$ and $(\hat{L}_z^k)^2 + (\hat{L}_x^k)^2 + (\hat{L}_y^k)^2 := (L^k)^2$. Moreover, $\hat{L}_i^1$ acts only on the first state and $\hat{L}_i^2$ on the second state

$$\hat{L}_i^1|L_1 m_1\rangle|L_2 m_2\rangle = [L_i|L_1 m_1\rangle]\,|L_2 m_2\rangle\,, \tag{2.0.1}$$

$$\hat{L}_i^2|L_1 m_1\rangle|L_2 m_2\rangle = |L_1 m_1\rangle\,[L_i|L_2 m_2\rangle]\,. \tag{2.0.2}$$

These definitions imply that $[L_i^1, L_k^2] = 0$, i.e. all the operators acting on the first particle commute with the operators acting on the second.

The states $|L_1 m_1\rangle|L_2 m_2\rangle$ are not eigenstates of the square of the total angular momentum $\hat{J}^2 := (L_x^1 + L_x^2)^2 + (L_y^1 + L_y^2)^2 + (L_z^1 + L_z^2)^2$ operator.

To obtain the eigenstates of $\hat{J}^2$ we need to make a change of basis, and the coefficients for this change of basis are called Clebsch-Gordan coefficients.

Because the procedure to obtain the Clebsch-Gordan coefficients is purely algebraic, the same procedure can be used to couple, for instance, the spin and orbital angular momenta. The spin is an intrinsic characteristic of some particles such as the electron, proton and neutron. Unlike the angular momentum operators $\hat{L}_x$ or $\hat{L}_y$ or $\hat{L}_z$ that can be written in terms of derivatives of spatial coordinates (r, θ, ϕ), the spin operators act in a purely abstract space, emphasizing that they must commute with the spatial degrees of freedom.

This section serves as good practice for the more complicated case of combining representations of SU(3), and the construction of tensor operators for this algebra.

2.1 Recursion Relation for the Clebsch-Gordan coefficients

We start with the addition of the orbital angular momentum and the spin

$$\hat{J} = \hat{S} + \hat{L} \quad . \tag{2.1.1}$$

where $\hat{J}$ has components

$$\hat{J}_x = \hat{S}_x + \hat{L}_x, \quad \hat{J}_y = \hat{S}_y + \hat{L}_y, \quad \hat{J}_z = \hat{S}_z + \hat{L}_z \quad . \tag{2.1.2}$$

The components of the operator $\hat{J}$ satisfy the commutation relations for angular momentum

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_z \quad \left[\hat{J}_y, \hat{J}_z\right] = i\hbar\hat{J}_x \quad \left[\hat{J}_z, \hat{J}_x\right] = i\hbar\hat{J}_y \quad . \tag{2.1.3}$$

Because the components of the operator $\hat{J}$ and the operator $\hat{J}^2$ commute we can choose a complete set of eigenstates that are simultaneous eigenstates of $\hat{J}^2$ and one component of $\hat{J}$. Traditionally, we choose this component to be $\hat{J}_z$. The operators $\hat{J}^2$ and $\hat{J}_z$ satisfy

$$\hat{J}^2 \left|J, m_J\right\rangle = \hbar^2 J(J+1) \left|J, m_J\right\rangle \quad \hat{J}_z \left|J, m_J\right\rangle = \hbar m_J \left|J, m_J\right\rangle \quad . \tag{2.1.4}$$

where $|J, m_J\rangle$ are the simultaneous eigenstates, and J and m_J satisfy $|S - L| \leq J \leq |S + L|$ and $m_J = -J, -J+1, \ldots, J-1, J$, respectively. In this manner, given the angular momentum J, there are $2J+1$ possible values for m_J, where the lowest is $-J$ and the highest is $+J$. These eigenstates are linearly expanded in function of the orbital angular momentum $|L, m_L\rangle$ and spin $|S, m_S\rangle$ eigenstates

$$\left|J, m_J\right\rangle = \sum_{m_S, m_L} \left\langle {S \atop m_S}; {L \atop m_L} \middle| {J \atop m_J} \right\rangle \left|S, m_S\right\rangle \left|L, m_L\right\rangle . \tag{2.1.5}$$

The expansion coefficients $\left\langle {S \atop m_S}; {L \atop m_L} \middle| {J \atop m_J} \right\rangle$, also denoted by $C^{Jm_J}_{Sm_s;Lm_L}$, are called Clebsch-Gordan coefficients.

Let us choose the state that has $m_J = J$, that is the highest eigenstate

$$\left|J, J\right\rangle = \sum_{m_S, m_L} \left\langle {S \atop m_S}; {L \atop m_L} \middle| {J \atop J} \right\rangle \left|S, m_S\right\rangle \left|L, m_L\right\rangle \quad . \tag{2.1.6}$$

Now, the raising operator is defined as

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \tag{2.1.7}$$

and it can be decomposed into

$$\hat{J}_+ = \hat{S}_+ + \hat{L}_+ \quad . \tag{2.1.8}$$

This operator raises the quantum number m_J by one unit, that is for a given eigenstate $|J, m_J\rangle$ we find

$$\hat{J}_+ \left|J, m_J\right\rangle = \hbar\sqrt{(J + m_J + 1)(J - m_J)} \left|J, m_J + 1\right\rangle \quad . \tag{2.1.9}$$

Since the quantum number J is related to the eigenvalue $J(J+1)$ of the operator $\hat{J}^2$, and since $\hat{J}_+$ commutes with $\hat{J}^2$, the action of $\hat{J}_+$ on an eigenstate of $\hat{J}^2$ transforms it to another eigenstate of $\hat{J}^2$ as it was shown in equation (2.1.9). In addition, the state $|J, J\rangle$ is killed by the operator $\hat{J}_+$

$$\hat{J}_+ \left|J, J\right\rangle = \sum_{m_S, m_L} \left\langle {S \atop m_S}; {L \atop m_L} \middle| {J \atop J} \right\rangle \left(\hat{S}_+ + \hat{L}_+\right) \left|S, m_S\right\rangle \left|L, m_L\right\rangle = 0 \quad . \tag{2.1.10}$$

The application of the operators $\hat{L}_+$ and $\hat{S}_+$ on the states $|L, m_L\rangle$ and $|S, m_S\rangle$ follow the same form of equation (2.1.9), but the spin operator only acts on the spin state $|S, m_S\rangle$ and the orbital angular momentum operator on the orbital state $|L, m_L\rangle$. Using this assumption, equation (2.1.10) becomes

$$\sum_{m_S, m_L} \left\langle \begin{smallmatrix} S \\ m_S \end{smallmatrix}; \begin{smallmatrix} L \\ m_L \end{smallmatrix} \middle| \begin{smallmatrix} J \\ J \end{smallmatrix} \right\rangle \sqrt{(S+m_S+1)(S-m_S)}\, |S, m_S+1\rangle\, |L, m_L\rangle \\ + \sum_{m_S, m_L} \left\langle \begin{smallmatrix} S \\ m_S \end{smallmatrix}; \begin{smallmatrix} L \\ m_L \end{smallmatrix} \middle| \begin{smallmatrix} J \\ J \end{smallmatrix} \right\rangle \sqrt{(L+m_L+1)(L-m_L)}\, |S, m_S\rangle\, |L, m_L+1\rangle = 0 \quad . \tag{2.1.11}$$

Hitting equation (2.1.11) with $\langle L, M| \langle S, M'|$ from the left and doing some mathematical manipulations, it yields

$$\left\langle \begin{smallmatrix} S \\ M' \end{smallmatrix}; \begin{smallmatrix} L \\ M-1 \end{smallmatrix} \middle| \begin{smallmatrix} J \\ J \end{smallmatrix} \right\rangle = -\sqrt{\frac{(S+M')(S-M'+1)}{(L+M)(L-M+1)}} \left\langle \begin{smallmatrix} S \\ M'-1 \end{smallmatrix}; \begin{smallmatrix} L \\ M \end{smallmatrix} \middle| \begin{smallmatrix} J \\ J \end{smallmatrix} \right\rangle \quad . \tag{2.1.12}$$

Now, let us set $M' = m'$ and $M = m + 1$ and rewrite equation (2.1.12) as

$$\left\langle \begin{smallmatrix} S \\ m' \end{smallmatrix}; \begin{smallmatrix} L \\ m \end{smallmatrix} \middle| \begin{smallmatrix} J \\ J \end{smallmatrix} \right\rangle = -\sqrt{\frac{(S+m')(S-m'+1)}{(L+m+1)(L-m)}} \left\langle \begin{smallmatrix} S \\ m'-1 \end{smallmatrix}; \begin{smallmatrix} L \\ m+1 \end{smallmatrix} \middle| \begin{smallmatrix} J \\ J \end{smallmatrix} \right\rangle \quad . \tag{2.1.13}$$

These are recursion relations for the Clebsch-Gordan coefficients and it allows us to calculate consecutive ratios for Clebsch-Gordan coefficients. Moreover, there is a normalization condition for the coefficients of equation (2.1.13)

$$\sum_{m_S, m_L} |\left\langle \begin{smallmatrix} S \\ m_S \end{smallmatrix}; \begin{smallmatrix} L \\ m_L \end{smallmatrix} \middle| \begin{smallmatrix} J \\ m_J \end{smallmatrix} \right\rangle|^2 = 1 \quad , \tag{2.1.14}$$

and this extra condition allows us to determine their final numerical values.

2.1.1 The Lowering Operator $\hat{J}_-$

Since we were able to calculate the Clebsch-Gordan coefficients for the highest state $|J, J\rangle$, we can apply the lowering operator $\hat{J}_-$, which can be decomposed in the same manner as in equation (2.1.8), on the highest state and find states with lower magnetic quantum number m_J. For instance, let us act $\hat{J}_-$ on the state $|J, J\rangle$:

$$\hat{J}_- |J, J\rangle = \sum_{m_S, m_L} \left\langle \begin{smallmatrix} S \\ m_S \end{smallmatrix}; \begin{smallmatrix} L \\ m_L \end{smallmatrix} \middle| \begin{smallmatrix} J \\ J \end{smallmatrix} \right\rangle (\hat{L}_- + \hat{S}_-)\, |S, m_S\rangle\, |L, m_L\rangle \quad . \tag{2.1.15}$$

The action of the lowering operator on any state $|K, m_K\rangle$ can be written in terms of factorials

$$\hat{K}_- |K, m_K\rangle = \hbar \sqrt{\frac{(K+m_K)!(K-m_K+1)!}{(K-m_K)!(K+m_K-1)!}}\, |K, m_K-1\rangle \quad , \tag{2.1.16}$$

therefore, after some mathematical manipulations, equation (2.1.15) becomes

$$\hat{J}_- |J,J\rangle = \sum_{m_S,m_L} \left\langle {S \atop m_S}; {L \atop m_L} \middle| {J \atop J} \right\rangle \left\{ \sqrt{\frac{(L+m_L)!(L-m_L+1)!}{(L-m_L)!(L+m_L-1)!}} |L,m_L-1\rangle |S,m_S\rangle \right.$$
$$\left. + \sqrt{\frac{(S+m_S)!(S-m_S+1)!}{(S-m_S)!(S+m_S-1)!}} |L,m_L\rangle |S,m_S-1\rangle \right\} \quad , \tag{2.1.17}$$

where we set $\hbar = 1$. We can lower the highest state $|J,J\rangle$ with the operator $\left(\hat{J}_-\right)^2$ to obtain the states $|J,J-n\rangle = (\hat{J}_-)^n |J,J\rangle$. After a few actions, one should be able to recognize that a general state is given by

$$|J,m_J\rangle = \sum_{m_L,m_S} \sum_{k=0}^{J-m_J} \left\langle {S \atop m_S}; {L \atop m_L} \middle| {J \atop J} \right\rangle \binom{J-m_J}{k} \sqrt{\frac{(L+m_L)!(L-m_L+J-m_J-k)!}{(L-m_L)!(L+m_L-J+m_J+k)!}}$$
$$\times \sqrt{\frac{(S+m_S)!(S-m_S+k)!}{(S-m_S)!(S+m_S-k)!}} |L,m_L-J+m_J+k\rangle |S,m_S-k\rangle \quad , \tag{2.1.18}$$

where $m_J = J - n$. Therefore, one can find the Clebsch-Gordan coefficients using the recursion relation of equation (2.1.13) and construct the highest state $|J,J\rangle$, and then act on this state with the lowering operator of equation (2.1.16) to find general states of the form of equation (2.1.18).

Example

Let us obtain the state $|1,1\rangle$ by using equation (2.1.18). In order to find this state, we first choose $S=1, L=1$ and notice that $J=1$. This brings equation (2.1.18) to the form

$$|1,1\rangle = \left\langle {1 \atop 1}; {1 \atop 0} \middle| {1 \atop 1} \right\rangle |1,0\rangle |1,1\rangle + \left\langle {1 \atop 0}; {1 \atop 1} \middle| {1 \atop 1} \right\rangle |1,1\rangle |1,0\rangle \quad . \tag{2.1.19}$$

The Clebsch-Gordan coefficients are found via the recursion relation of equation (2.1.13) and assuming that the state $|1,1\rangle$ should be normalized. Therefore,

$$\left\langle {1 \atop 0}; {1 \atop 1} \middle| {1 \atop 1} \right\rangle = \frac{1}{\sqrt{2}} \qquad \left\langle {1 \atop 1}; {1 \atop 0} \middle| {1 \atop 1} \right\rangle = -\frac{1}{\sqrt{2}} \quad , \tag{2.1.20}$$

and the state of equation (2.1.19) is

$$|1,1\rangle = \frac{1}{\sqrt{2}} \Big(|1,1\rangle |1,0\rangle - |1,0\rangle |1,1\rangle \Big) \quad . \tag{2.1.21}$$

We are able to find more states by using equation (2.1.18) or even the lowering operator. The other two states are written as

$$|1,0\rangle = \frac{1}{\sqrt{2}} \Big(|1,1\rangle |1,-1\rangle - |1,-1\rangle |1,1\rangle \Big)$$
$$|1,-1\rangle = \frac{1}{\sqrt{2}} \Big(|1,0\rangle |1,-1\rangle - |1,-1\rangle |1,0\rangle \Big) \quad . \tag{2.1.22}$$

The states of equations (2.1.21) and (2.1.22) are orthonormal.

2.2 Matrix Representation of Rotations

A rotation $\hat{R}(\Omega)$ is obtained as

$$\begin{aligned}\hat{R}(\Omega) &= \mathrm{e}^{-i\hat{J}\cdot\hat{\Omega}} \\ &= \mathbb{1} - i(\hat{J}_x\hat{\Omega}_x + \hat{J}_y\hat{\Omega}_y + \hat{J}_z\hat{\Omega}_z) + \frac{1}{2!}(-i)^2(\hat{J}_x\hat{\Omega}_x + \hat{J}_y\hat{\Omega}_y + \hat{J}_z\hat{\Omega}_z)^2 + \ldots \end{aligned} \tag{2.2.1}$$

A rotation does not change the length of a vector; this is reflected in the commutation relation

$$\left[\hat{R}(\Omega), \hat{J}^2\right] = 0. \tag{2.2.2}$$

this leads to

$$\hat{J}^2\hat{R}(\Omega)\left|J, m_J\right\rangle = \hat{R}(\Omega)\hat{J}^2\left|J, m_J\right\rangle = J(J+1)\hat{R}(\Omega)\left|J, m_J\right\rangle \quad . \tag{2.2.3}$$

Thus, the rotation $\hat{R}(\Omega)$ does not change the quantum number J.

The rotation of equation (2.2.1) geometrically rotates the component $\hat{J}_z$, so this operator will be transformed into a linear combination of $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$. Finally, the eigenstates $\left|J, m_J\right\rangle$ of the original operator $\hat{J}_z$ will no longer be eigenstates of the transformed operator $\hat{R}(\Omega)\hat{J}_z\hat{R}^\dagger(\Omega)$. The transformed eigenstates $\hat{R}(\Omega)\left|J, m_J\right\rangle$ can be expanded as

$$\hat{R}(\Omega)\left|J, m_J\right\rangle = \sum_{J', m'_J}\left|J', m'_J\right\rangle\left\langle J', m'_J\right|\hat{R}(\Omega)\left|J, m_J\right\rangle = \sum_{m'_J=-J}^{J} D^J_{m'_J, m_J}(\Omega)\left|J, m'_J\right\rangle \tag{2.2.4}$$

where $D^J_{m'_J, m_J}(\Omega)$ is a $SU(2)$ Wigner D-function.

While it is possible to write the rotation $\hat{R}(\Omega)$ in the form of equation (2.2.1), a more convenient form of writing rotations was devised by Euler (see page 361 of [1]):

$$\hat{R}(\Omega) = e^{-i\hat{J}\cdot\hat{\Omega}} = e^{-i\alpha\hat{J}_z}e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z} \tag{2.2.5}$$

where the triple (α, β, γ) are known as Euler angles. The Euler angles are complicated functions of the $(\Omega_x, \Omega_y, \Omega_z)$ and vice versa. The advantage of using this factorization is tied to the choice of basis states $|J, m_J\rangle$: since $J_z|J, m_J\rangle = m_J|J, m_J\rangle$, it follows that

$$\langle J, m'_J|\hat{R}(\Omega)|J, m_J\rangle = \langle J, m'_J|e^{-i\alpha\hat{J}_z}e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}|J, m_J\rangle\,, \tag{2.2.6}$$

$$= e^{-i\alpha m'_J}\langle J, m'_J|e^{-i\beta\hat{J}_y}|J, m_J\rangle e^{-i\gamma m_J} \tag{2.2.7}$$

so that only the exponential of the simpler J_y matrix needs to be evaluated. In fact,

$$\begin{aligned}D^J_{m'_J, m_J}(\Omega) &= \langle J, m'_J|\hat{R}(\Omega)|J, m_J\rangle \\ &:= e^{-i\alpha m'_J}d^J_{m'_J, m_J}(\beta)e^{-i\gamma m_J}\,. \end{aligned} \tag{2.2.8}$$

There are several techniques to evaluate the rotation functions $d^{(J)}_{m'_J, m_J}(\beta)$ and the simplest ones are also tabulated [27].

2.3 Spherical Tensor Operators

A unitary transformation of an operator $\hat{A}$ is written as

$$\hat{A}' = \hat{R}(\Omega)\hat{A}\hat{R}^{-1}(\Omega) \quad , \tag{2.3.1}$$

where by unitary we mean that $\hat{R}^{\dagger}(\Omega) = \hat{R}^{-1}(\Omega)$ and the expectation value of the operator $\hat{A}$ in a basis $\{|\Phi\rangle\}$ is conserved after a rotation $\hat{R}(\Omega)$ in the transformed basis $\{|\Phi'\rangle\}$. Mathematically, we can write

$$\langle\Phi|\,\hat{A}\,|\Phi\rangle = \langle\Phi|\,\hat{R}^{-1}(\Omega)\hat{R}(\Omega)\hat{A}\hat{R}^{-1}(\Omega)\hat{R}(\Omega)\,|\Phi\rangle \quad , \tag{2.3.2}$$

then

$$\langle\Phi|\,\hat{A}\,|\Phi\rangle = \langle\Phi'|\,\hat{A}'\,|\Phi'\rangle \quad . \tag{2.3.3}$$

The type of unitary transformation present in this chapter and chapter 3 is the $SU(2)$ rotation, but for chapter 5, I will present the $SU(3)$ rotation, since that chapter is devoted to quasi distributions in $SU(3)$. There are types of operators called *scalars* which do not change under rotation $\hat{R}(\Omega)$. Scalar operators commute with the angular momentum operator $\hat{J}$ and are unchanged by equation (2.3.1). Different types of operators, which have simple transformation properties under rotation, are called tensor operators. If a set of operators $\hat{T}^{k}_{q}$, where $q = -k, -k+1, \ldots, k-1, k$, transform among themselves under unitary rotations according to the transformation law

$$\hat{R}(\Omega)\hat{T}^{(k)}_{q}\hat{R}^{-1}(\Omega) = \sum_{q'=-k}^{k} \hat{T}^{(k)}_{q'} D^{(k)}_{q',q}(\Omega) \quad , \tag{2.3.4}$$

they are said to be irreducible tensor operators as defined on page 368 of [1] . The coefficients $D^{(k)}_{q',q}(\Omega)$ of this linear expansion are the matrix elements of the irreducible representation of the rotation group of dimension $2k+1$.

Now, suppose we have an infinitesimal rotation

$$\hat{R}(\Omega) = e^{-i\hat{J}\cdot\hat{\Omega}} \approx \mathbb{1} - i\hat{J}\cdot\hat{\Omega} \quad . \tag{2.3.5}$$

Using this approximation on the left hand side of equation (2.3.4) and keeping only the linear terms in Ω, we get

$$\begin{aligned} R(\Omega)\hat{T}^{(k)}_{q}R^{-1}(\Omega) &= \left(\mathbb{1} - i\hat{J}\cdot\hat{\Omega}\right)\hat{T}^{(k)}_{q}\left(\mathbb{1} + i\hat{J}\cdot\hat{\Omega}\right) \\ &= \hat{T}^{(k)}_{q} - i\Omega\cdot\left[\hat{J},\hat{T}^{(k)}_{q}\right] \quad . \end{aligned} \tag{2.3.6}$$

Now, we can evaluate the right hand side of equation (2.3.4) with the help of the approximation of equation (2.3.5)

$$\begin{aligned} R(\Omega)\hat{T}^{(k)}_{q}R^{-1}(\Omega) &= \sum_{q'=-k}^{k} \hat{T}^{(k)}_{q'}\,\langle k,q'|\left(\mathbb{1} - i\hat{J}\cdot\hat{\Omega}\right)|k,q\rangle \\ &= \hat{T}^{(k)}_{q} - i\hat{\Omega}\cdot\sum_{q'=-k}^{k} \hat{T}^{(k)}_{q'}\,\langle k,q'|\,\hat{J}\,|k,q\rangle \quad , \end{aligned} \tag{2.3.7}$$

and compare it with equation (2.3.6) to obtain

$$\left[\hat{J},\hat{T}_q^{(k)}\right]=\sum_{q'=-k}^{k}\hat{T}_{q'}^{(k)}\left\langle k,q'\right|\hat{J}\left|k,q\right\rangle \quad . \tag{2.3.8}$$

Equation (2.3.8) is valid for all three components of the total angular momentum. For instance, for the z component we find

$$\left[\hat{J}_z,\hat{T}_q^{(k)}\right]=\sum_{q'=-k}^{k}\hat{T}_{q'}^{(k)}\left\langle k,q'\right|\hat{J}_z\left|k,q\right\rangle=q\hat{T}_q^{(k)} \quad , \tag{2.3.9}$$

where $\hbar$ was omitted from equation (2.3.9). Instead of using the $\hat{J}_x$ and $\hat{J}_y$ components of the total angular momentum in equation (2.3.8), let us use the raising $\hat{J}_+$ and lowering $\hat{J}_-$ operators

$$\left[\hat{J}_\pm,\hat{T}_q^{(k)}\right]=\sqrt{\left(k\mp q\right)\left(k\pm q+1\right)}\hat{T}_{q\pm1}^{(k)} \tag{2.3.10}$$

The commutation relations of equations (2.3.9) and (2.3.10) encapsulate the properties of irreducible tensor operators under infinitesimal rotations.

Example: An application of the commutation relations of angular momentum and tensor operators

The components of a vector $\vec{V}$ obey the following commutation relations with the components of the total angular momentum operator $\hat{J}$

$$\left[\hat{J}_i,\hat{V}_j\right]=i\epsilon_{ijk}\hat{V}_k \quad . \tag{2.3.11}$$

Let us define the $q=0$ component of the rank 1 tensor operator to be

$$\hat{T}_{q=0}^{(1)}=\hat{V}_z \quad . \tag{2.3.12}$$

We can now use the commutation relations of equation (2.3.10) to obtain

$$\begin{aligned}\hat{T}_{q=1}^{(1)}&=-\frac{1}{\sqrt{2}}\hat{V}_+\\ \hat{T}_{q=-1}^{(1)}&=\frac{1}{\sqrt{2}}\hat{V}_-\end{aligned} \tag{2.3.13}$$

Let us make a connection between the components of some tensor operators $\hat{T}_q^{(1)}$, where $q=-1,0,1$, and the spherical harmonics $Y_{L,M}(\theta,\phi)$. Let us assume

$$\hat{V}_+=\hat{r}_1 \quad ,\hat{V}_0=\hat{r}_0 \quad ,\hat{V}_-=\hat{r}_{-1} \tag{2.3.14}$$

and then

$$\hat{r}_1=x+iy, \quad \hat{r}_0=z, \quad \hat{r}_{-1}=x-iy \quad . \tag{2.3.15}$$

Now, let us transform these coordinates into spherical coordinates and substitute them into equations (2.3.12) and (2.3.13)

$$\hat{T}^{(1)}_{q=1} = -\frac{r\sin\theta}{\sqrt{2}}\mathrm{e}^{i\phi}, \quad \hat{T}^{(1)}_{q=0} = r\cos\theta, \quad \hat{T}^{(1)}_{q=-1} = \frac{r\sin\theta}{\sqrt{2}}\mathrm{e}^{-i\phi} \quad . \tag{2.3.16}$$

Recalling that the spherical harmonics for $L = 1$ are given by

$$Y_{1,1}(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin(\theta), \quad Y_{1,0}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta), \quad Y_{1,-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}\sin(\theta) \quad , \tag{2.3.17}$$

then

$$\hat{T}^{(1)}_{q=1} = \sqrt{\frac{4\pi}{3}}rY_{1,1}(\theta,\phi), \quad \hat{T}^{(1)}_{q=0} = \sqrt{\frac{4\pi}{3}}rY_{1,0}(\theta,\phi), \quad \hat{T}^{(1)}_{q=-1} = \sqrt{\frac{4\pi}{3}}rY_{1,-1}(\theta,\phi) \tag{2.3.18}$$

or more generally,

$$r_q = \sqrt{\frac{4\pi}{3}}rY_{1,q}(\theta,\phi) \tag{2.3.19}$$

2.4 The Wigner-Eckart theorem

The Wigner-Eckart theorem is an important result which states that a matrix element can be factorized as[1]

$$\langle\alpha',j',m'|\,\hat{T}^k_q\,|\alpha,j,m\rangle = \frac{\langle\alpha',j'\,\|\hat{T}^k\|\alpha,j\rangle}{\sqrt{2j'+1}}\left\langle \begin{smallmatrix} j \\ m \end{smallmatrix} ; \begin{smallmatrix} k \\ q \end{smallmatrix} \middle| \begin{smallmatrix} j' \\ m' \end{smallmatrix} \right\rangle \quad , \tag{2.4.1}$$

where $\langle\alpha',j'\,\|\hat{T}^k\|\alpha,j\rangle$ is a reduced matrix element of the tensor operator $\hat{T}^k_q$. One may notice that this reduced matrix element does depend only on the angular momentum of the two states, but also on the rank k of the tensor operator and other parameters that are represented by α' and α.

Example: The three dimensional Harmonic Oscillator

Let us assume that the wave functions that describe the three dimensional Harmonic Oscillator are already known. Now, define the state

$$|n,\ell,m\rangle = R_{n,\ell}Y_{\ell,m}(\theta,\phi) \quad , \tag{2.4.2}$$

where $Y_{\ell,m}(\theta,\phi)$ are the spherical harmonics and $R_{n,\ell}$ are the radial solutions (properly normalized) for the 3-d harmonic oscillator. Let us look at matrix elements of the the type

$$\langle n'\ell'm'|\hat{T}^k_q|n\ell m\rangle\,. \tag{2.4.3}$$

For the purpose of this discussion, we will use the quantum numbers in the table below for the wave functions in equation (2.4.2). These quantum numbers were chosen to represent the independence of the

[1]Details on the derivation of this theorem can be found in [1].

reduced matrix elements on the quantum numbers m and m'.

Table 2.1: Quantum numbers for the 3-D Harmonic Oscillator

Quantum numbers					
n	ℓ	m	n'	ℓ'	m'
4	4	4	4	4	2
4	4	3	4	4	1
4	4	1	4	4	-1
4	4	0	4	4	-2

The tensor operator that will be used in this example is

$$\hat{T}_2^2 = \hat{Q}_2^2 = (x+iy)^2 = 4\sqrt{\frac{2\pi}{15}}r^2 Y_{2,2}(\theta,\phi) \quad . \tag{2.4.4}$$

Let us start with the substitution of the information of the first row of table (2.1) into equation (2.4.1)

$$\langle 4,4,4|\,\hat{Q}_2^2\,|4,4,2\rangle = \frac{\langle 4,4\,\|\hat{Q}^2\|4,4\rangle}{3}\left\langle {4 \atop 2};{2 \atop 2}\,\middle|\,{4 \atop 4}\right\rangle \quad . \tag{2.4.5}$$

The matrix element $\langle 4,4,4|\,\hat{Q}_2^2\,|4,4,2\rangle$ is given by

$$\begin{aligned}\langle 4,4,4|\,\hat{Q}_2^2\,|4,4,2\rangle =& 4\sqrt{\frac{2\pi}{15}}\int_0^\infty r^4 R_{4,4}^2(r)dr \\ &\times \int_0^{2\pi}\int_0^\pi \Big(Y_{4,4}(\theta,\phi)\Big)^* Y_{2,2}(\theta,\phi)Y_{4,2}(\theta,\phi)d\theta d\phi\end{aligned} \tag{2.4.6}$$

$$\langle 4,4,4|\,\hat{Q}_2^2\,|4,4,2\rangle = -\frac{54}{11\sqrt{7}} \quad . \tag{2.4.7}$$

By solving for the reduced matrix element, one can find that

$$\langle 4,4\,\|\hat{Q}^2\|4,4\rangle = -27\sqrt{\frac{30}{77}} \quad . \tag{2.4.8}$$

As mentioned before, the reduced matrix element does not depend on the quantum numbers m and m'. Therefore, the reduced matrix element will be the same for every row (pair of states) in table (2.1).

Example

Let us show that the operator represented by

$$\hat{A}_{k,q}^{\ell} = \sum_{mm'}\left\langle {\ell \atop m};{\ell \atop -m'}\,\middle|\,{k \atop q}\right\rangle(-1)^{\ell-m'}\,|\ell m\rangle\,\langle \ell m'| \tag{2.4.9}$$

is a tensor operator. This is done by setting $\ell = 1$ and $k = 0, 1$ and 2.

For $\ell = 1$ and $k = 0$, the equation above gives the following operator

$$\hat{A}^1_{0,0} = \frac{1}{\sqrt{3}}\Big(\left|1,-1\right\rangle \left\langle 1,-1\right| + \left|1,0\right\rangle \left\langle 1,0\right| + \left|1,1\right\rangle \left\langle 1,1\right| \Big) \tag{2.4.10}$$

Let us check the commutation relations that characterize the tensor operators, which are written in equations (2.3.9) and (2.3.10):

$$\left[\hat{J}_z, \hat{A}^1_{0,0}\right] = \hat{J}_z \hat{A}^1_{0,0} - \hat{A}^1_{0,0}\hat{J}_z = 0 \tag{2.4.11}$$

and

$$\left[\hat{J}_\pm, \hat{A}^1_{0,0}\right] = 0 \tag{2.4.12}$$

Therefore, this tensor operator has rank zero and is a scalar. Using the Wigner-Ekcart theorem, we can evaluate the reduced matrix element to be

$$\langle 1 \,\|\hat{A}^0\| 1\rangle = \frac{\sqrt{3}\,\left\langle 1,1\right| \hat{A}^1_{0,0} \left|1,1\right\rangle}{\left\langle {0 \atop 0}; {1 \atop 1} \middle| {1 \atop 1} \right\rangle} = 1 \quad . \tag{2.4.13}$$

Table (2.2) summarizes the tensor operators and reduced matrix elements for $\ell = 1$ and $k = 0, 1$ and 2. As one can notice, the reduced matrix element does only depend on the values of k and ℓ.

Table 2.2: Spherical Tensor Operator and Reduced Matrix Elements.

(ℓ,k,q)	$\hat{A}^\ell_{k,q}$	$\langle \ell \,\|\hat{A}^k\| \ell\rangle$
(0,0,0)	$\left\|0,0\right\rangle \left\langle 0,0\right\|$	1
(1,0,0)	$\frac{1}{\sqrt{3}}\Big(\left\|1,1\right\rangle \left\langle 1,1\right\| + \left\|1,0\right\rangle \left\langle 1,0\right\| + \left\|1,-1\right\rangle \left\langle 1,-1\right\| \Big)$	1
(1,1,0)	$\frac{1}{\sqrt{2}}\Big(\left\|1,1\right\rangle \left\langle 1,1\right\| - \left\|1,-1\right\rangle \left\langle 1,-1\right\| \Big)$	$-\sqrt{3}$
(1,1,1)	$-\frac{1}{\sqrt{2}}\Big(\left\|1,1\right\rangle \left\langle 1,0\right\| - \left\|1,0\right\rangle \left\langle 1,-1\right\| \Big)$	$-\sqrt{3}$
(1,1,-1)	$\frac{1}{\sqrt{2}}\Big(\left\|1,0\right\rangle \left\langle 1,1\right\| + \left\|1,-1\right\rangle \left\langle 1,0\right\| \Big)$	$-\sqrt{3}$
(1,2,0)	$\frac{1}{\sqrt{6}} \left\|1,1\right\rangle \left\langle 1,1\right\|$	$\sqrt{5}$
(1,2,1)	$\frac{1}{\sqrt{2}}\Big(\left\|1,0\right\rangle \left\langle 1,-1\right\| - \left\|1,1\right\rangle \left\langle 1,0\right\| \Big)$	$\sqrt{5}$
(1,2,2)	$\left\|1,1\right\rangle \left\langle 1,-1\right\|$	$\sqrt{5}$
(1,2,-1)	$\frac{1}{\sqrt{2}}\Big(\left\|1,0\right\rangle \left\langle 1,1\right\| - \left\|1,-1\right\rangle \left\langle 1,0\right\| \Big)$	$\sqrt{5}$
(1,2,-2)	$\left\|1,-1\right\rangle \left\langle 1,1\right\|$	$\sqrt{5}$

Chapter 3

Quasi-distributions in SU(2)

3.1 Evolution of Quantum Systems in the Wigner Function Formalism

The motivation is in obtaining the evolution of a particle under the action of a Hamiltonian $\hat{H}$. For this evolution, there are two popular representations in quantum mechanics: The Heisenberg picture and Schrödinger picture. In the first scheme, the operators associated with meaningful physical quantities evolve over time, say $\hat{A} = \hat{A}(t)$, while the density matrix $\hat{\rho}$ remains unchanged in time. In contrast, the Schrödinger picture is based on the evolution of the density matrix over time, $\hat{\rho} = \hat{\rho}(t)$, while the operators remain constant in time [23].

3.2 Wigner Function of Finite Dimensional Systems

Let us consider a Hilbert space $\mathcal{H}$ of dimension $(2S+1)$, that carries a unitary irreducible representation of the group $SU(2)$ and is spanned by the orthonormal basis $\{|S,m\rangle, m = -S,...S\}$. The operators $\{\hat{S}_i, i = x,y,z\}$ are generators of the algebra $su(2)$. The quantum states of this system are chosen to be the simultaneous eigenstates of the operators $\hat{S}_z$ and $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ as it was presented in chapter 2

$$\hat{S}_z |S,m\rangle = m\,|S,m\rangle\,, \qquad \hat{S}^2 |S,m\rangle = S(S+1)\,|S,m\rangle \quad . \tag{3.2.1}$$

We now introduce the $SU(2)$ quantization kernel [14]

$$\hat{w}(\Omega) = \sqrt{\frac{4\pi}{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} \hat{T}^S_{L,M} Y^*_{L,M}(\Omega) \qquad \Omega := (\theta,\phi) \quad , \tag{3.2.2}$$

where $Y^*_{L,M}(\Omega)$ are the spherical harmonics and $\hat{T}^S_{L,M}$ are irreducible tensor operators defined by

$$\hat{T}^S_{L,M} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-S}^{S} \left\langle \begin{matrix} S \\ m \end{matrix} ; \begin{matrix} L \\ M \end{matrix} \middle| \begin{matrix} S \\ m' \end{matrix} \right\rangle |S,m'\rangle \langle S,m| \tag{3.2.3}$$

where $L = 0,1,...,2S$,$M = -L,...,L$ and the coefficients $\left\langle \begin{smallmatrix} S \\ m \end{smallmatrix} ; \begin{smallmatrix} L \\ M \end{smallmatrix} \middle| \begin{smallmatrix} S \\ m' \end{smallmatrix} \right\rangle$ were already introduced in chapter 2. These tensor operators form an orthogonal basis of matrices of size $(2S+1)\times(2S+1)$ that act on the states of the Hilbert space defined previously in this section. Following equation (2.3.4), the tensor operators of equation (3.2.3) are transformed under similarity operation by the operator

$$\hat{R}_z(\phi)\hat{R}_y(\theta)\hat{R}_z(-\phi) = \hat{D}(\theta,\phi) = \exp\Big[-\frac{1}{2}\theta(\hat{S}_+e^{-i\phi} - \hat{S}_-e^{i\phi})\Big] \tag{3.2.4}$$

as

$$\hat{D}(\theta,\phi)\hat{T}^S_{L,M}\hat{D}^\dagger(\theta,\phi) = \sum_{M'=-L}^{L} D^L_{M',M}(\theta,\phi)\hat{T}^S_{L,M'} \tag{3.2.5}$$

where $\hat{D}(\theta,\phi)$ is the displacement operator and $D^L_{M',M}$ is a Wigner D-function, which are defined as

$$D^L_{M',M} = \langle L, M'|\, \hat{D}(\theta,\phi)\, |L, M\rangle\,, \tag{3.2.6}$$

Therefore, one can write the Wigner symbol for an operator $\hat{f}$ acting in this space as

$$W_{\hat{f}}(\Omega) = \mathrm{Tr}\Big(\hat{w}(\Omega)\hat{f}\Big) \tag{3.2.7}$$

and the Wigner function for a density operator $\hat{\rho} = |\psi\rangle\langle\psi|$ as

$$W_{\hat{\rho}}(\Omega) = \mathrm{Tr}(\hat{w}(\Omega)\hat{\rho}) \tag{3.2.8}$$

where $\hat{w}$ is given in equation(3.2.2).

As an example, consider the $SU(2)$ coherent state $|\theta_0, \phi_0\rangle$ defined by the action of the displacement operator on the highest weight state

$$|\theta_0,\phi_0\rangle = \hat{D}(\theta_0,\phi_0)\, |S, S\rangle \tag{3.2.9}$$

and it can be brought to the form

$$|\theta_0,\phi_0\rangle = \sum_{m=-S}^{S} \Lambda_m(\theta_0,\phi_0)\, |S, m\rangle \tag{3.2.10}$$

with

$$\Lambda_m(\theta_0,\phi_0) = \sqrt{\frac{(2S)!}{(S-m)!(S+m)!}}\left(\sin\frac{\theta_0}{2}\right)^{S-m}\left(\cos\frac{\theta_0}{2}\right)^{S+m} e^{i(S-m)\phi_0} \tag{3.2.11}$$

Thus, the density operator for a coherent state can be constructed as $\hat{\rho} = |\theta_0,\phi_0\rangle\langle\theta_0,\phi_0|$ or explicitly

$$\hat{\rho} = \sum_{m',m=-S}^{S} \Lambda_m(\theta_0,\phi_0)\Lambda^*_{m'}(\theta_0,\phi_0)\, |S, m\rangle\langle S, m'| \tag{3.2.12}$$

As an example, we can construct the Wigner function for a coherent state using equations (3.2.2), (3.2.8) and (3.2.12). For simplicity, let us choose $S = 1$ and $(\theta_0, \phi_0) = (\frac{\pi}{2}, 0)$. The matrix representation of the density operator becomes

$$\hat{\rho} = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} \tag{3.2.13}$$

and the Wigner function $W_{\hat{\rho}}(\Omega)$ is written as

$$W_{\hat{\rho}}(\Omega) = \frac{1}{48}\left(6\sqrt{10}\sin^2\theta\cos 2\phi + 24\sqrt{2}\sin\theta\cos\phi - 3\sqrt{10}\cos 2\theta - \sqrt{10} + 16\right) \tag{3.2.14}$$

and graphically depicted in figure (3.1).

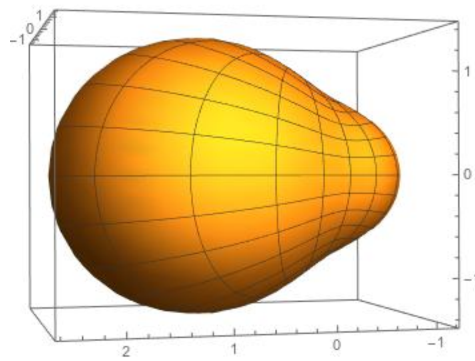

Figure 3.1: Wigner function of the quasi-distribution of equation (3.2.14).

3.2.1 Example: The Lipkin-Meshkov-Glick Model

This model was first proposed by Lipkin *et al* [16] in the context of nuclear physics. However, applications of this model have been introduced in the context of entanglement and phase transitions [19, 28].

The Hamiltonian for the LMG model is given by

$$\hat{H} = -g\hat{S}_x - \frac{1}{2S}\hat{S}_z^2 \quad , \tag{3.2.15}$$

here, g is a parameter. Let us choose $g = 1$ for simplicity and use the coherent states defined in equation (3.2.10).

The evolution of a coherent state $|\theta_0, \phi_0\rangle$ under the Hamiltonian of equation (3.2.15) is given by

$$|\psi(t)\rangle = \mathrm{e}^{-i\hat{H}t}|\theta_0, \phi_0\rangle \tag{3.2.16}$$

and the expectation value of the operator $\hat{S}_x$ is written as

$$\langle\psi(t)|\,\hat{S}_x\,|\psi(t)\rangle = \langle\theta_0, \phi_0|\,\mathrm{e}^{i\hat{H}t}\hat{S}_x\mathrm{e}^{-i\hat{H}t}\,|\theta_0, \phi_0\rangle \quad . \tag{3.2.17}$$

By locating the coherent state at the equator of the sphere $\left|\theta_0 = \frac{1}{2}\pi, \phi_0 = 0\right\rangle$ and choosing the spin variable to be $S = 5$, I was able to evaluate the exact evolution of equation (3.2.17) and produce figure (3.2).

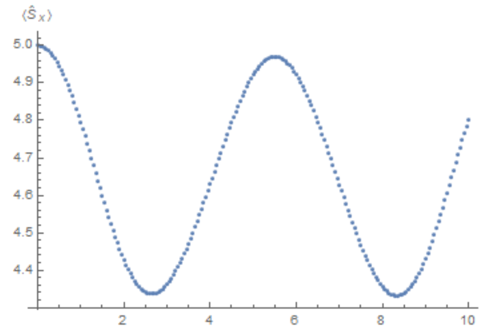

Figure 3.2: Expectation value of S_x calculated via quantum mechanics for $S = 5$ and a coherent state located at the equator $\left|\theta_0 = \frac{1}{2}\pi, \phi_0 = 0\right\rangle$

.

Now, using the Wigner function formalism, it is possible to describe the same system using the tools of statistical mechanics in phase space. First, let us construct the density operator

$$\hat{\rho} = |\psi(t)\rangle \langle\psi(t)| = e^{-i\hat{H}t} |\theta_0, \phi_0\rangle \langle\theta_0, \phi_0| e^{i\hat{H}t} \quad . \tag{3.2.18}$$

Now, it is possible to calculate the Wigner function of this density operator as

$$W_{\hat{\rho}}(\theta, \phi) = \mathrm{Tr}(\hat{w}(\theta, \phi)\hat{\rho}) = \sum_q \langle S, q| \hat{w}(\theta, \phi)e^{-i\hat{H}t} |\theta_0, \phi_0\rangle \langle\theta_0, \phi_0| e^{i\hat{H}t} |S, q\rangle \quad . \tag{3.2.19}$$

Using equations (3.2.2) and (3.2.10), it is possible to rewrite equation (3.2.19) as

$$W_{\hat{\rho}}(\theta, \phi; t) = \sqrt{\frac{4\pi}{2S+1}} \sum_{L=0}^{2S} \sqrt{\frac{2L+1}{2S+1}} \sum_{M=-L}^{L} Y^*_{LM}(\Omega) f(L, M, t) \tag{3.2.20}$$

where

$$f(L, M, t) = \sum_{q,q',\mu,\mu'} \Lambda_\mu(\theta_0, \phi_0)\Lambda^*_{q'}(\theta_0, \phi_0) \langle S, \mu'| e^{-i\hat{H}t} |S, \mu\rangle \langle S, q'| e^{i\hat{H}t} |S, q\rangle \left\langle \begin{matrix} S \\ \mu' \end{matrix}; \begin{matrix} L \\ M \end{matrix} \middle| \begin{matrix} S \\ q \end{matrix} \right\rangle \quad . \tag{3.2.21}$$

The symbol for the operator $\hat{S}_x$ is [14]

$$W_{\hat{S}_x}(\theta, \phi) = \sqrt{S(S+1)} \sin\theta \cos\phi \tag{3.2.22}$$

and the expectation value of this operator, which is time dependent since the Wigner function of equation (3.2.20) is time dependent, in phase space is calculated via integration

$$\langle\hat{S}_x\rangle = \frac{2S+1}{4\pi} \int_0^\pi \int_0^{2\pi} d\phi d\theta W_{\hat{S}_x}(\theta, \phi) W_{\hat{\rho}}(\theta, \phi; t) \quad . \tag{3.2.23}$$

By substituting equations (3.2.20) and (3.2.22) into equation (3.2.23) and after some mathematical effort, it is possible to find a final expression for the expectation value of the operator $\hat{S}_x$

$$\begin{aligned} \langle\hat{S}_x\rangle = & \frac{\sqrt{S(S+1)}}{4} \sum_{L=0}^{2S} (2L+1) \Big[\frac{f(L,1,t)}{\sqrt{L(L+1)}} \int_{-1}^{1} dx \sqrt{1-x^2} P_L^1(x) \\ & + \sqrt{L(L+1)} f(L,-1,t) \int_{-1}^{1} dx \sqrt{1-x^2} P_L^{-1}(x) \Big] \quad , \end{aligned} \tag{3.2.24}$$

where the functions $P_L^1(x)$ are the famous associate Legendre Polynomials.

The expectation value of the operator $\hat{S}_x$ over time is shown in figure (3.3) for a spin number $S = 5$. As one can see, this plot is exactly the same as the graphic of figure (3.2).

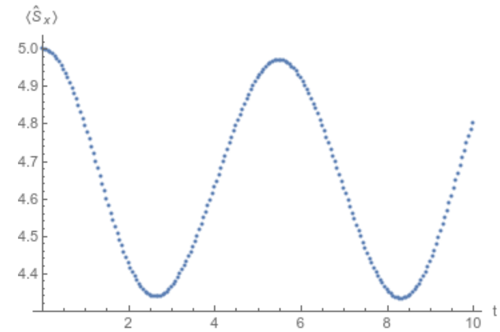

Figure 3.3: Expectation value of the operator $\hat{S}_x$ via Wigner function formalism

.

One can compare equations (3.2.17) and (3.2.24). Whereas the former requires the evaluation of a $(2S+1) \times (2S+1)$ matrix, the latter is a sum from 0 to $L = 2S$ of integrals which does not depend on the size $(2S+1)$ of the system. This illustrates how phase space methods become powerful tools in the limit of large S, *i.e.* in the semi-classical limit.

3.2.2 The $\star$-Product and Correspondence Rules in $SU(2)$

The correspondence rules were introduced in chapter 1 in the context of infinite-dimensional Hilbert space of harmonic oscillator systems. Although the concept of these rules remains the same as stated in chapter 1, the finite and (q,p) systems are different and therefore, the correspondence rules for finite dimensional systems will differ from the (q,p) counterpart.

For spin systems, Klimov and Espinoza [10] found the $\star$-product

$$\begin{aligned} W_{\hat{A}\hat{B}}(\theta,\phi) &:= W_{\hat{A}}(\theta,\phi) \star W_{\hat{B}}(\theta,\phi) \\ &= \sqrt{2S+1}\sum_j a_j \tilde{F}^{-1}(\mathcal{L}^2)(\hat{\mathbb{S}}^{+(j)}\tilde{F}(\mathcal{L}^2)W_{\hat{A}}(\theta,\phi))(\hat{\mathbb{S}}^{-(j)}\tilde{F}(\mathcal{L}^2)W_{\hat{B}}(\theta,\phi)) \end{aligned} \tag{3.2.25}$$

where $\tilde{F}(\mathcal{L}^2)$ is a function of the Casimir $\mathcal{L}^2$ operator on the sphere. It acts on the Spherical Harmonics functions $Y_{LM}(\theta,\phi)$ as it follows

$$\tilde{F}(\mathcal{L}^2)Y_{LM}(\theta,\phi) = F(L)Y_{LM}(\theta,\phi), \tag{3.2.26}$$

with $F(L) = \sqrt{(2S+L+1)!(2S-L)!}$. The expansion coefficients are easily calculated as

$$a_j = \frac{(-1)^j}{j!(2S+j+1)!}. \tag{3.2.27}$$

In addition, Klimov and Espinoza introduced the symbolic powers $\hat{\mathbb{S}}^{\pm(j)}$ as

$$\hat{\mathbb{S}}^{\pm(j)} = \prod_{k=0}^{j-1}\left(k\cot\theta - \frac{\partial}{\partial\theta} \mp \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right) \quad . \tag{3.2.28}$$

In chapter 5, I derive the ⋆-product for finite systems in $SU(3)$ and remarkably this product has a form very similar to the one introduced in equation (3.2.25).

3.2.3 Example

Let us give an application of the correspondence rules. For simplicity, let us choose the raising $\hat{S}_+$ and lowering $\hat{S}_-$ operators to be $\hat{A}$ and $\hat{B}$, respectively. The Wigner symbols of these operators are [10]

$$\begin{aligned} W_{\hat{S}_+}(\theta,\phi) &= \sqrt{S(S+1)}\mathrm{e}^{i\phi}\sin\theta \\ W_{\hat{S}_-}(\theta,\phi) &= \sqrt{S(S+1)}\mathrm{e}^{-i\phi}\sin\theta \end{aligned} \tag{3.2.29}$$

These symbols can be expressed in terms of the Spherical Harmonics $Y_{LM}(\theta,\phi)$:

$$\begin{aligned} W_{\hat{S}_+}(\theta,\phi) &= -\sqrt{\frac{8\pi}{3}S(S+1)}Y_{1,1}(\theta,\phi) \\ W_{\hat{S}_-}(\theta,\phi) &= \sqrt{\frac{8\pi}{3}S(S+1)}Y_{1,-1}(\theta,\phi) \end{aligned} \tag{3.2.30}$$

and due to the property of the operator $\tilde{F}(\mathcal{L}^2)$ given in equation (3.2.26), we can find

$$\tilde{F}(\mathcal{L}^2)Y_{1,M}(\theta,\phi) = F(1)Y_{1,M}(\theta,\phi) \quad . \tag{3.2.31}$$

It is possible now to write equation (3.2.25) in terms of the Wigner symbols of the raising and lowering operators

$$W_{\hat{S}_+}(\theta,\phi)\star W_{\hat{S}_-}(\theta,\phi) = \sqrt{2S+1}F^2(1)\tilde{F}^{-1}(\mathcal{L}^2)\Big[\frac{W_{\hat{S}_+}(\theta,\phi)W_{\hat{S}_-}(\theta,\phi)}{(2S+1)!} - \frac{\hat{\mathbb{S}}^{+(1)}W_{\hat{S}_+}(\theta,\phi)\hat{\mathbb{S}}^{-(1)}W_{\hat{S}_-}(\theta,\phi)}{(2S+2)!}\Big] \tag{3.2.32}$$

which is an example of correspondence rules, since one can substitute the right hand side of this equation by an operator that acts on the symbol $W_{\hat{S}_-}(\theta,\phi)$.

After some mathematical effort on equation (3.2.32), one can find

$$W_{\hat{S}_+}(\theta,\phi)\star W_{\hat{S}_-}(\theta,\phi) = \frac{\sqrt{S(S+1)}}{2}\Big[\frac{4\sqrt{S(S+1)}}{3} + 2\cos\theta - \sqrt{(2S+3)(2S-1)}(\cos^2\theta - \frac{1}{3})\Big] \tag{3.2.33}$$

Let us now choose $S=3$ to obtain

$$W_{\hat{S}_+}(\theta,\phi)\star W_{\hat{S}_-}(\theta,\phi) = 8 + 2\sqrt{3}\cos\theta - 3\sqrt{15}(\cos^2\theta - \frac{1}{3}) \tag{3.2.34}$$

The next step is to compare equation (3.2.34) to the direct calculation of $W_{\hat{S}_+\hat{S}_-}(\theta,\phi)$, which is written as

$$W_{\hat{S}_+\hat{S}_-}(\theta,\phi) = \mathrm{Tr}\Big(\hat{w}(\theta,\phi)\hat{S}_+\hat{S}_-\Big) \tag{3.2.35}$$

and easily evaluated resulting in

$$W_{\hat{S}_+\hat{S}_-}(\theta,\phi) = 8 + 2\sqrt{3}\cos\theta - 3\sqrt{15}(\cos^2\theta - \frac{1}{3}) \quad . \tag{3.2.36}$$

This expression is exactly the same result of equation (3.2.34).

Klimov and Espinoza point out that there is another viable approach to find the expression of equation (3.2.34) because we chose the raising and lowering operators to evaluate equation (3.2.25). Since the product $\hat{S}_+\hat{S}_-$ is expanded as

$$\begin{aligned}\hat{S}_+\hat{S}_- &= (\hat{S}_x + i\hat{S}_y)(\hat{S}_x - i\hat{S}_y)\\ &= \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z\end{aligned} \tag{3.2.37}$$

and the Casimir $\mathcal{L}^2$ operator written as

$$\mathcal{L}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \quad , \tag{3.2.38}$$

one can express the product $\hat{S}_+\hat{S}_-$ as

$$\hat{S}_+\hat{S}_- = \mathcal{L}^2 - \hat{S}_z^2 + \hat{S}_z \tag{3.2.39}$$

and find the Wigner symbol of the product

$$W_{\hat{S}_+\hat{S}_-}(\theta,\phi) = W_{\hat{\mathcal{L}}^2}(\theta,\phi) - W_{\hat{S}_z^2}(\theta,\phi) + W_{\hat{S}_z}(\theta,\phi) \quad . \tag{3.2.40}$$

The expressions for the symbols on the right hand side of equation (3.2.40) are

$$\begin{aligned}W_{\mathcal{L}^2}(\theta,\phi) &= S(S+1)\\ W_{\hat{S}_z^2}(\theta,\phi) &= \frac{1}{2}\sqrt{S(S+1)(2S+3)(2S-1)}(\cos^2\theta - \frac{1}{3}) + \frac{S(S+1)}{3}\\ W_{\hat{S}_z}(\theta,\phi) &= \sqrt{S(S+1)}\cos\theta\end{aligned} \tag{3.2.41}$$

and by choosing $S = 3$ we recover the expressions of equation (3.2.34).

Chapter 4

Some results on $SU(3)$ Clebsch-Gordan Coefficients and Tensor Operators

In chapter 2, I showed the importance of the Clebsch-Gordan coefficients and tensor operators in the construction of the quantization kernel for $SU(2)$. The same concepts are again important in $SU(3)$. This chapter is dedicated to the construction of the $SU(3)$ Clebsch-Gordan coefficients and some examples with particular emphasis on constructing tensors and other coefficients directly relevant to this thesis.

This section is based on a forthcoming paper which I contributed. The basic algorithm was developed by Dr. de Guise a few years ago and my task was to verify the details and implement it with application to correspondence rules with emphasis on symbolic rather than numerical results.

4.1 Basis states in SU(3)

$SU(3)$ states are constructed following Rowe *et al* in [20] where these states are obtained by coupling three $SU(2)$ states. The generators of $SU(3)$ are defined as

$$\hat{C}_{ij} = \hat{a}^{\dagger}_{i1}\hat{a}_{j1} + \hat{a}^{\dagger}_{i2}\hat{a}_{j2} \tag{4.1.1}$$

where $i, j = 1, 2, 3$ and the operators satisfy the commutation relations $\left[\hat{C}_{ij}, \hat{C}_{k\ell}\right] = \delta_{jk}\hat{C}_{i\ell} - \delta_{i\ell}\hat{C}_{kj}$.

We can find two diagonal operators $\hat{H}_1$ and $\hat{H}_2$ that act diagonally like $\hat{J}_z$ in the theory of angular momentum (see equation (2.1.4))

$$\hat{H}_1 = \hat{C}_{11} - \hat{C}_{22} \qquad \hat{H}_2 = \hat{C}_{22} - \hat{C}_{33}\,. \tag{4.1.2}$$

The remaining operators are classified as raising and lowering operators

Table 4.1: Generators of the $su(3)$ algebra.

Raising	Lowering	Diagonal
$\hat{C}_{13}$	$\hat{C}_{21}$	$\hat{H}_1$
$\hat{C}_{12}$	$\hat{C}_{31}$	$\hat{H}_2$
$\hat{C}_{23}$	$\hat{C}_{32}$	

These generators are displayed in the root diagram as shown in figure (4.1).

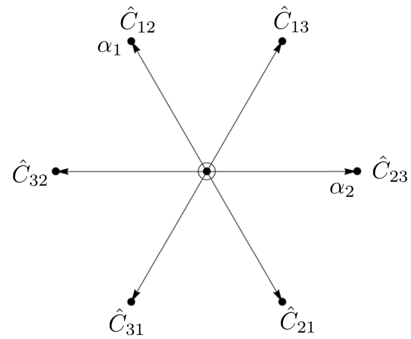

Figure 4.1: The su(3) root diagram that shows the two fundamental weights α_1 and α_2 and the eight generators of this algebra.

This diagram encapsulates important properties of the commutation relations: to each $\hat{C}_{ij}$ is associated a vector as shown. If the sum of two of these is another vector of the root diagram, then $[\hat{C}_{ij}, \hat{C}_{k\ell}] \neq 0$; if the sum is not another vector, then $[\hat{C}_{ij}, \hat{C}_{k\ell}] = 0$. Thus, for instance, one immediately sees that $[\hat{C}_{13}, \hat{C}_{23}] = 0$ since the vectors $\alpha_1 + \alpha_2$ and α_2 associated with $\hat{C}_{13}$ and $\hat{C}_{23}$ respectively do not add to another vector in the diagram. On the other hand, $[\hat{C}_{12}, \hat{C}_{23}]$ will be proportional to $\hat{C}_{13}$ since the vector sum of $\alpha_1 + \alpha_2$ is precisely the vector associated with $\hat{C}_{13}$.

For fixed i, one writes the states of the $su(2)$ algebra in terms of two harmonic oscillator creation operators $\{\hat{a}^\dagger_{i1}, \hat{a}^\dagger_{i2}\}$ acting on the vacuum state $|0\rangle$

$$|s_i m_i\rangle = \frac{(\hat{a}^\dagger_{i1})^{s_i+m_i}(\hat{a}^\dagger_{i2})^{s_i-m_i}}{\sqrt{(s_i+m_i)!(s_i-m_i)!}}\,|0\rangle \quad . \tag{4.1.3}$$

One can also define an $su(2)$ algebra that is spanned by

$$\hat{B}_{rs} = \sum_i \hat{a}^\dagger_{ir}\hat{a}_{is} \tag{4.1.4}$$

These $\hat{B}_{rs}$ operators commute with the $\hat{C}_{ij}$ operators of equation (4.1.1), and the $su(3)$ basis states are constructed from $su(2)$ states of the $\hat{B}_{rs}$ operators.

The $su(3)$ state $|(\lambda,\mu)\nu_1\nu_2\nu_3; I_{23}\rangle$, which is labeled by the three occupation numbers ν_1, ν_2 and ν_3 and constrained to the condition $\nu_1+\nu_2+\nu_3 = \lambda+2\mu$, is found as the coupling of $su(2)$ states [18]

$$\begin{aligned} |(\lambda,\mu)\nu; I\rangle :&= |(\lambda,\mu)\nu_1,\nu_2,\nu_3; I\rangle \\ &= \sum_{\substack{m_1,m_2\\ m_3,N}} \left\langle \begin{matrix}\frac{1}{2}\nu_3\\ m_3\end{matrix}; \begin{matrix}\frac{1}{2}\nu_2\\ m_2\end{matrix} \,\middle|\, \begin{matrix}I_{23}\\ N\end{matrix}\right\rangle \left\langle \begin{matrix}I_{23}\\ N\end{matrix}; \begin{matrix}\frac{1}{2}\nu_1\\ m_1\end{matrix} \,\middle|\, \begin{matrix}\frac{1}{2}\lambda\\ \frac{1}{2}\lambda\end{matrix}\right\rangle \left|\tfrac{1}{2}\nu_1 m_1\right\rangle \left|\tfrac{1}{2}\nu_2 m_2\right\rangle \left|\tfrac{1}{2}\nu_3 m_3\right\rangle \end{aligned} \tag{4.1.5}$$

where the expansion coefficients are SU(2) Clebsch-Gordan coefficients and for convenience I write the components of this state in a short notation $\nu = (\nu_1\nu_2\nu_3)$. Moreover, the states of equation (4.1.5) are eigenstates of the diagonal operators of equation (4.1.2); the eigenvalues are known as the *weight* of the state, and this weight is just $(\nu_1-\nu_2, \nu_2-\nu_3)$. In addition, some weights may occur more than once (i.e.

degenerate eigenvalues). To distinguish states having the same weight, one may further specify an SU(2) angular momentum as the states of equation (4.1.5) are also states of good "angular momentum", with the $su(2) \oplus u(1)$ subalgebra spanned by the operators $\{\hat{C}_{23}, \hat{C}_{32}, \hat{H}_1, \hat{H}_2\}$.

All the states of an $su(2) \oplus u(1)$ multiplet possess the same occupation number ν_1, but the states may be degenerate in eigenvalues. This is shown in figure (4.2) where one can see that the states $|(4,2)431;1\rangle$ and $|(4,2)431;2\rangle$ have the same weight but their angular momenta are different. Degenerate states in the weight diagram are represented as black circles with as many rings as many degenerate states.

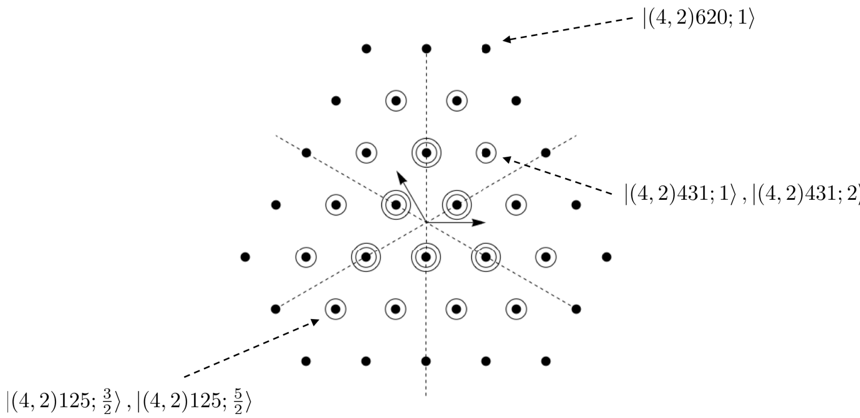

Figure 4.2: The weight diagram for the irrep $(4,2)$.

The highest weight state is defined as the state killed by all the raising operators. One can verify that this state has the general form

$$|(\lambda,\mu)\text{hws}\rangle = \left|(\lambda,\mu)\lambda+\mu,\mu,0;\tfrac{1}{2}\mu\right\rangle \quad , \tag{4.1.6}$$

and it is easily established that the sum of occupation numbers is $\lambda + 2\mu$. Since the operators $\hat{C}_{ij}$ do not change the *total* number of harmonic oscillator excitations in the system, all states in the irrep (λ,μ) will have $\lambda + 2\mu$ excitations.

4.2 Tensor Operators

To proceed with the evaluation of CG coefficients, it is useful to introduce the $su(2)$ tensor operators

$$\begin{aligned}\hat{T}^L_M &= \left\langle \begin{matrix}\frac{1}{2}(L+M)\\ \frac{1}{2}(L+M)\end{matrix};\begin{matrix}\frac{1}{2}(L-M)\\ -\frac{1}{2}(L-M)\end{matrix}\,\middle|\,\begin{matrix}L\\ M\end{matrix}\right\rangle \frac{(2L)!}{(L+M)!(L-M)!}(\hat{C}_{31})^{L-M}(\hat{C}_{21})^{L+M}\\ &= \sqrt{\frac{(2L)!}{(L+M)!(L-M)!}}(\hat{C}_{31})^{L-M}(C_{21})^{L+M} \quad ,\end{aligned} \tag{4.2.1}$$

The $su(2)$ subalgebra is again spanned by $\{\hat{C}_{23}, \hat{C}_{32}, \hat{h}_2\}$. A few tensors are given below:

Table 4.2: Construction of $SU(2)$ tensor operators in terms of $SU(3)$ raising and lowering operators

L	M	$\hat{T}^L_M$	L	M	$\hat{T}^L_M$
1	1	$\hat{C}^2_{21}$	$\frac{3}{2}$	$\frac{3}{2}$	$\hat{C}^3_{21}$
1	0	$\sqrt{2}\hat{C}_{31}\hat{C}_{21}$	$\frac{3}{2}$	$\frac{1}{2}$	$\sqrt{3}\hat{C}_{31}\hat{C}^2_{21}$
1	-1	$\hat{C}^2_{31}$	$\frac{3}{2}$	$-\frac{1}{2}$	$\sqrt{3}\hat{C}^2_{31}\hat{C}_{21}$
			$\frac{3}{2}$	$-\frac{3}{2}$	$\hat{C}^3_{31}$

Table 4.3: Commutation relations between $\hat{T}^1_1$ and the operators of the $su(2)$ subalgebra

	$\hat{T}^1_1$
$\hat{H}_2$	$2\hat{T}^1_1$
$\hat{C}_{23}$	0
$\hat{C}_{32}$	$\sqrt{2}\hat{T}^1_0$

From table (4.3), we can notice that the generator $\hat{H}_2$ corresponds to $2\hat{J}_z$. Meanwhile, $\hat{C}_{23}$ and $\hat{C}_{32}$ correspond to the raising and lowering operators, respectively.This result can be compared with the definitions of equations (2.3.9) and (2.3.10) of chapter 2 of this thesis.

We see that these $SU(2)$ tensors of table (4.2) are polynomials in the $SU(3)$ lowering operators. If *one* operator acts on *one* state of the form given in equation (4.1.5), one will obtain a *linear combinations* of states. For instance

$$\hat{C}_{13}\,|(\lambda,\mu)\nu_1,\nu_2,\nu_3;I\rangle = \sum_{I'} |(\lambda,\mu)\nu_1+1,\nu_2,\nu_3-1;I'\rangle \times \langle(\lambda,\mu)\nu_1+1,\nu_2,\nu_3-1;I'|\,\hat{C}_{13}\,|(\lambda,\mu)\nu_1,\nu_2,\nu_3;I\rangle \tag{4.2.2}$$

Now, let $L=\frac{1}{2}p$ and let us consider the specific linear combination of operators and states

$$\begin{aligned}&\sum_{M_I}\left\langle \begin{smallmatrix} I \\ M_I\end{smallmatrix}; \begin{smallmatrix} L \\ M\end{smallmatrix} \middle| \begin{smallmatrix} J \\ M_J\end{smallmatrix}\right\rangle \hat{T}^L_M\,|(\Lambda,\mu)\eta_1\eta_2\eta_3;I\rangle \\ &= |(\Lambda,\mu)\eta_1-2L,\eta_2+L+M_J-M_I,\eta_3+L-M_J+M_I;J\rangle \\ &\quad\times\frac{\langle(\Lambda,\mu)\eta_1-2L;J\,\|\hat{T}^L\|(\Lambda,\mu)\eta_1;I\rangle}{\sqrt{2J+1}}\quad,\end{aligned} \tag{4.2.3}$$

where J and M_J are fixed and J is one of the coupling $J+I, J+I-1,\ldots,|I-J|$ and $M_I=\frac{1}{2}(\eta_2-\eta_3)$.

In addition, the factor $\langle(\Lambda,\mu)\eta_1-2L;J\,\|\hat{T}^L\|(\Lambda,\mu)\eta_1;I\rangle$ is a reduced matrix element and the analytical form of this element is [18]

$$\begin{aligned}
&\langle(\lambda,\mu)\lambda+\mu-p-2k;J'\,\|\hat{T}^{k}\|(\lambda,\mu)\lambda+\mu-p;J\rangle\\
&=(2J+1)\sqrt{2k+p+1}(-1)^{\frac{1}{2}(\mu+p)+k+J'}\begin{Bmatrix} k & \frac{1}{2}p & k+\frac{1}{2}p\\ \frac{1}{2}\mu & J' & J\end{Bmatrix}\\
&\times\frac{\langle(\lambda,\mu)\lambda+\mu-p-2k;J'\,\|\hat{T}^{k+\frac{1}{2}p}\|(\lambda,\mu)\lambda+\mu;\frac{1}{2}\mu\rangle}{\langle(\lambda,\mu)\lambda+\mu-p;J\,\|\hat{T}^{\frac{1}{2}p}\|(\lambda,\mu)\lambda+\mu;\frac{1}{2}\mu\rangle}
\end{aligned}\tag{4.2.4}$$

where

$$\begin{aligned}
&\langle(\lambda,\mu)\lambda+\mu-p;J\,\|\hat{T}^{\frac{1}{2}p}\|(\lambda,\mu)\lambda+\mu;\tfrac{1}{2}\mu\rangle\\
&=(-1)^{(p-2J+\mu)/2}\sqrt{\frac{(2J+1)(\lambda+\mu+1)!\lambda!p!}{(\lambda-J+\frac{1}{2}(\mu-p))!(\lambda+J+\frac{1}{2}(\mu-p)+1)!}}\,.
\end{aligned}\tag{4.2.5}$$

Also, the factor $\begin{Bmatrix} k & \frac{1}{2}p & k+\frac{1}{2}p\\ \frac{1}{2}\mu & J' & J\end{Bmatrix}$ in equation (4.2.4) is a 6-j symbol. The action of the operator of equation (4.2.3) on a specific state shifts this state down on the weight diagram by p layers to a *specific state* instead of a linear combination of states.

4.2.1 Example

Let us choose the irrep $(\lambda,\mu)=(2,1)$. The weight diagram of this irrep is shown in figure (4.3). Suppose we start with the states $|(2,1)2\nu_2\nu_3;I\rangle$ illustrated in red on the diagram, and wish to reach the target state $|(2,1)022;1\rangle$. This requires going down two layers on the diagram so $p=2$ and $L=p/2=1$. We can expand the left hand side of equation (4.2.3) and find

$$\begin{aligned}
&\left\langle {}^{1}_{1};{}^{1}_{-1}\,\middle|\,{}^{1}_{0}\right\rangle\hat{T}^{1}_{-1}\,|(2,1)220;1\rangle+\left\langle {}^{1}_{0};{}^{1}_{0}\,\middle|\,{}^{1}_{0}\right\rangle\hat{T}^{1}_{0}\,|(2,1)211;1\rangle+\left\langle {}^{1}_{-1};{}^{1}_{1}\,\middle|\,{}^{1}_{0}\right\rangle\hat{T}^{1}_{1}\,|(2,1)202;1\rangle\\
&=-2\sqrt{2}\,|(2,1)022;1\rangle\quad.
\end{aligned}\tag{4.2.6}$$

The reduced matrix element for this example is

$$\langle(2,1)0;1\,\|\hat{T}^{1}\|(2,1)2;1\rangle=-2\sqrt{6}\tag{4.2.7}$$

and since we already chose the target state, which is $|(2,1)022;1\rangle$, we can recover the result of equation (4.2.6). Figure (4.3) provides a geometric example of the calculations above.

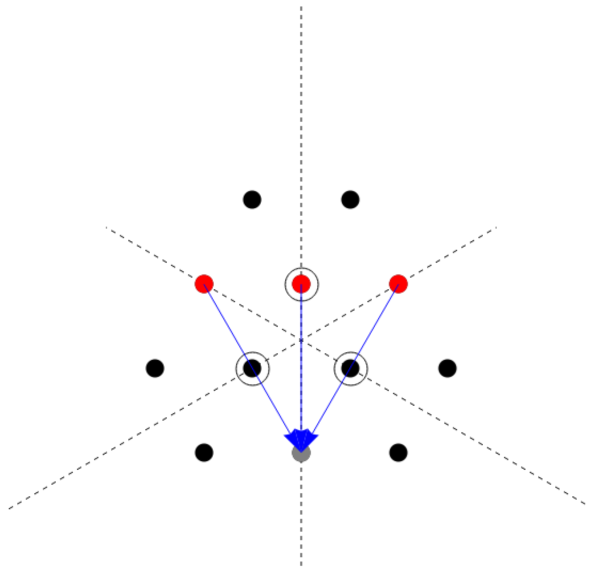

Figure 4.3: Reaching the specific state $|(2,1)022;1\rangle$ of the irrep $(2,1)$ on the weight diagram by the action of the operator of equation (4.2.3).

4.3 The Construction of SU(3) Clebsch-Gordan Coefficients

4.3.1 Highest Weight State SU(3) CGs

If (p_2, q_2) is a copy of an irrep $(\bar{p}_2, \bar{q}_2)$ occurring in the tensor product $(p_1, q_1) \otimes (\lambda, \mu)$, we can write the highest weight state as a linear combination of coupled $su(3)$ states of the irreps (p_1, q_1) and (λ, μ)

$$|(p_2, q_2)\text{hws}\rangle = \sum_{\substack{\nu J \\ (n) I_n}} \left\langle \begin{smallmatrix} (p_1,q_1) \\ \nu;J \end{smallmatrix} ; \begin{smallmatrix} (\lambda,\mu) \\ n;I_n \end{smallmatrix} \middle| \begin{smallmatrix} (p_2,q_2) \\ p_2+q_2,q_2,0;\frac{1}{2}q_2 \end{smallmatrix} \right\rangle |(p_1, q_1)\nu; J\rangle \, |(\lambda, \mu)n; I_n\rangle \quad . \tag{4.3.1}$$

This highest weight state has the same form as the $SU(2)$ counterpart presented in equation (2.1.6). There are some constraints for the indices ν and n for the SU(3) CGs of equation (4.3.1). For instance, the total number of excitations in the composite system is $p_1 + 2q_1 + \lambda + 2\mu$ and must equal $p_2 + 2q_2 + 3k$, where k is an integer and satisfies

$$k = \frac{1}{3}(p_1 + \lambda - p_2 + 2(q_1 + \mu - q_2)) \tag{4.3.2}$$

and

$$\nu_1 + n_1 = p_2 + q_2 + k, \quad \nu_2 + n_2 = q_2 + k, \quad \nu_3 + n_3 = k \quad . \tag{4.3.3}$$

The reason for these constraints is that the highest weight of the irrep (p_2, q_2) can be multiplied by a 3×3 determinant

$$\begin{vmatrix} a^\dagger_{11} & a^\dagger_{12} & a^\dagger_{13} \\ a^\dagger_{21} & a^\dagger_{22} & a^\dagger_{23} \\ a^\dagger_{31} & a^\dagger_{32} & a^\dagger_{33} \end{vmatrix}^k \quad , \tag{4.3.4}$$

which transforms by the $SU(3)$ irrep $(0,0)$ but adds $3k$ excitations to the system without changing the irrep label (p_2,q_2). Therefore, the constraint

$$p_1 + 2q_1 + \lambda + 2\mu = p_2 + q_2 + 3k \quad , \tag{4.3.5}$$

guarantees that the number of excitations of the separate systems, irreps (p_1,q_1) and (λ,μ), is the same as that in the resulting irrep (p_2,q_2). For this thesis, I made the choice of using the label k implicitly in the expressions of the CGs, although this label is important in understanding the Weyl symmetries of the CGs [18].

Now, let us find the highest weight CGs for a given decomposition $(p_1,q_1)\otimes(\lambda,\mu)\to(p_2,q_2)$. Following the construction of $SU(2)$ CGs in chapter 2, we can act on equation (4.3.1) with any raising operator of the $su(3)$ algebra. Let us choose $\hat{C}_{12}$ for this calculation. Then,

$$\begin{aligned} 0 &= \hat{C}_{12}\left|(p_2,q_2)\text{hws}\right\rangle \\ &= \sum_{\substack{\nu J\\(n)I_n}} \left\langle \begin{smallmatrix}(p_1,q_1)\\ \nu;J\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n;I_n\end{smallmatrix}\middle|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2,q_2,0;\frac{1}{2}q_2\end{smallmatrix}\right\rangle[\hat{C}_{12}\left|(p_1,q_1)\nu;J\right\rangle]\left|(\lambda,\mu)nI_n\right\rangle \\ &\qquad + \sum_{\substack{\nu J\\(n)I_n}} \left\langle \begin{smallmatrix}(p_1,q_1)\\ \nu;J\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n;I_n\end{smallmatrix}\middle|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2,q_2,0;\frac{1}{2}q_2\end{smallmatrix}\right\rangle\left|(p_1,q_1)\nu;J\right\rangle[\hat{C}_{12}\left|(\lambda,\mu)nI_n\right\rangle] \quad . \end{aligned} \tag{4.3.6}$$

We can multiply both sides of equation (4.3.6) by $\langle(p_1,q_1)\nu' J'|\,\langle(\lambda,\mu)n'|$ and get the basic recursion relation

$$\begin{aligned} 0 &= \sum_{(\nu)J} \left\langle \begin{smallmatrix}(p_1,q_1)\\ \nu;J\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n';I'_n\end{smallmatrix}\middle|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2,q_2,0;\frac{1}{2}q_2\end{smallmatrix}\right\rangle\left\langle(p_1,q_1)\nu;J\right|\hat{C}_{12}\left|(p_1,q_1)\nu';J'\right\rangle \\ &\qquad + \sum_{(n)I_n} \left\langle \begin{smallmatrix}(p_1,q_1)\\ \nu';J'\end{smallmatrix};\begin{smallmatrix}\lambda,\mu\\ n;I_n\end{smallmatrix}\middle|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2,q_2,0;\frac{1}{2}q_2\end{smallmatrix}\right\rangle\left\langle(\lambda,\mu)n;I_n\right|\hat{C}_{21}\left|(\lambda,\mu)n';I'_n\right\rangle \quad . \end{aligned} \tag{4.3.7}$$

We can rewrite this recursion relation as

$$\begin{aligned} 0 = &\sum_{(\nu_1)m_\nu J} \left\langle \begin{smallmatrix}(p_1,q_1)\\ \nu_1;J\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n'_1;I'_n\end{smallmatrix}\middle\|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2;\frac{1}{2}q_2\end{smallmatrix}\right\rangle\left\langle \begin{smallmatrix}J\\ m_\nu\end{smallmatrix};\begin{smallmatrix}I'_n\\ m'_n\end{smallmatrix}\middle|\begin{smallmatrix}\frac{1}{2}q_2\\ \frac{1}{2}q_2\end{smallmatrix}\right\rangle\left\langle \begin{smallmatrix}J'\\ m'_\nu\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\middle|\begin{smallmatrix}J\\ m_\nu\end{smallmatrix}\right\rangle \\ &\qquad\times\frac{\langle(p_1,q_1)\nu_1;J\,\|\hat{T}^{\frac{1}{2}}\|(p_1,q_1)\nu'_1;J'\rangle}{\sqrt{2J+1}} \\ +&\sum_{(n_1)I_n m_n} \left\langle \begin{smallmatrix}(p_1,q_1)\\ \nu'_1;J'\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n_1;I_n\end{smallmatrix}\middle\|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2;\frac{1}{2}q_2\end{smallmatrix}\right\rangle\left\langle \begin{smallmatrix}J'\\ m'_\nu\end{smallmatrix};\begin{smallmatrix}I_n\\ m_n\end{smallmatrix}\middle|\begin{smallmatrix}\frac{1}{2}q_2\\ \frac{1}{2}q_2\end{smallmatrix}\right\rangle\left\langle \begin{smallmatrix}I_n\\ m_n\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\middle|\begin{smallmatrix}I'\\ m'_n\end{smallmatrix}\right\rangle \\ &\qquad\times\frac{\langle(\lambda,\mu)n_1;I_n\,\|\hat{T}^{\frac{1}{2}}\|(\lambda,\mu)n'_1;I'_n\rangle}{\sqrt{2I_n+1}} \end{aligned} \tag{4.3.8}$$

where $m_\nu = \frac{1}{2}(\nu_2-\nu_3)$, $m_n = \frac{1}{2}(n_2-n_3)$, etc.

Multiplying equation (4.3.8) by $\left\langle \begin{smallmatrix}J'\\ m'_\nu\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\ m\end{smallmatrix}\middle|\begin{smallmatrix}\tilde{J}\\ \tilde{m}_\nu\end{smallmatrix}\right\rangle$ followed by summation over m'_ν and m, and using the orthogonality property of $SU(2)$ CGs under summation, produces

$$0 = \sum_{\nu_1} \frac{\langle (p_1,q_1)\nu_1;\tilde{J}\,\|\hat{T}^{\frac{1}{2}}\|(p_1,q_1)\nu_1';J'\rangle}{\sqrt{2\tilde{J}+1}} \left\langle {(p_1,q_1) \atop \nu_1;\tilde{J}} ; {(\lambda,\mu) \atop n_1';I_n'} \right\| \left. {(p_2,q_2) \atop p_2+q_2;\frac{1}{2}q_2} \right\rangle \left\langle {\tilde{J} \atop \tilde{m}_\nu} ; {I_n' \atop k} \right| \left. {\frac{1}{2}q_2 \atop \frac{1}{2}q_2} \right\rangle$$
$$+ \sum_{(n_1)I_n} \frac{\langle (\lambda,\mu)n_1;I_n\,\|\hat{T}^{\frac{1}{2}}\|(\lambda,\mu)n_1';I_n'\rangle}{\sqrt{2I_n+1}} \left\langle {(p_1,q_1) \atop \nu_1';\tilde{J}'} ; {(\lambda,\mu) \atop n_1;I_n} \right\| \left. {(p_2,q_2) \atop p_2+q_2;\frac{1}{2}q_2} \right\rangle$$
$$\times \sum_{km_\nu'm_n} \left\langle {I_n' \atop m_n'} ; {\frac{1}{2} \atop k} \right| \left. {I_n \atop m_n} \right\rangle \left\langle {J' \atop m_\nu'} ; {I_n \atop m_n} \right| \left. {\frac{1}{2}q_2 \atop \frac{1}{2}q_2} \right\rangle \left\langle {J' \atop m_\nu'} ; {\frac{1}{2} \atop k} \right| \left. {\tilde{J} \atop \tilde{m}_\nu} \right\rangle \quad . \tag{4.3.9}$$

It is possible to rearrange the arguments of the $SU(2)$ CGs so the result of the sum of triple product is a $SU(2)$ CG multiplied by a 6-j symbol leading to a direct recursion relation for the reduced CGs

$$\left\langle {(p_1,q_1) \atop \nu_1'-1;\tilde{J}} ; {(\lambda,\mu) \atop n_1';I_n'} \right\| \left. {(p_2,q_2) \atop p_2+q_2;\frac{1}{2}q_2} \right\rangle$$
$$= (2\tilde{I}_n+1) \sum_J \frac{\langle (p_1,q_1)\nu_1'-1;J\,\|\hat{T}^{\frac{1}{2}}\|(p_1,q_1)\nu_1';J'\rangle}{\langle (\lambda,\mu)n_1'-1;\tilde{I}_n\,\|\hat{T}^{\frac{1}{2}}\|(\lambda,\mu)n_1';I_n'\rangle}$$
$$\times (-1)^{J'-\tilde{I}_n+\frac{1}{2}q_2+2I_n'} \left\langle {(p_1,q_1) \atop \nu_1'-1;J} ; {(\lambda,\mu) \atop n_1';I_n'} \right\| \left. {(p_2,q_2) \atop p_2+q_2;\frac{1}{2}q_2} \right\rangle \left\{ \begin{array}{ccc} \frac{1}{2} & J' & \tilde{J} \\ \frac{1}{2}q_2 & I_n' & I_n \end{array} \right\} \quad . \tag{4.3.10}$$

In this recursion relation, the steps in the angular momenta are at most $J' = J \pm \frac{1}{2}$ and $I_n' = \tilde{I}_n \pm \frac{1}{2}$. Moreover, the recursion relation of equation (4.3.7) does not depend on the order of the irreps (p_1,q_1) and (λ,μ). The CGs $\left\langle {(p_1,q_1) \atop \nu;J} ; {(\lambda,\mu) \atop n';I_n'} \right| \left. {(p_2,q_2) \atop p_2+q_2,q_2,0;\frac{1}{2}q_2} \right\rangle$ and $\left\langle {(\lambda,\mu) \atop n';I_n'} ; {(p_1,q_1) \atop \nu;J} \right| \left. {(p_2,q_2) \atop p_2+q_2,q_2,0;\frac{1}{2}q_2} \right\rangle$ have the same numerical value, since they follow the same recursion relation of equation (4.3.10). However, they differ by at most a phase, which depends on the seed coefficient of the recursion relation. The phase convention that de Guise and I used in [18] and that will also be used in this thesis is to take

$$\left\langle {(p_1,q_1) \atop \text{hws}} ; {(\lambda,\mu) \atop n';\tilde{I}_n'} \right| \left. {(p_2,q_2) \atop \text{hws}} \right\rangle \geq 0 \quad , \tag{4.3.11}$$

where $\tilde{I}_n'$ is the largest value of I_n' compatible with n'. For more information concerning this phase choice, refer to [18].

4.3.2 General expression for $SU(3)$ CGs using $9j$-symbols

The job at hand is to construct the Clebch-Gordan coefficients for the decomposition $(p_1,q_1)\otimes(\lambda,\mu) \to (p_2,q_2)$ for a fixed irrep (p_2,q_2), *i.e.* if the irrep (p_2,q_2) occurs more than once in the decomposition of $(p_1,q_1)\otimes(\lambda,\mu)$ then we have selected a particular copy. My assumption here is that I already have all the highest weight CGs obtained via recursion relation (see equation (4.3.10)) for a given decomposition, then I can construct the any highest weight state of equation (4.3.1).

In order to construct the recursion relation for the general CGs, we start by a target state

$$\begin{aligned}
&\left|(p_2,q_2)p_2+q_2-p,\tfrac{1}{2}(p_2+q_2+p)+I,\tfrac{1}{2}(p_2+q_2+p)-I;I\right\rangle \\
&\quad\times\frac{\langle(p_2,q_2)p_2+q_2-p;I\,\|\hat{T}^{\frac{1}{2}p}\|(p_2,q_2)p_2+q_2;\frac{1}{2}q_2\rangle}{\sqrt{2I+1}} \\
&=\sum_{s(m_p)}\left\langle\begin{smallmatrix}\frac{1}{2}q_2\\ \frac{1}{2}q_2-s\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}p\\ m_p\end{smallmatrix}\middle|\begin{smallmatrix}I\\ I\end{smallmatrix}\right\rangle\sum_{\nu_1J(n_1)I_n}\left\langle\begin{smallmatrix}(p_1,q_1)\\ \nu_1;J\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n_1;I_n\end{smallmatrix}\middle\|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2;\frac{1}{2}q_2\end{smallmatrix}\right\rangle \\
&\quad\times\sum_{m_\nu m_n}\left\langle\begin{smallmatrix}J\\ m_\nu\end{smallmatrix};\begin{smallmatrix}I_n\\ m_n\end{smallmatrix}\middle|\begin{smallmatrix}\frac{1}{2}q_2\\ \frac{1}{2}q_2-s\end{smallmatrix}\right\rangle\sum_{j_a}\frac{p!}{(2j_a)!(p-2j_a)!}\sum_{m_am_b}\left\langle\begin{smallmatrix}j_a\\ m_a\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}p-j_a\\ m_b\end{smallmatrix}\middle|\begin{smallmatrix}\frac{1}{2}p\\ m_p\end{smallmatrix}\right\rangle \\
&\quad\times\left[\hat{T}^{j_a}_{m_a}\left|(p_1,q_1)\nu;J\right\rangle\right]\left[\hat{T}^{\frac{1}{2}p-j_a}_{m_b}\left|(\lambda,\mu)n;I_n\right\rangle\right].
\end{aligned}\tag{4.3.12}$$

From this expression, it is possible to write

$$\begin{aligned}
&\left\langle\begin{smallmatrix}(p_1,q_1)\\ \nu_1';J'\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n_1';I_n'\end{smallmatrix}\middle\|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2-p;I\end{smallmatrix}\right\rangle\left\langle\begin{smallmatrix}J'\\ m_\nu'\end{smallmatrix};\begin{smallmatrix}I_n'\\ m_n'\end{smallmatrix}\middle|\begin{smallmatrix}I\\ I\end{smallmatrix}\right\rangle \\
&\quad\times\frac{\langle(p_2,q_2)p_2+q_2-p;I\,\|\hat{T}^{\frac{1}{2}p}\|(p_2,q_2)p_2+q_2;\frac{1}{2}q_2\rangle}{\sqrt{2I+1}} \\
&=\sum_{s(m_p)}\left\langle\begin{smallmatrix}\frac{1}{2}q_2\\ \frac{1}{2}q_2-s\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}p\\ m_p\end{smallmatrix}\middle|\begin{smallmatrix}I\\ I\end{smallmatrix}\right\rangle\sum_{\nu_1J(n_1)I_n}\left\langle\begin{smallmatrix}(p_1,q_1)\\ \nu_1;J\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n_1;I_n\end{smallmatrix}\middle\|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2;\frac{1}{2}q_2\end{smallmatrix}\right\rangle \\
&\quad\times\sum_{m_\nu m_n}\left\langle\begin{smallmatrix}J\\ m_\nu\end{smallmatrix};\begin{smallmatrix}I_n\\ m_n\end{smallmatrix}\middle|\begin{smallmatrix}\frac{1}{2}q_2\\ \frac{1}{2}q_2-s\end{smallmatrix}\right\rangle\sum_{j_a}\frac{p!}{(2j_a)!(p-2j_a)!}\sum_{m_am_b}\left\langle\begin{smallmatrix}j_a\\ m_a\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}p-j_a\\ m_b\end{smallmatrix}\middle|\begin{smallmatrix}\frac{1}{2}p\\ m_p\end{smallmatrix}\right\rangle \\
&\quad\times\langle(p_1,q_1)\nu';J'\,\|\hat{T}^{j_a}_{m_a}\|(p_1,q_1)\nu;J\rangle\langle(\lambda,\mu)n';I_n'\,\|\hat{T}^{\frac{1}{2}p-j_a}_{m_b}\|(\lambda,\mu)n;I_n\rangle
\end{aligned}\tag{4.3.13}$$

By inserting the expressions for the reduced matrix elements in equation (4.3.13) will produce an expression containing a quadruple product of $SU(2)$ CGs, which can be written as an expression containing an $SU(2)$ $9j$-symbol. After some mathematical effort, one can find an expression for the reduced Clebsch-Gordan coefficients

$$\begin{aligned}
&\left\langle\begin{smallmatrix}(p_1,q_1)\\ \nu_1';J'\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n_1';I_n'\end{smallmatrix}\middle\|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2-p;I\end{smallmatrix}\right\rangle \\
&=\frac{(-1)^{p-I-J'-I_n'}\sqrt{(2I+1)(q_2+1)(p+1)}}{\langle(p_2,q_2)p_2+q_2-p;I\,\|\hat{T}^{\frac{1}{2}p}\|(p_2,q_2)p_2+q_2;\frac{1}{2}q_2\rangle}\sum_{\nu_1(n_1)JI_n}\left\langle\begin{smallmatrix}(p_1,q_1)\\ \nu_1;J\end{smallmatrix};\begin{smallmatrix}(\lambda,\mu)\\ n_1;I_n\end{smallmatrix}\middle\|\begin{smallmatrix}(p_2,q_2)\\ p_2+q_2;\frac{1}{2}q_2\end{smallmatrix}\right\rangle \\
&\quad\times\binom{p}{\nu_1-\nu_1'}(-1)^{\frac{1}{2}q_2+J+I_n}\begin{Bmatrix}I_n & J & \frac{1}{2}q_2\\ \frac{1}{2}(p-\nu_1+\nu_1') & \frac{1}{2}(\nu_1-\nu_1') & \frac{1}{2}p\\ I_n' & J' & I\end{Bmatrix} \\
&\quad\times\langle(p_1,q_1)\nu_1';J'\,\|\hat{T}^{\frac{1}{2}(\nu_1-\nu_1')}\|(p_1,q_1)\nu_1;J\rangle\langle(\lambda,\mu)n_1';I_n'\,\|\hat{T}^{\frac{1}{2}(p-\nu_1+\nu_1')}\|(\lambda,\mu)n_1;I_n\rangle\quad.
\end{aligned}\tag{4.3.14}$$

This expression can be compared with its $SU(2)$ counterpart described in equation (2.1.18), since both expressions depend on the highest weight Clebsch-Gordan coefficients. I am capable of making this connection between these expressions because I used the same approach to obtain both expressions, which was finding the highest weight CGs and then applying lowering operators to find a targeted state.

Therefore, we now can construct all Clebsch-Gordan coefficients for a given decomposition $(p_1, q_1) \otimes (\lambda, \mu) \to (p_2, q_2)$. The most important steps for the derivation of equation (4.3.14) were the identification of the targeted state in equation (4.3.12), the expansion of equation (4.2.3) in terms of $\hat{C}_{j1} = \hat{C}_{j1}^{(1)} + \hat{C}_{j1}^{(2)}$, where $\hat{C}_{j1}^{(i)}$ acts on states in the irrep (p_1, q_1) for $i = 1$ and in the irrep (λ, μ) for $i = 2$. The action of these lowering operators on the highest weight state made it feasible to find the expression of equation (4.3.14).

The greatest advantage of having equation (4.3.14) is that one can now find analytical expressions for the Clebsch-Gordan coefficients. The next section is devoted to finding some interesting coefficients that will be used in applications in the next chapter.

4.4 Some coupling coefficients needed in this thesis

This basic algorithm can be implemented to the evaluation of CG coefficients for the coupling $(1,1) \otimes (\sigma, \sigma) \to (\tau, \tau)$. For this thesis I am specifically interested in analytical expressions for the cases $(\tau, \tau) = (\sigma+1, \sigma+1), (\sigma-1, \sigma-1)$, and for the two copies of (σ, σ) that occur in $(1,1) \otimes (\sigma, \sigma)$ because these CGs are present in the derivation of the correspondence rules of the next chapter. Since $(\sigma \pm 1, \sigma \pm 1)$ occurs once in $(1,1) \otimes (\sigma, \sigma)$, the highest weight of the irrep is uniquely determined by the recursion relation and the CG can be obtained by direct application of the formalism given above. For $(\tau, \tau) = (\sigma, \sigma)$, the two copies must be handled separately and the distinction between copies will be managed by introducing the index ρ, which can assume values 1 and 2 for the first and second copies, respectively.

4.4.1 Analytical expressions for coefficients in $(\tau, \tau) = (\sigma \pm 1, \sigma \pm 1)$.

These two decompositions have different k value. For instance, the decomposition $(1,1) \otimes (\sigma, \sigma) \to (\sigma-1, \sigma-1)$ has $k = 2$ whereas the decomposition $(1,1) \otimes (\sigma, \sigma) \to (\sigma+1, \sigma+1)$ has $k = 0$. Direct implementation of equation (4.3.10) yields:

$$\begin{aligned}
\left\langle \begin{smallmatrix}(1,1)\\ 2;\frac{1}{2}\end{smallmatrix};\ \begin{smallmatrix}(\sigma,\sigma)\\ 2(\sigma-1);\end{smallmatrix}\tfrac{1}{2}\sigma - 1 \ \middle\|\ \begin{smallmatrix}(\sigma-1,\sigma-1)\\ 2(\sigma-1);\frac{1}{2}(\sigma-1)\end{smallmatrix} \right\rangle &= -\tfrac{1}{\sigma+1}\sqrt{\tfrac{(\sigma-1)}{(2\sigma+1)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 2;\frac{1}{2}\end{smallmatrix};\ \begin{smallmatrix}(\sigma,\sigma)\\ 2(\sigma-1);\end{smallmatrix}\tfrac{1}{2}\sigma \ \middle\|\ \begin{smallmatrix}(\sigma-1,\sigma-1)\\ 2(\sigma-1);\frac{1}{2}(\sigma-1)\end{smallmatrix} \right\rangle &= \tfrac{1}{\sigma+1}\sqrt{\tfrac{\sigma+2}{2\sigma+1}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 1;1\end{smallmatrix};\ \begin{smallmatrix}(\sigma,\sigma)\\ 2\sigma-1;\end{smallmatrix}\tfrac{1}{2}(\sigma-1) \ \middle\|\ \begin{smallmatrix}(\sigma-1,\sigma-1)\\ \end{smallmatrix}\tfrac{1}{2}(\sigma-1) \right\rangle &= \tfrac{\sigma}{\sigma+1}\sqrt{\tfrac{(\sigma-1)}{2(\sigma+1)(2\sigma+1)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 1;1\end{smallmatrix};\ \begin{smallmatrix}(\sigma,\sigma)\\ 2\sigma-1;\end{smallmatrix}\tfrac{1}{2}(\sigma+1) \ \middle\|\ \begin{smallmatrix}(\sigma-1,\sigma-1)\\ 2(\sigma-1);\end{smallmatrix}\tfrac{1}{2}(\sigma-1) \right\rangle &= -\tfrac{1}{\sigma+1}\sqrt{\tfrac{\sigma(\sigma+2)}{(\sigma+1)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 1;0\end{smallmatrix};\ \begin{smallmatrix}(\sigma,\sigma)\\ 2\sigma-1;\end{smallmatrix}\tfrac{1}{2}(\sigma-1) \ \middle\|\ \begin{smallmatrix}(\sigma-1,\sigma-1)\\ 2(\sigma-1);\end{smallmatrix}\tfrac{1}{2}(\sigma-1) \right\rangle &= \tfrac{\sigma}{\sigma+1}\sqrt{\tfrac{3}{2(2\sigma+1)}}
\end{aligned} \tag{4.4.1}$$

It so happens that the decomposition $(1,1) \otimes (\sigma, \sigma) \to (\sigma+1, \sigma+1)$ has only one highest weight CG. Since $k = 0$, the highest states of the irreps $(1,1)$ and (σ, σ) have $(\nu_1; J) = (2; \frac{1}{2})$ and $(n_1; I_n) = (2\sigma; \frac{1}{2}\sigma)$, respectively, which adds up to $(p_2 + q_2 + k, J + I_n) = (2(\sigma+1); \frac{1}{2}(\sigma+1))$. Therefore, we can choose the reduced Clebsch-Gordan coefficient for this decomposition to be

$$\left\langle \begin{smallmatrix}(1,1)\\ 2;\frac{1}{2}\end{smallmatrix};\ \begin{smallmatrix}(\sigma,\sigma)\\ (2\sigma;\frac{1}{2}\sigma)\end{smallmatrix} \ \middle\|\ \begin{smallmatrix}(\sigma+1,\sigma+1)\\ 2(\sigma+1);\frac{1}{2}(\sigma+1)\end{smallmatrix} \right\rangle = 1 \tag{4.4.2}$$

Using these highest weight CGs and equation (4.3.14), I was able to calculate the following CGs which are useful in the derivation of the $SU(3)$ correspondence rules that will be presented in the next chapter.

$$\begin{aligned}
\left\langle \begin{smallmatrix}(1,1)\\ 2;\frac{1}{2}\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma+1,\sigma+1)\\ \sigma+2;\frac{1}{2}\end{smallmatrix} \right\rangle &= \frac{(\sigma+2)}{2(\sigma+1)}\sqrt{\frac{(\sigma+3)}{(2\sigma+3)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 1;1\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma+1,\sigma+1)\\ \sigma+1;1\end{smallmatrix} \right\rangle &= \frac{(\sigma+2)(\sigma+3)}{2(\sigma+1)}\sqrt{\frac{1}{3(\sigma+1)(2\sigma+3)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 1;0\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma+1,\sigma+1)\\ \sigma+1;0\end{smallmatrix} \right\rangle &= \frac{(\sigma+2)}{2}\sqrt{\frac{3}{(2\sigma+3)(\sigma+1)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 0;\frac{1}{2}\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma+1,\sigma+1)\\ \sigma;\frac{1}{2}\end{smallmatrix} \right\rangle &= \frac{(\sigma+2)}{2(\sigma+1)}\sqrt{\frac{\sigma+3}{(2\sigma+3)}}
\end{aligned} \tag{4.4.3}$$

$$\begin{aligned}
\left\langle \begin{smallmatrix}(1,1)\\ 2;\frac{1}{2}\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma-1,\sigma-1)\\ \sigma;\frac{1}{2}\end{smallmatrix} \right\rangle &= -\frac{\sigma}{2(\sigma+1)}\sqrt{\frac{(\sigma-1)}{(2\sigma+1)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 1;1\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma-1,\sigma-1)\\ \sigma-1;1\end{smallmatrix} \right\rangle &= \frac{\sigma(\sigma-1)}{2(\sigma+1)\sqrt{3(\sigma+1)(2\sigma+1)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 0;\frac{1}{2}\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma-1,\sigma-1)\\ \sigma-2;\frac{1}{2}\end{smallmatrix} \right\rangle &= -\frac{\sigma}{2(\sigma+1)}\sqrt{\frac{(\sigma-1)}{(2\sigma+1)}} \\
\left\langle \begin{smallmatrix}(1,1)\\ 1;0\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ \sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma-1,\sigma-1)\\ \sigma-1;0\end{smallmatrix} \right\rangle &= \frac{\sigma}{2}\sqrt{\frac{3}{(2\sigma+1)(\sigma+1)}}
\end{aligned} \tag{4.4.4}$$

The advantage of equation (4.3.14) is that it is possible to find analytical expression for some CGs. For instance, I was able to find the "general" analytical forms below:

$$\begin{aligned}
\left\langle \begin{smallmatrix}(1,1)\\ 1;0\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ 2\sigma-1-p;I\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma-1,\sigma-1)\\ 2(\sigma-1)-p;I\end{smallmatrix} \right\rangle &= \frac{(-1)^{p-2I+\sigma+1}}{8(\sigma+1)}\sqrt{\frac{3(3+2I+p+\sigma)(1-2I+p+\sigma)}{(\sigma+1)(2\sigma+1)}} \\
&\quad\times\sqrt{(1+2I-p+3\sigma)(-1-2I-p+3\sigma)} \\
\left\langle \begin{smallmatrix}(1,1)\\ 1;0\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\ 2(\sigma+1)-p-1;I\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma+1,\sigma+1)\\ 2(\sigma+1)-p;I\end{smallmatrix} \right\rangle &= \frac{(-1)^{p-2I+\sigma+1}}{2(\sigma+1)}\sqrt{\frac{3(3+2I+p+\sigma)(1-2I+p+\sigma)}{(\sigma+1)(2\sigma+3)}} \\
&\quad\times\sqrt{(3-2I-p+3\sigma)(5+2I-p+3\sigma)}
\end{aligned} \tag{4.4.5}$$

4.4.2 Analytical expressions for coefficients in $(\tau,\tau)=(\sigma,\sigma)_\rho$.

Table (4.4) gives the highest weight CGs of the decomposition $(1,1)\otimes(\sigma,\sigma)\to(\sigma,\sigma)$, where $k=1$.

Table 4.4: The copy labeled $\rho = 1$ is chosen using the usual convention that the $SU(3)$ CGs agree with the Wigner-Eckart theorem when the generators are considered as $SU(3)$ tensors transforming by the $(1,1)$ representation. The copy labeled $\rho = 2$ is chosen to be orthogonal to the $\rho = 1$ copy.

	$\left\| \begin{matrix} (\sigma,\sigma) \\ 2\sigma; \frac{1}{2}\sigma \end{matrix} \right\rangle_1$	$\left\| \begin{matrix} (\sigma,\sigma) \\ 2\sigma; \frac{1}{2}\sigma \end{matrix} \right\rangle_2$
$\left\langle \begin{matrix} (1,1) \\ 1;1 \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ 2\sigma; \frac{1}{2}\sigma \end{matrix} \right\|$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}\sqrt{\frac{2\sigma+1}{2\sigma+3}}$
$\left\langle \begin{matrix} (1,1) \\ 1;0 \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ 2\sigma; \frac{1}{2}\sigma \end{matrix} \right\|$	$\frac{1}{2}\sqrt{\frac{3\sigma}{\sigma+2}}$	$\frac{1}{2}\sqrt{\frac{\sigma(2\sigma+1)}{(\sigma+2)(2\sigma+3)}}$
$\left\langle \begin{matrix} (1,1) \\ 2; \frac{1}{2} \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ 2\sigma-1; \frac{1}{2}(\sigma+1) \end{matrix} \right\|$	$\sqrt{\frac{\sigma+2}{2(\sigma+1)(\sigma+2)}}$	$\sqrt{\frac{3(2\sigma+1)}{2(\sigma+1)(2\sigma+3)}}$
$\left\langle \begin{matrix} (1,1) \\ 2; \frac{1}{2} \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ 2\sigma-1; \frac{1}{2}(\sigma-1) \end{matrix} \right\|$	$-\sqrt{\frac{(2\sigma+1)}{2(\sigma+1)(\sigma+2)}}$	$\sqrt{\frac{3}{2(\sigma+1)(\sigma+2)(2\sigma+3)}}$

With the aid of the highest weight CGs of table (4.4) and equation (4.3.14), I was able to calculate the following CGs

$$
\begin{aligned}
\left\langle \begin{matrix} (1,1) \\ 2;\frac{1}{2} \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma+1;\frac{1}{2} \end{matrix} \right\rangle_{\rho=1} &= -\tfrac{1}{2} &
\left\langle \begin{matrix} (1,1) \\ 2;\frac{1}{2} \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma+1;\frac{1}{2} \end{matrix} \right\rangle_{\rho=2} &= \sqrt{\tfrac{3}{4(2\sigma+1)(2\sigma+3)}} \\
\left\langle \begin{matrix} (1,1) \\ 1;0 \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \right\rangle_{\rho=1} &= 0 &
\left\langle \begin{matrix} (1,1) \\ 1;0 \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \right\rangle_{\rho=2} &= \sqrt{\tfrac{\sigma(\sigma+2)}{(2\sigma+1)(2\sigma+3)}} \\
\left\langle \begin{matrix} (1,1) \\ 1;1 \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma;1 \end{matrix} \right\rangle_{\rho=1} &= 0 &
\left\langle \begin{matrix} (1,1) \\ 1;1 \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma;1 \end{matrix} \right\rangle_{\rho=2} &= -\sqrt{\tfrac{\sigma(\sigma+2)}{(2\sigma+1)(2\sigma+3)}} \\
\left\langle \begin{matrix} (1,1) \\ 0;\frac{1}{2} \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma-1;1/2 \end{matrix} \right\rangle_{\rho=1} &= \tfrac{1}{2} &
\left\langle \begin{matrix} (1,1) \\ 0;\frac{1}{2} \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ \sigma;0 \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ \sigma-1;1/2 \end{matrix} \right\rangle_{\rho=2} &= \tfrac{\sqrt{3}}{2}\tfrac{1}{\sqrt{(2\sigma+1)(2\sigma+3)}}
\end{aligned}
\tag{4.4.6}
$$

Some extra analytical expressions for the decomposition $(1,1) \otimes (\sigma,\sigma) \to (\sigma,\sigma)$

$$
\begin{aligned}
\left\langle \begin{matrix} (1,1) \\ 1;0 \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ 2\sigma-p;I \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ 2\sigma-p;I \end{matrix} \right\rangle_{\rho=1} &= \tfrac{\sqrt{3}(-1)^{p+\sigma-2I}}{2(\sigma+1)\sqrt{\sigma(\sigma+2)}}\Big(\sigma(\sigma+1) - \tfrac{1}{4}(p-\sigma+2I) \\
&\quad \times(\sigma-p+2I+2) - \tfrac{1}{4}(p+\sigma-2I)(p+\sigma+2I+2)\Big) \\
\left\langle \begin{matrix} (1,1) \\ 1;0 \end{matrix} ; \begin{matrix} (\sigma,\sigma) \\ 2\sigma-p;I \end{matrix} \middle\| \begin{matrix} (\sigma,\sigma) \\ 2\sigma-p;I \end{matrix} \right\rangle_{\rho=2} &= \tfrac{(-1)^{p+\sigma-2I}\sqrt{2\sigma+1}}{2(\sigma+1)\sqrt{\sigma(\sigma+2)(2\sigma+3)}}\Big(\sigma(\sigma+1) - \tfrac{3}{4}(p-\sigma+2I) \\
&\quad \times(\sigma-p+2I+2) + \tfrac{3}{4}\tfrac{(p+\sigma-2I)(p+\sigma+2I+2)}{(2\sigma+1)}\Big)
\end{aligned}
\tag{4.4.7}
$$

Chapter 5

Quasi-distributions in $SU(3)$

5.1 The kernel for Wigner Functions

5.1.1 Definition of $SU(3)$ tensor operators

In [12], Klimov and de Guise provided an algorithm for a general operational form for the quantization kernel $\hat{w}(\Omega)$ in a system with $SU(n)$ symmetry. In addition, the same authors together with José L. Romero wrote the review paper [14] where they discuss the generalization of quantum mechanics in phase space to $SU(2)$ and $SU(3)$ symmetry. I will base my definition of the quantization kernel in $SU(3)$ according to their work.

The notation in this chapter will follow the definitions of the previous chapter. For instance, a general state of the irrep $(\lambda, 0)$ can be shortly written as

$$|\lambda; \nu\rangle := |(\lambda, 0)\nu_1\nu_2\nu_3; I_{23}\rangle \quad , \tag{5.1.1}$$

where we can write the triple $\nu = (\nu_1, \nu_2, \nu_3)$ and any weight $(\nu_1 - \nu_2, \nu_2 - \nu_3)$ in this irrep only occurs once and the components of this state satisfy $\nu_1 + \nu_2 + \nu_3 = \lambda$.

The tensor operators acting on a Hilbert space carrying the irrep λ will transform according to the irrep $\lambda \otimes \lambda^*$, where $\lambda^* = (0, \lambda)$ is the conjugate of the irrep $(\lambda, 0)$. This is an operator of the form

$$\hat{T}^{\lambda}_{\sigma;\gamma I_\gamma} = \sum_{\alpha\beta} \left|(\lambda, 0)\alpha_1\alpha_2\alpha_3; \tfrac{1}{2}(\lambda - \alpha_1)\right\rangle \left\langle(\lambda, 0)\beta_1\beta_2\beta_3; \tfrac{1}{2}(\lambda - \beta_1)\right| \tilde{C}^{\sigma\gamma I_\gamma}_{\lambda\alpha I_\alpha;\lambda^*\beta^* I_\beta}(-1)^{\lambda-\beta_2} \tag{5.1.2}$$

where the indices in equation (5.1.2) mean that λ^* is the conjugate of the irrep λ as stated previously and $\beta^* = (\lambda - \beta_1, \lambda - \beta_2, \lambda - \beta_3)$ is the weight conjugate to β, and σ is an irrep of the decomposition of $(\lambda, 0) \otimes (0, \lambda)$

$$(\lambda, 0) \otimes (0, \lambda) = \sum_{\sigma=0}^{\lambda} (\sigma, \sigma). \tag{5.1.3}$$

The expansion coefficients $\tilde{C}^{\sigma\gamma I_\gamma}_{\lambda\alpha I_\alpha;\lambda^*\beta^* I_{\beta^*}}$ are proportional to $SU(3)$ Clebsch-Gordan coefficients and they follow the orthogonality condition

$$\sum_{\alpha I_\alpha \beta I_\beta} (\tilde{C}^{\sigma'\nu' I'_\nu}_{\lambda\alpha I_\alpha;\lambda^*\beta^* I_{\beta^*}})^* \tilde{C}^{\sigma\nu I_\nu}_{\lambda\alpha I_\alpha;\lambda^*\beta I_\beta} = \delta_{\sigma\sigma'}\delta_{\nu\nu'}\delta_{I_\nu I_{\nu'}} \quad . \tag{5.1.4}$$

The irreducible tensor operators of equation (5.1.2) satisfy the commutation relation

$$\left[\hat{H}_i, \hat{T}^{\lambda}_{\sigma;\alpha I_\alpha}\right] = \alpha_i \hat{T}^{\lambda}_{\sigma;\alpha I_\alpha} \tag{5.1.5}$$

and also have an orthogonality condition represented by the trace of these tensors

$$\mathrm{Tr}\Big((\hat{T}^{\lambda}_{\sigma;\alpha I_\alpha})^\dagger \hat{T}^{\lambda}_{\sigma';\alpha' I_{\alpha'}}\Big) = \delta_{\sigma\sigma'}\delta_{\alpha\alpha'}\delta_{I_\alpha I_{\alpha'}} \tag{5.1.6}$$

5.1.2 Definition of the quantization kernel $\hat{w}_\lambda(\Omega)$

In the previous chapter, I introduced the general $SU(3)$ states as a coupling of three $SU(2)$ states and found the Clebsch-Gordan coefficients. Moreover, these coefficients were used in the construction of the tensor operators of equation (5.1.2) which are the fundamental ingredients for the realization of the quantization kernel $\hat{w}_\lambda(\Omega)$. Although these irreducible tensor operators are expressed as given in equation (5.1.2) for a given irrep (σ, σ), one can recall chapter 2 of this thesis that irreducible tensor operators are transformed by a group operation of a parametrized rotation operator $\hat{R}(\Omega)$, and this transformation mixes elements of the basis of irreducible tensor operators of the given irrep (σ, σ).

The parametrization used in this thesis was first introduced by de Guise and Klimov in [12]. Therefore, following their ideas, the transformations $\hat{R}(\Omega)$ are written as

$$\begin{aligned}\hat{R}(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2) &= \hat{R}_{23}(\alpha_1, \beta_1, -\alpha_1)\hat{R}_{12}(\alpha_2, \beta_2, -\alpha_2) \\ &\times \hat{R}_{23}(\alpha_3, \beta_3, -\alpha_3)\mathrm{e}^{-i\gamma_1(\hat{C}_{11}-\hat{C}_{22})}\mathrm{e}^{-i\gamma_2(\hat{C}_{22}-\hat{C}_{33})}\end{aligned} \tag{5.1.7}$$

where the angles of the full transformation are written as $\tilde{\Omega} := (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2)$ and

$$\hat{R}_{23}(\alpha_1, \beta_1, -\alpha_1)\hat{R}_{12}(\alpha_2, \beta_2, -\alpha_2)\hat{R}_{23}(\alpha_3, \beta_3, -\alpha_3)$$

is the $SU(3)$ version of the displacement operator of equation (3.2.4). Moreover, a transformation of the tensor operators of equation (5.1.2) is written as

$$\hat{R}(\tilde{\Omega})\hat{T}^{\lambda}_{\tau;\mu I}\hat{R}^\dagger(\tilde{\Omega}) = \sum_{\nu J} D^{(\tau,\tau)}_{\nu J;\mu I}(\tilde{\Omega})\hat{T}^{\lambda}_{\tau;\nu J} \tag{5.1.8}$$

where the functions $D^{(\tau,\tau)}_{\nu J;\mu I}(\tilde{\Omega})$ are the $SU(3)$ Wigner D-functions.

Now, we can define the quantization kernel $\hat{w}_\lambda(\tilde{\Omega})$ in $SU(3)$ as

$$\hat{w}_\lambda(\tilde{\Omega}) = \hat{R}(\tilde{\Omega})\hat{w}_\lambda(0)\hat{R}(\tilde{\Omega})^\dagger \tag{5.1.9}$$

where $\hat{R}(0)$ is the identity transformation and $\hat{w}_\lambda(0)$ is the quantization kernel at $\tilde{\Omega} = 0$. This kernel is constructed by a linear combination of diagonal tensor operators

$$\hat{w}_\lambda(0) = \sum_{\sigma=0}^{\lambda} F^\lambda_\sigma \hat{T}^\lambda_{\sigma;(\sigma\sigma\sigma)0}, \quad F^\lambda_\sigma = \sqrt{\frac{2(\sigma+1)^3}{(\lambda+1)(\lambda+2)}} \quad . \tag{5.1.10}$$

It is important to point out that the transformation

$$\hat{R}_{23}(\alpha_3, \beta_3, -\alpha_3)\mathrm{e}^{-i\gamma_1(\hat{C}_{11}-\hat{C}_{22})}\mathrm{e}^{-i\gamma_2(\hat{C}_{22}-\hat{C}_{33})} \tag{5.1.11}$$

acting on equation (5.1.10) leaves this equation invariant. This leads to a reduction of the number of angles in equation (5.1.7)

$$\tilde{\Omega} = (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2) \quad \rightarrow \quad \Omega = (\alpha_1, \beta_1, \alpha_2, \beta_2) \tag{5.1.12}$$

Therefore, substituting equations (5.1.8) and (5.1.10) into equation (5.1.9), it is possible to find

$$\hat{w}_\lambda(\Omega) = \sum_{\sigma=0}^{\lambda} F_\sigma^\lambda \sum_{\nu J} D^{(\sigma,\sigma)}_{\nu J;(\sigma\sigma\sigma)0}(\Omega) \hat{T}^\lambda_{\sigma;\nu J} \tag{5.1.13}$$

where the second summation is over all the states of the irrep (σ, σ).

5.2 Examples of some Wigner Symbols

In this section, I will derive some Wigner symbols that may be used in the applications of the $\star$-product and correspondence rules of this chapter. A $SU(3)$ Wigner symbol follows the same definition of equation (3.2.7) of the third chapter of this thesis. That is

$$W_{\hat{A}}(\Omega) = \mathrm{Tr}\Big(\hat{w}_\lambda(\Omega)\hat{A}\Big) \tag{5.2.1}$$

where the quantization kernel $\hat{w}_\lambda(\Omega)$ is given in equation (5.1.13). For simplicity, let us choose $\hat{A}$ to be one of the generators of the $su(3)$ algebra, say $\hat{C}_{12}$, and calculate the Wigner symbol of this operator.

First, the operator $\hat{C}_{12}$ can be written as a $SU(3)$ tensor operator

$$\hat{C}_{12} = -\frac{N}{2\sqrt{6}} \hat{T}^\lambda_{1;(201)\frac{1}{2}} \quad . \tag{5.2.2}$$

Substituting equations (5.2.2) and (5.1.13) into equation (5.2.1) we get

$$W_{\hat{C}_{12}}(\Omega) = -\frac{N}{2\sqrt{6}} \sum_{\sigma=0}^{\lambda} F_\sigma^\lambda \sum_{\nu J} D^{(\sigma,\sigma)}_{\nu J;(\sigma\sigma\sigma)0}(\Omega)\, \mathrm{Tr}\Big(\hat{T}^\lambda_{\sigma;\nu J}\hat{T}^\lambda_{1;(201)\frac{1}{2}}\Big) \tag{5.2.3}$$

where $N = \sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}$. We can rewrite the tensor operators $\hat{T}^\lambda_{\sigma;\nu J}$ of the above expression as

$$\hat{T}^\lambda_{\sigma;\nu J} = (-1)^{\sigma+\nu_2} (\hat{T}^\lambda_{\sigma;\nu^* J})^\dagger \tag{5.2.4}$$

with $\nu^* = (2\sigma - \nu_1, 2\sigma - \nu_2, 2\sigma - \nu_3)$. Substituting equation (5.2.4) into the expression of equation (5.2.2), we get

$$W_{\hat{C}_{12}}(\Omega) = \frac{N}{2\sqrt{6}} F_1^\lambda D^{(1,1)}_{(021)\frac{1}{2};(111)0}(\Omega) \tag{5.2.5}$$

and finally

$$W_{\hat{C}_{12}}(\Omega) = \frac{1}{2}\sqrt{\lambda(\lambda+3)} \mathrm{e}^{i\alpha_2} \cos\left(\frac{\beta_1}{2}\right) \sin\beta_2 \quad . \tag{5.2.6}$$

Using the same approach, I calculated the Wigner symbol of the remaining seven generators of the $su(3)$ algebra. They are summarized in the table below.

Table 5.1: Summary of the $SU(3)$ Wigner symbol of the generators

Generator	Tensor Operator	Wigner Symbol
$\hat{C}_{12}$	$-\frac{N}{2\sqrt{6}}\hat{T}^{\lambda}_{1;(201)\frac{1}{2}}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{i\alpha_2}\cos\left(\frac{1}{2}\beta_1\right)\sin\beta_2$
$\hat{C}_{13}$	$\frac{N}{2\sqrt{6}}\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{i(\alpha_1+\alpha_2)}\sin\left(\frac{1}{2}\beta_1\right)\sin\beta_2$
$\hat{C}_{23}$	$\frac{N}{2\sqrt{6}}\hat{T}^{\lambda}_{1;(120)1}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{i\alpha_1}\sin\beta_1\sin^2(\frac{1}{2}\beta_2)$
$\hat{C}_{21}$	$-\frac{N}{2\sqrt{6}}\left(\hat{T}^{\lambda}_{1;(201)\frac{1}{2}}\right)^{\dagger}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{-i\alpha_2}\cos\left(\frac{1}{2}\beta_1\right)\sin\beta_2$
$\hat{C}_{31}$	$\frac{N}{2\sqrt{6}}\left(\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}\right)^{\dagger}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{1}{2}\beta_1\right)\sin\beta_2$
$\hat{C}_{32}$	$\frac{N}{2\sqrt{6}}\left(\hat{T}^{\lambda}_{1;(120)1}\right)^{\dagger}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}\mathrm{e}^{-i\alpha_1}\sin\beta_1\sin^2(\frac{1}{2}\beta_2)$
$\hat{H}_1$	$\frac{N}{2}\hat{T}^{\lambda}_{1;(111)0}$	$\frac{1}{2}\sqrt{\lambda(\lambda+3)}(1+3\cos\beta_2)$
$\hat{H}_2$	$-\frac{N}{4\sqrt{3}}\hat{T}^{\lambda}_{1;(111)1}$	$\sqrt{\lambda(\lambda+3)}\cos\beta_1\sin^2(\frac{1}{2}\beta_2)$

5.3 The Action of a Generator on the Wigner Kernel

The main idea of this section is to derive the action of a generator $\hat{T}^{\lambda}_{\sigma;\mu J}$ on the quantization kernel $\hat{w}_{\lambda}(\Omega)$. This type of calculation is important for the future derivation of the correspondence rules, since I will write the action of a generator on the kernel as a differential operator that depends on the generator only.

Let us start by writing the Wigner Symbol of an operator $\hat{B}$:

$$W_{\hat{B}}(\Omega) = \mathrm{Tr}\Big(\hat{B}\hat{w}_{\lambda}(\Omega)\Big) \quad . \tag{5.3.1}$$

We can take, without loss of generality, the operator $\hat{B}$ as

$$\hat{B} = \hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}} \tag{5.3.2}$$

and substitute it, together with equation (5.1.13), into equation (5.3.1) to get

$$\begin{aligned} W_{\hat{B}}(\Omega) &= \sum_{\tau} F^{\lambda}_{\tau} \sum_{\mu J} D^{(\tau,\tau)}_{\mu J;(\tau\tau\tau)0}(\Omega)\mathrm{Tr}\left(\hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}\hat{T}^{\lambda}_{\tau;\mu J}\right) \\ &= \sum_{\tau} F^{\lambda}_{\tau} \sum_{\mu J} D^{(\tau,\tau)}_{\mu J;(\tau\tau\tau)0}(\Omega)\mathrm{Tr}\left(\hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}(-1)^{\tau+\mu_2}\left(\hat{T}^{\lambda}_{\tau;\mu^* J}\right)^{\dagger}\right) \\ &= F^{\lambda}_{\sigma} D^{(\sigma,\sigma)}_{\bar{\mu}^*\bar{J};(\sigma\sigma\sigma)0}(\Omega)(-1)^{\sigma-\bar{\mu}_2} = \sqrt{\frac{2(\sigma+1)^3}{(\lambda+1)(\lambda+2)}}\left(D^{(\sigma,\sigma)}_{\bar{\mu}\bar{J};(\sigma\sigma\sigma)0}(\Omega)\right)^{*} \end{aligned} \tag{5.3.3}$$

where I wrote $\hat{T}^{\lambda}_{\tau;\mu J} = (-1)^{\tau+\mu_2}(\hat{T}^{\lambda}_{\sigma;\mu^* J})^\dagger$ and made use of the orthogonality property of the tensor operators as given in equation (5.1.6).

The $\star$-product was defined as

$$W_{\hat{A}}(\Omega) \star W_{\hat{B}}(\Omega) = W_{\hat{A}\hat{B}}(\Omega) \quad , \tag{5.3.4}$$

but it is also possible to write

$$W_{\hat{B}}(\Omega) \star W_{\hat{A}}(\Omega) = W_{\hat{B}\hat{A}}(\Omega) \tag{5.3.5}$$

and from the definition of a Wigner symbol, these two equations are given by

$$W_{\hat{A}}(\Omega) \star W_{\hat{B}}(\Omega) = \mathrm{Tr}\Big(\hat{w}_\lambda(\Omega)\hat{A}\hat{B}\Big) \tag{5.3.6}$$

and

$$W_{\hat{B}}(\Omega) \star W_{\hat{A}}(\Omega) = \mathrm{Tr}\Big(\hat{A}\hat{w}_\lambda(\Omega)\hat{B}\Big). \tag{5.3.7}$$

I would like to consider now the case where the operator $\hat{A}$ is $\hat{T}^{\lambda}_{1;\mu J}$, that is $\hat{A}$ is one of the generators of the $su(3)$ algebra, and evaluate the action of this operator on the quantization kernel $\hat{w}_\lambda(\Omega)$ in both cases of the $\star$-products of equations (5.3.6) and (5.3.7). Mathematically, I want to calculate the actions $\hat{T}^{\lambda}_{1;\mu J}\hat{w}_\lambda(\Omega)$ and $\hat{w}_\lambda(\Omega)\hat{T}^{\lambda}_{1;\mu J}$ and show how, in the limit of large λ, I can substitute these actions by differential operators $\hat{\mathcal{A}}^L_{\nu I}(\Omega)$ and $\hat{\mathcal{A}}^R_{\nu I}(\Omega)$, respectively, depending on the generators only.

We can write equations (5.3.6) and (5.3.7) as

$$W_{\hat{T}^{\lambda}_{1;\nu I}}(\Omega) \star W_{\hat{B}}(\Omega) = \mathrm{Tr}\left(\hat{w}_\lambda(\Omega)\hat{T}^{\lambda}_{1;\nu I}\hat{B}\right) \tag{5.3.8}$$

and

$$W_{\hat{B}}(\Omega) \star W_{\hat{T}^{\lambda}_{1;\nu I}}(\Omega) = \mathrm{Tr}\Big(\hat{w}_\lambda(\Omega)\hat{B}\hat{T}^{\lambda}_{1;\nu I}\Big) = \mathrm{Tr}\Big(\hat{T}^{\lambda}_{1;\nu I}\hat{w}_\lambda(\Omega)\hat{B}\Big) \tag{5.3.9}$$

I present the specific mathematical developments of the correspondence rules in the next sections.

5.3.1 The Action of a Generator from the Left on the Wigner Kernel

We want an operational form for

$$\begin{aligned}
\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_\lambda(\Omega) &= \hat{T}^{\lambda}_{1;\alpha J}\hat{R}(\Omega)\hat{w}_\lambda(0)\hat{R}^\dagger(\Omega) \\
&= \hat{R}(\Omega)\left[\hat{R}^\dagger(\Omega)\hat{T}^{\lambda}_{1;\alpha J}\hat{R}(\Omega)\right]\hat{w}_\lambda(0)\hat{R}^\dagger(\Omega) \\
&= \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1})\left[\hat{R}(\Omega)\,\hat{T}^{\lambda}_{1;\nu I}\hat{w}_\lambda(0)\,\hat{R}^\dagger(\Omega)\right]
\end{aligned} \tag{5.3.10}$$

by first writing $\hat{R}(\Omega)\left(\hat{T}^{\lambda}_{1;\nu I}\hat{w}_\lambda(0)\right)\hat{R}^\dagger(\Omega)$ as an operator equation.

From the definition of equation (5.1.13) we have

$$\begin{aligned}\hat{T}^{\lambda}_{1;\nu I}\,\hat{w}_{\lambda}(0) &= \hat{T}^{\lambda}_{1;\nu I}\sum_{\sigma} F^{\sigma}_{\lambda}\,\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma)0} \\ &= \sqrt{\frac{16}{(\lambda+1)(\lambda+2)}}\sum_{\tau} F^{\tau}_{\lambda} \\ &\times \left(\sum_{\sigma,\rho}\frac{F^{\sigma}_{\lambda}}{F^{\tau}_{\lambda}}\left\langle {}^{(1,1)}_{\nu_1 I};{}^{(\sigma,\sigma)}_{\sigma;0}\parallel{}^{(\tau,\tau)}_{\bar{\nu}_1;I}\right\rangle_{\rho} U_{\rho}\left[(1,1)(\lambda,0)(\tau,\tau)(0,\lambda);(\lambda,0)(\sigma,\sigma)\right]_{\rho}\right)\hat{T}^{\lambda}_{\tau;\bar{\nu}I} \\ &= \sum_{\tau} F^{\tau}_{\lambda}\,a^{L}_{\bar{\nu}_1 I}(\lambda;\tau)\hat{T}^{\lambda}_{\tau;\bar{\nu}I} \qquad (5.3.11)\end{aligned}$$

where (for fixed λ):

$$a^{L}_{\tau;\nu I}(\lambda;\bar{\nu},I) = \sqrt{\frac{16}{(\lambda+1)(\lambda+2)}}\left(\sum_{\sigma=\tau-1}^{\tau+1}\frac{F^{\sigma}_{\lambda}}{F^{\tau}_{\lambda}}\sum_{\rho}\left\langle {}^{(1,1)}_{\nu_1 I};{}^{(\sigma,\sigma)}_{\sigma;0}\parallel{}^{(\tau,\tau)}_{\bar{\nu}_1;I}\right\rangle_{\rho} U_{\rho}\left[(1,1)(\lambda,0)(\tau,\tau)(0,\lambda);(\lambda,0)(\sigma,\sigma)\right]_{\rho}\right). \qquad (5.3.12)$$

The coefficients $U_{\rho}\left[(1,1)(\lambda,0)(\tau,\tau)(0,\lambda);(\lambda,0)(\sigma,\sigma)\right]_{\rho}$ are the Racah coefficient and the four necessary coefficients that enter equation (5.3.12) are given in the following table:

Table 5.2: Summary of the $SU(3)$ Wigner symbol of the generators

τ	ρ	$U_{\rho}\left[(1,1)(\lambda,0)(\tau,\tau)(0,\lambda);(\lambda,0)(\sigma,\sigma)\right]_{\rho}$
$\sigma+1$		$\frac{(\sigma+1)}{2}\sqrt{\frac{3(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+3)(\sigma+2)(2\sigma+3)}}$
σ	1	$\frac{\sqrt{3}}{2}\sqrt{\frac{\sigma(\sigma+2)}{\lambda(\lambda+3)}}$
σ	2	$\frac{(2\lambda+3)}{2}\sqrt{\frac{\sigma(\sigma+2)}{\lambda(\lambda+3)(2\sigma+1)(2\sigma+3)}}$
$\sigma-1$		$-\frac{(\sigma+1)}{2}\sqrt{\frac{3(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+3)\sigma(2\sigma+1)}}$

The labels ν in $T^{\lambda}_{1;\nu I}$ and $\bar{\nu}$ in $T^{\lambda}_{\tau;\bar{\nu}I}$, and in $a^{L}_{\bar{\nu}_1 I}(\lambda;\tau)$, are related by

$(\nu_1\nu_2\nu_3)$	$(\bar{\nu}_1\bar{\nu}_2\bar{\nu}_3)$
(210)	$(\tau+1,\tau,\tau-1)$
(021)	$(\tau-1,\tau+1,\tau)$
(201)	$(\tau+1,\tau-1,\tau)$
(012)	$(\tau-1,\tau,\tau+1)$
(111)	(τ,τ,τ)
(120)	(τ,τ,τ)
(102)	(τ,τ,τ)

(5.3.13)

and the $SU(3)$ CGs that appear in equation (5.3.12) were presented in section (4.4). Thus, the coefficients $a^L_{\tau;\nu I}(\lambda;\bar{\nu},I)$ are given explicitly as

$$
\begin{aligned}
a^L_{\tau;2,1/2}(\lambda;\tau+1) &= \sqrt{\frac{3\tau(\tau+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \\
&\quad\times\left(\frac{\tau\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}}{(\tau+1)(2\tau+1)} - \frac{(\tau+2)\sqrt{\lambda-\tau}\sqrt{\lambda+\tau+3}}{(\tau+1)(2\tau+3)} + \frac{2(\lambda-2\tau(\tau+2))}{4\tau(\tau+2)+3}\right) \\
a^L_{\tau;1,1}(\lambda;\tau) &= \frac{\tau(\tau+2)}{(\tau+1)(2\tau+1)(2\tau+3)\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \\
&\quad\times\left(-2(2\lambda+3)(\tau+1)+(2\tau+1)\sqrt{(\lambda-\tau)(\lambda+\tau+3)}+(2\tau+3)\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}\right) \\
a^L_{\tau;1,0}(\lambda;\tau) &= \frac{1}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \\
&\quad\times\left(\frac{3\tau^2\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}}{(\tau+1)(2\tau+1)} + \frac{2(2\lambda+3)(\tau+2)\tau}{4\tau(\tau+2)+3} + \frac{3(\tau+2)^2\sqrt{\lambda-\tau}\sqrt{\lambda+\tau+3}}{(\tau+1)(2\tau+3)}\right) \\
a^L_{\tau;0,1/2}(\lambda;\tau-1) &= \sqrt{\frac{3\tau(\tau+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \\
&\quad\times\left(\frac{\tau\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}}{(\tau+1)(2\tau+1)} - \frac{(\tau+2)\sqrt{\lambda-\tau}\sqrt{\lambda+\tau+3}}{(\tau+1)(2\tau+3)} + \frac{2(\lambda+2\tau(\tau+2)+3)}{4\tau(\tau+2)+3}\right) \qquad (5.3.14)
\end{aligned}
$$

where, for simplicity, it is only necessary to give ν_1, since $a^L_{\nu_1\nu_2\nu_3 I}(\lambda;\bar{\nu}I) = a^L_{\nu_1\nu_3\nu_2 I}(\lambda;\bar{\nu}I)$.

Substituting these coefficients together with equation (5.3.11) into equation (5.3.10), I was able to find

$$
\begin{aligned}
\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_{\lambda}(\Omega) &= \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1})\left[\hat{R}(\Omega)\sum_{\tau} F^{\tau}_{\lambda}\, a^L_{\tau;\nu I}(\lambda;\bar{\nu},I)\hat{T}^{\lambda}_{\tau;\bar{\nu}I}\hat{R}^{\dagger}(\Omega)\right] \\
&= \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1})\sum_{\tau\nu' I'} F^{\tau}_{\lambda}\, a^L_{\tau;\nu I}(\lambda;\bar{\nu},I) D^{(\tau,\tau)}_{\nu' I',\bar{\nu}I}(\Omega)\hat{T}^{\lambda}_{\tau;\nu' I'}, \qquad (5.3.15)
\end{aligned}
$$

which is the final form for the action of a generator from the left on the quantization kernel.

5.3.2 The Action of a Generator from the Right on the Wigner Kernel

Consider the action of a generator $\hat{T}^{\lambda}_{1;\alpha J}$ from the right on the quantization kernel

$$
\begin{aligned}
\hat{w}_{\lambda}(\Omega)\hat{T}^{\lambda}_{1;\alpha J} &= \hat{R}(\Omega)\hat{w}_{\lambda}(0)\hat{R}^{\dagger}(\Omega)\hat{T}^{\lambda}_{1;\alpha J} \\
&= \sum_{\nu I} D^{(1,1)}_{\nu I,\alpha J}(\Omega^{-1})\left(\hat{R}(\Omega)\hat{w}_{\lambda}(0)\hat{T}^{\lambda}_{1;\nu I}\hat{R}^{\dagger}(\Omega)\right). \qquad (5.3.16)
\end{aligned}
$$

Let us focus on the tensor product $\hat{w}_{\lambda}(0)\hat{T}^{\lambda}_{1;\nu I}$ and write

$$\hat{w}_{\lambda}(0)\hat{T}^{\lambda}_{1;\nu I} = \sum_{\tau} F^{\tau}_{\lambda}\, a^R_{\tau;\nu I}(\lambda;\bar{\nu},I)\hat{T}^{\lambda}_{\tau;\bar{\nu}I}, \qquad (5.3.17)$$

with the coefficients $a^R_{\tau;\nu_1 I}(\lambda;\bar\nu_1,I)$ given by

$$
\begin{aligned}
a^R_{\tau;0,1/2}(\lambda;\tau-1,\frac{1}{2}) &= a^L_{\tau;0,1/2}(\lambda;\tau-1,\frac{1}{2}) - \frac{2\sqrt{3\tau(\tau+2)}}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\\
a^R_{\tau;1,1}(\lambda;\tau,1) &= a^L_{\tau;1,1}(\lambda;\tau,1)\\
a^R_{\tau;1,0}(\lambda;\tau,0) &= a^L_{\tau;1,0}(\lambda;\tau,0)\\
a^R_{\tau;2,1/2}(\lambda;\tau+1,\frac{1}{2}) &= a^L_{\tau;2,1/2}(\lambda;\tau+1,\frac{1}{2}) + \frac{2\sqrt{3\tau(\tau+2)}}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\,.
\end{aligned}
\tag{5.3.18}
$$

and substituting these coefficients together with equation (5.3.17) into equation (5.3.16), I could find the action of a generator on the quantization kernel from the right

$$
\begin{aligned}
\hat w_\lambda(\Omega)\hat T^\lambda_{1;\alpha J} &= \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1})\left[\hat R(\Omega)\sum_\tau F^\tau_\lambda a^R_{\tau;\nu_1 I}(\lambda;\bar\nu_1,I)\hat T^\lambda_{\tau;\bar\nu I}\hat R^\dagger(\Omega)\right]\\
&= \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1})\sum_{\tau\nu' I'} F^\tau_\lambda a^R_{\tau;\nu_1 I}(\lambda;\bar\nu_1,I)D^{(\tau,\tau)}_{\nu' I',\bar\nu I}(\Omega)\hat T^\lambda_{\tau;\nu' I'}\quad .
\end{aligned}
\tag{5.3.19}
$$

5.3.3 Commutator Action

I am now interested in calculating the difference between equations (5.3.15) and (5.3.19), that is calculating the commutator

$$
\left[\hat T^\lambda_{1;\alpha J},\hat w_\lambda(\Omega)\right] = \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1})\hat R(\Omega)\left[\sum_\tau F^\tau_\lambda(a^L_{\tau;\nu_1 I}(\lambda;\bar\nu_1,I) - a^R_{\tau;\nu_1 I}(\lambda;\bar\nu_1,I))T^\lambda_{\tau;\bar\nu I}\right]\hat R^\dagger(\Omega)\,. \tag{5.3.20}
$$

and considering the relations of equation (5.3.18), the difference between $a^L_{\nu_1 I}(\lambda;\tau)$ and $a^R_{\nu_1 I}(\lambda;\tau)$ is

$$
a^L_{\tau;\nu_1 I}(\lambda;\bar\nu_1,I) - a^R_{\tau;\nu_1 I}(\lambda;\bar\nu_1,I) = \begin{cases} \dfrac{(-1)^{\nu_1/2}2\sqrt{3\tau(\tau+2)}}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} & \text{if } \nu_1 = 0,2\,,\\ 0 & \text{if } \nu_1 = 1\,. \end{cases} \tag{5.3.21}
$$

Thus, the commutator will contain only terms with $\nu_1 = 2$ or $\nu_1 = 0$, with $I = 1/2$ in both cases. Using this we can rewrite the commutator as

$$
\begin{aligned}
[T^\lambda_{1;\alpha J},\hat w_\lambda(\Omega)] &= \sum_\nu D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\sum_\tau F^\tau_\lambda \frac{(-1)^{\nu_1/2}2\sqrt{3\tau(\tau+2)}}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat R(\Omega)\hat T^\lambda_{\tau;\bar\nu\frac{1}{2}}\hat R^\dagger(\Omega)\\
&= \sum_\nu D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\sum_\tau F^\tau_\lambda \frac{(-1)^{\nu_1/2}2\sqrt{3\tau(\tau+2)}}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\sum_{\mu j} D^{(\tau,\tau)}_{\mu j;\bar\nu\frac{1}{2}}(\Omega)\hat T^\lambda_{\tau;\mu j}\,.
\end{aligned}
\tag{5.3.22}
$$

5.4 Differential Realizations on Group Functions

We want to replace the functions $D^{(\tau,\tau)}_{\mu j;\bar\nu\frac{1}{2}}(\Omega)$ of equation (5.3.22) with differential operators $\hat{\mathbb{S}}_{\bar\nu;\frac{1}{2}}$ acting on functions of the type $D^{(\tau,\tau)}_{\mu j;(\tau\tau\tau)0}(\Omega)$, *i.e.* we need to find differential operators $\hat{\mathbb{S}}_\nu$ acting on these functions so that

$$\hat{\mathbb{S}}_{\nu;I} D^{(\tau,\tau)}_{\mu J;(\tau\tau\tau)0}(\Omega) \propto D^{(\tau,\tau)}_{\mu J;\bar{\nu} I}(\Omega)\,. \tag{5.4.1}$$

First, we can recast this as follows. Let $\theta_k \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ and let us start with

$$\begin{aligned}\frac{\partial}{\partial\theta_k} D^{(\tau,\tau)}_{\mu J;(\tau,\tau\tau)0}(\Omega) &= \frac{\partial}{\partial\theta_k}\langle(\tau,\tau)\mu J|\hat{R}(\Omega)|(\tau,\tau)\tau\tau\tau;0\rangle \\ &= \langle(\tau,\tau)\mu J|\frac{\partial}{\partial\theta_k}\hat{R}(\Omega)|(\tau,\tau)\tau\tau\tau;0\rangle \\ &= \langle(\tau,\tau)\mu J|\hat{R}(\Omega)\left(\sum_{\nu I} c_{\nu I}(\theta_k)\hat{\mathcal{C}}_{\nu I}\right)|(\tau,\tau)\tau\tau\tau;0\rangle \quad ,\end{aligned} \tag{5.4.2}$$

where the table bellow gives the $\hat{\mathcal{C}}_{\nu I}$ in function of the $\hat{C}_{ij}$

Table 5.3: The relation between $\hat{\mathcal{C}}_{\nu I}$ and generators

$\hat{\mathcal{C}}_{210;1/2}$	$\hat{C}_{13}$	$\hat{\mathcal{C}}_{201;1/2}$	$-\hat{C}_{12}$
$\hat{\mathcal{C}}_{120;1}$	$\hat{C}_{23}$	$\hat{\mathcal{C}}_{111;1}$	$-\frac{1}{\sqrt{2}}\left(\hat{C}_{22}-\hat{C}_{33}\right)$
$\hat{\mathcal{C}}_{102;1}$	$-\hat{C}_{32}$	$\hat{\mathcal{C}}_{111;0}$	$\frac{1}{\sqrt{6}}\left(2\hat{C}_{11}-\hat{C}_{22}-C_{33}\right)$
$\hat{\mathcal{C}}_{021;1/2}$	$\hat{C}_{21}$	$\hat{\mathcal{C}}_{012;1/2}$	$\hat{C}_{31}$

Defined in this way, the operators $\hat{\mathcal{C}}_{\nu I}$ differ from the generators $\hat{C}_{ij}$ by at most a sign and from the tensor operators $\hat{T}^{\lambda}_{1;\nu I}$ by a normalization that is a function of the $su(3)$ quadratic Casimir invariant and the dimension of the irrep on which the tensors act.

From above we now have the general relation

$$\frac{\partial}{\partial\theta_k}\hat{R}(\Omega) = \sum_{\nu I} c_{\nu I}(\theta_k)\hat{R}(\Omega)\hat{\mathcal{C}}_{\nu I} \tag{5.4.3}$$

It is important to notice that this relation does not depend on the $su(3)$ irrep so the coefficients $c_{\nu I}(\theta_k)$ can be found using any irrep. The most expeditious choice is the 3×3 irrep $(1,0)$. For this representation the operators $\hat{\mathcal{C}}_{\nu I}$ are orthonormal under trace:

$$\mathrm{Tr}\left((\hat{\mathcal{C}}_{\nu' I'})^\dagger\hat{\mathcal{C}}_{\nu I}\right) = \delta_{\nu'\nu}\delta_{I' I} \tag{5.4.4}$$

so we can easily write

$$c_{\nu' I'}(\theta_k) = \mathrm{Tr}\left((\hat{\mathcal{C}}_{\nu' I'})^\dagger\hat{R}^\dagger(\Omega)\frac{\partial}{\partial\theta_k}\hat{R}(\Omega)\right)\,. \tag{5.4.5}$$

The coefficients $c_{\nu I}(\theta_k)$ are given in Table 5.4.

Table 5.4: The $c_{\nu I}(\theta_k)$ coefficients of equation (5.4.3).

νI	$c_{\nu I}(\alpha_1)$	$c_{\nu I}(\beta_1)$
$210; \frac{1}{2}$	$ie^{-i(\alpha_1+\alpha_2)} \sin\left(\frac{\beta_1}{2}\right)\left(\cos^2\left(\frac{\beta_2}{4}\right)+\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\right)\sin\left(\frac{\beta_2}{2}\right)$	$-\frac{1}{2}e^{-i(\alpha_1+\alpha_2)}\cos\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)$
$201; \frac{1}{2}$	$ie^{-i\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\sin\left(\frac{\beta_2}{2}\right)$	$-\frac{1}{2}e^{-i\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)$
$120; 1$	$-ie^{-i\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\left(\cos^2\left(\frac{\beta_2}{4}\right)+\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\right)$	$e^{-i\alpha_1}\sin^2\left(\frac{\beta_2}{4}\right)$
$111; 1$	$-2i\sqrt{2}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{4}\right)\left(\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)+1\right)$	0
$102; 1$	$ie^{i\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\left(\cos^2\left(\frac{\beta_2}{4}\right)+\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\right)$	$e^{i\alpha_1}\sin^2\left(\frac{\beta_2}{4}\right)$
$111; 0$	$i\sqrt{\frac{3}{2}}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{2}\right)$	0
$021; \frac{1}{2}$	$-ie^{i\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\sin\left(\frac{\beta_2}{2}\right)$	$-\frac{1}{2}e^{i\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)$
$012; \frac{1}{2}$	$ie^{i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\left(\cos^2\left(\frac{\beta_2}{4}\right)+\cos(\beta_1)\sin^2\left(\frac{\beta_2}{4}\right)\right)\sin\left(\frac{\beta_2}{2}\right)$	$\frac{1}{2}e^{i(\alpha_1+\alpha_2)}\cos\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)$
νI	$c_{\nu I}(\alpha_2)$	$c_{\nu I}(\beta_2)$
$210; \frac{1}{2}$	$\frac{1}{2}ie^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_2)$	$-\frac{1}{2}e^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$
$201\frac{1}{2}$	$-\frac{1}{2}ie^{-i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\sin(\beta_2)$	$\frac{1}{2}e^{-i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)$
$120; 1$	$-\frac{1}{2}ie^{-i\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right)$	0
$111; 1$	$\frac{i\cos(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right)}{\sqrt{2}}$	0
$102; 1$	$\frac{1}{2}ie^{i\alpha_1}\sin(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right)$	0
$111; 0$	$i\sqrt{\frac{3}{2}}\sin^2\left(\frac{\beta_2}{2}\right)$	0
$021; \frac{1}{2}$	$\frac{1}{2}ie^{i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\sin(\beta_2)$	$\frac{1}{2}e^{i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)$
$012; \frac{1}{2}$	$\frac{1}{2}ie^{i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin(\beta_2)$	$\frac{1}{2}e^{i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$

To continue, it is convenient to divide the generators in two sets. The first contains elements in the $u(2)$ subalgebra: $\{\hat{\mathcal{C}}_{120;1}, \hat{\mathcal{C}}_{111;1}, \hat{\mathcal{C}}_{102;1}, \hat{\mathcal{C}}_{111;0}\}$ and will be labeled by roman letters $a, b, c \ldots$. The second contains the remaining operators $\{\hat{\mathcal{C}}_{210;1/2}, \hat{\mathcal{C}}_{201;1/2}\hat{\mathcal{C}}_{021;1/2}, \hat{\mathcal{C}}_{012;1/2}\}$ and will be labeled using Greek letters $\alpha, \beta \ldots$.

Consider now

$$\sum_k d_\beta(\theta_k)\frac{\partial}{\partial\theta_k}\hat{R}(\Omega) = \sum_{k\alpha} d_\beta(\theta_k)c_\alpha(\theta_k)\hat{R}(\Omega)\hat{\mathcal{C}}_\alpha + \sum_{ak} d_\beta(\theta_k)c_a(\theta_k)\hat{R}(\Omega)\hat{\mathcal{C}}_a \tag{5.4.6}$$

and choose $d_\beta(\theta_k)$ so that

$$\sum_k d_\beta(\theta_k)c_\alpha(\theta_k) = \delta_{\beta\alpha} \tag{5.4.7}$$

$$\sum_k d_\beta(\theta_k)\frac{\partial}{\partial\theta_k}\hat{R}(\Omega) = \hat{R}(\Omega)\hat{\mathcal{C}}_\beta + \sum_a\left(\sum_k d_\beta(\theta_k)c_a(\theta_k)\right)\hat{R}(\Omega)\hat{\mathcal{C}}_a\,. \tag{5.4.8}$$

Using equations (5.4.7) and (5.4.8), I was able to find the expressions for the $d_\beta(\theta_k)$ coefficients. These coefficients are given in Table 5.5.

Table 5.5: The $d_{\nu I}(\theta_k)$ coefficients.

	$\nu I = 210; \frac{1}{2}$	$\nu I = 201; \frac{1}{2}$
$d_{\nu I}(\alpha_1)$	$-\frac{1}{2}ie^{i(\alpha_1+\alpha_2)}\csc\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$	$-\frac{i}{2}e^{i\alpha_2}\csc\left(\frac{\beta_2}{2}\right)\sec\left(\frac{\beta_1}{2}\right)$
$d_{\nu I}(\beta_1)$	$-e^{i(\alpha_1+\alpha_2)}\cos\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$	$-e^{i\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$
$d_{\nu I}(\alpha_2)$	$-2ie^{i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{4}\right)\csc(\beta_2)$	$-\frac{i}{2}\sin\left(\frac{\beta_1}{2}\right)e^{i\alpha_2}\left(\cot(\beta_1)\csc\left(\frac{\beta_2}{2}\right)\right.$ $\left.-2\left(\cos(\beta_1)+\cos\left(\frac{\beta_2}{2}\right)+1\right)\csc(\beta_1)\csc(\beta_2)\right)$
$d_{\nu I}(\beta_2)$	$-e^{i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$	$e^{i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)$
	$\nu I = 012; \frac{1}{2}$	$\nu I = 021; \frac{1}{2}$
$d_{\nu I}(\alpha_1)$	$-\frac{1}{2}ie^{-i(\alpha_1+\alpha_2)}\csc\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$	$\frac{i}{2}e^{-i\alpha_2}\csc\left(\frac{\beta_2}{2}\right)\sec\left(\frac{\beta_1}{2}\right)$
$d_{\nu I}(\beta_1)$	$e^{-i(\alpha_1+\alpha_2)}\cos\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$	$-e^{-i\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\csc\left(\frac{\beta_2}{2}\right)\csc\left(\frac{\beta_2}{2}\right)$
$d_{\nu I}(\alpha_2)$	$-2ie^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\sin^2\left(\frac{\beta_2}{4}\right)\csc(\beta_2)$	$\frac{1}{2}ie^{-i\alpha_2}\sin\left(\frac{\beta_1}{2}\right)\left(\cot(\beta_1)\csc\left(\frac{\beta_2}{2}\right)\right.$ $\left.-2\left(\cos(\beta_1)+\cos\left(\frac{\beta_2}{2}\right)+1\right)\csc(\beta_1)\csc(\beta_2)\right)$
$d_{\nu I}(\beta_2)$	$e^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$	$e^{-i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)$

With this, I can write

$$\hat{\mathbb{S}}_{\nu\frac{1}{2}}D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega) = \sum_k d_{\nu\frac{1}{2}}(\theta_k)\frac{\partial}{\partial\theta_k}D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega) \tag{5.4.9}$$

where $\hat{\mathbb{S}}_{\nu\frac{1}{2}}$ is a differential operator that shifts the function $D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega)$ into $D^{(\tau,\tau)}_{\mu J;\bar{\nu}\frac{1}{2}}(\Omega)$ up to a proportionality term. As one can notice, this operator is written in function of the $d_{\nu\frac{1}{2}}(\theta_k)$ coefficients given on Table 5.5 and first order differentials.

If we substitute equation (5.4.6) into equation (5.4.9) we find

$$\begin{aligned}
\hat{\mathbb{S}}_{\nu\frac{1}{2}}D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega) &= \langle(\tau,\tau)\mu J|\hat{R}(\Omega)\left(\hat{\mathcal{C}}_{\nu\frac{1}{2}} + \sum_{ak} d_{\nu\frac{1}{2}}(\theta_k)c_a(\theta_k)\hat{\mathcal{C}}_a\right)|(\tau,\tau)\tau\tau\tau;0\rangle \\
&= \langle(\tau,\tau)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle \\
&= \langle(\tau,\tau)\mu J|\hat{R}(\Omega)|(\tau,\tau)\bar{\nu};\tfrac{1}{2}\rangle\langle(\tau,\tau)\bar{\nu};\tfrac{1}{2}|\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle \\
&= D^{(\tau,\tau)}_{\mu J;\bar{\nu}\frac{1}{2}}(\Omega)\langle(\tau,\tau)\bar{\nu};\tfrac{1}{2}|\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle
\end{aligned} \tag{5.4.10}$$

since $\hat{\mathcal{C}}_a|(\tau,\tau)\tau\tau\tau;0\rangle = 0$.

Finally, we can evaluate $\langle(\tau,\tau)\bar{\nu};\frac{1}{2}|\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle$. It turns out that this expression is quite simply expressed in terms of ν:

$$\langle(\tau,\tau)\bar{\nu};\tfrac{1}{2}|\hat{\mathcal{C}}_{\nu\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle = (-1)^{\bar{\nu}_1/2}\sqrt{\tfrac{\tau(\tau+2)}{2}}\,, \tag{5.4.11}$$

where the relations between ν and $\bar{\nu}$ are given in equation (5.3.13). Combining this with equation (5.4.10), we now have

$$\hat{\$}_{\nu\frac{1}{2}} D^{(\tau,\tau)}_{\mu J;(\tau\tau\tau)0}(\Omega) = (-1)^{\bar{\nu}_1/2}\sqrt{\frac{\tau(\tau+2)}{2}}\,D^{(\tau,\tau)}_{\mu J;\bar{\nu}\frac{1}{2}}(\Omega) \tag{5.4.12}$$

$$= \langle(\tau,\tau)\mu J|R(\Omega)\hat{\mathcal{C}}_{\nu;\frac{1}{2}}|(\tau,\tau)\tau\tau\tau;0\rangle \quad , \tag{5.4.13}$$

where the $\hat{\$}_{\nu\frac{1}{2}}$ operators are of first order only.

Using this latter result in equation (5.3.22) produces

$$\begin{aligned}
[\hat{T}^{\lambda}_{1;\alpha J},\hat{w}_\lambda(\Omega)] &= \sum_\nu D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\sum_\tau F^\tau_\lambda \frac{(-1)^{\nu_1/2}2\sqrt{3\tau(\tau+2)}}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\\
&\quad\times\sum_{\mu j}(-1)^{\bar{\nu}_1/2}\sqrt{\frac{2}{\tau(\tau+2)}}\hat{\$}_{\nu\frac{1}{2}}D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega)\hat{T}^{(\tau,\tau)}_{\mu j}\\
&= \sqrt{\frac{24}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\sum_\nu D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\$}_{\nu\frac{1}{2}}\sum_\tau F^\tau_\lambda\sum_{\mu j}D^{(\tau,\tau)}_{\mu J,(\tau\tau\tau)0}(\Omega)\hat{T}^{(\tau,\tau)}_{\mu j}\\
&= \sqrt{\frac{24}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\sum_\nu D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\$}_{\nu\frac{1}{2}}\hat{w}_\lambda(\Omega)\\
&= \sqrt{\frac{24}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\sum_\nu D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\left(\sum_k d_{\nu\frac{1}{2}}(\theta_k)\frac{\partial}{\partial\theta_k}\right)\hat{w}_\lambda(\Omega)\,.
\end{aligned} \tag{5.4.14}$$

for the commutator, where equations (5.1.13) and (5.4.9) have been used.

5.4.1 The Moyal Bracket

We are now able to evaluate the Moyal bracket between the Wigner symbol of a generator $W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)$ and Wigner symbol of an arbitrary operator $W_{\hat{B}}$

$$\{W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega),W_{\hat{B}}(\Omega)\}_{\mathcal{M}} = W_{\hat{T}^{\lambda}_{1;\alpha J}\hat{B}}(\Omega) - W_{\hat{B}\hat{T}^{\lambda}_{1;\alpha J}}(\Omega), \tag{5.4.15}$$

since

$$\begin{aligned}
W_{\hat{T}^{\lambda}_{1;\alpha J}\hat{B}}(\Omega) - W_{\hat{B}\hat{T}^{\lambda}_{1;\alpha J}}(\Omega) &= \mathrm{Tr}\Big(\hat{w}_\lambda(\Omega)\hat{T}^{\lambda}_{1;\alpha J}\hat{B}\Big) - \mathrm{Tr}\Big(\hat{w}_\lambda(\Omega)\hat{B}\hat{T}^{\lambda}_{1;\alpha J}\Big)\\
&= \mathrm{Tr}\Big(\hat{w}_\lambda(\Omega)\hat{T}^{\lambda}_{1;\alpha J}\hat{B}\Big) - \mathrm{Tr}\Big(\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_\lambda(\Omega)\hat{B}\Big)
\end{aligned} \tag{5.4.16}$$

which by linearity of the trace it becomes

$$\begin{aligned}\{W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega), W_{\hat{B}}(\Omega)\}_{\mathcal{M}} &= \mathrm{Tr}\Big(\Big(\hat{w}_{\lambda}(\Omega)\hat{T}^{\lambda}_{1;\alpha J} - \hat{T}^{\lambda}_{1;\alpha J}\hat{w}_{\lambda}(\Omega)\Big)\hat{B}\Big) \\ &= \mathrm{Tr}\Big(\Big[\hat{w}_{\lambda}(\Omega), \hat{T}^{\lambda}_{1;\alpha J}\Big]\hat{B}\Big).\end{aligned} \tag{5.4.17}$$

Now we can substitute the result of equation (5.4.14) into equation (5.4.17) to find

$$\begin{aligned}\{W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega), W_{\hat{B}}(\Omega)\}_{\mathcal{M}} &= \sqrt{\frac{24}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\sum_{\nu} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\left(\sum_k d_{\nu\frac{1}{2}}(\theta_k)\frac{\partial}{\partial\theta_k}\right)\mathrm{Tr}(-\hat{B}\hat{w}_{\lambda}(\Omega)) \\ &= -\sqrt{\frac{24}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\sum_{\nu} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\left(\sum_k d_{\nu\frac{1}{2}}(\theta_k)\frac{\partial}{\partial\theta_k}\right)W_{\hat{B}}(\Omega)\end{aligned} \tag{5.4.18}$$

which holds for arbitrary operators $\hat{B}$.

I would like to point out that the expression of equation (5.4.18) only depends on first order derivatives and it is an example of correspondence rules, since one could rewrite equation (5.4.18) as

$$\{W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega), W_{\hat{B}}(\Omega)\}_{\mathcal{M}} = \hat{\mathcal{T}}_{\alpha J}(\Omega)W_{\hat{B}}(\Omega) \tag{5.4.19}$$

where the differential operator $\hat{\mathcal{T}}_{\alpha J}(\Omega)$ is

$$\hat{\mathcal{T}}_{\alpha J}(\Omega) = -\sqrt{\frac{24}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\sum_{\nu} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\left(\sum_k d_{\nu\frac{1}{2}}(\theta_k)\frac{\partial}{\partial\theta_k}\right). \tag{5.4.20}$$

5.4.2 Comparison of the Moyal Bracket with the Poisson Bracket

The Poisson bracket is given by [13]

$$\begin{aligned}&\{W_{\hat{T}^{\lambda}_{1;\alpha J}}, W_{\hat{B}}\}_{\mathcal{P}} \\ &\quad= \frac{4}{\sin\beta_1\sin^2\frac{1}{2}\beta_2}\left(\frac{\partial W_{\hat{T}^{\lambda}_{1;\alpha J}}}{\partial\alpha_1}\frac{\partial W_{\hat{B}}}{\partial\beta_1} - \frac{\partial W_{\hat{T}^{\lambda}_{1;\alpha J}}}{\partial\beta_1}\frac{\partial W_{\hat{B}}}{\partial\alpha_1}\right) \\ &\quad- \frac{2\tan\frac{1}{2}\beta_1}{\sin^2\frac{1}{2}\beta_2}\left(\frac{\partial W_{\hat{T}^{\lambda}_{1;\alpha J}}}{\partial\alpha_2}\frac{\partial W_{\hat{B}}}{\partial\beta_1} - \frac{\partial W_{\hat{T}^{\lambda}_{1;\alpha J}}}{\partial\beta_1}\frac{\partial W_{\hat{B}}}{\partial\alpha_2}\right) \\ &\quad+ \frac{4}{\sin\beta_2}\left(\frac{\partial W_{\hat{T}^{\lambda}_{1;\alpha J}}}{\partial\alpha_2}\frac{\partial W_{\hat{B}}}{\partial\beta_2} - \frac{\partial W_{\hat{T}^{\lambda}_{1;\alpha J}}}{\partial\beta_2}\frac{\partial W_{\hat{B}}}{\partial\alpha_2}\right) \quad .\end{aligned} \tag{5.4.21}$$

One verifies that, for any operator $\hat{B}$,

$$\{W_{T^{\lambda}_{1;\alpha J}}(\Omega), W_B(\Omega)\}_{\mathcal{M}} = -\frac{i}{\epsilon}\{W_{T^{\lambda}_{1;\alpha J}}(\Omega), W_B(\Omega)\}_{\mathcal{P}} \tag{5.4.22}$$

where

$$\epsilon = 2\sqrt{\lambda(\lambda+3)} \tag{5.4.23}$$

is the semi-classical parameter.

We can bring the Moyal bracket in the form of equation (5.4.18) closer to the form of the Poisson bracket of equation (5.4.21). Let $\hat{B} = \hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}$. If we recall equation (5.3.3) and note that

$$\begin{aligned}
D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1}) &= \left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^{*} \\
&= (-1)^{\nu_1/2}\sqrt{\frac{2}{3}}\left(\hat{\$}_{\nu\frac{1}{2}} D^{(1,1)}_{\alpha J;(111)0}(\Omega)\right)^{*} \\
&= (-1)^{\nu_1/2}\sqrt{\frac{2}{3}}\hat{\$}^{*}_{\nu\frac{1}{2}}(-1)^{1+\alpha_2} D^{(1,1)}_{\alpha^{*} J;(111)0}(\Omega) \\
&= (-1)^{\nu_1/2}\sqrt{\frac{2}{3}}\hat{\$}^{*}_{\nu\frac{1}{2}}(-1)^{1+\alpha_2} D^{(1,1)}_{\alpha^{*} J;(111)0}(\Omega) \\
&= (-1)^{\nu_1/2}(-1)^{1+\alpha_2}\sqrt{\frac{2}{3}}\left(\sum_r (d_{\nu\frac{1}{2}}(\theta_r))^{*}\frac{\partial}{\partial\theta_r}\right) D^{(1,1)}_{\alpha^{*} J;(111)0}(\Omega) \\
&= (-1)^{\nu_1/2}\sqrt{\frac{2}{3}}\left(\sum_r (d_{\nu\frac{1}{2}}(\theta_r))^{*}\frac{\partial}{\partial\theta_r}\right)\frac{\sqrt{(\lambda+1)(\lambda+2)}}{4} W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\,.
\end{aligned} \tag{5.4.24}$$

Inserting this into equation (5.4.18) produces

$$\{W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega), W_{\hat{B}}(\Omega)\}_{\mathscr{M}} = -\frac{1}{\sqrt{\lambda(\lambda+3)}}\sum_{r,k}\left(\sum_{\nu}(-1)^{\nu_1/2}(d_{\nu\frac{1}{2}}(\theta_r))^{*} d_{\nu\frac{1}{2}}(\theta_k)\right)\frac{\partial W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)}{\partial\theta_r}\frac{\partial W_{\hat{B}}(\Omega)}{\partial\theta_k} \tag{5.4.25}$$

For each pair (r,k) we can evaluate $\sum_{\nu}(-1)^{\nu_1/2}(d_{\nu\frac{1}{2}}(\theta_r))^{*} d_{\nu\frac{1}{2}}(\theta_k)$ to obtain

r	k	$\sum_{\nu}(-1)^{\nu_1/2}(d_{\nu\frac{1}{2}}(\theta_r))^{*} d_{\nu\frac{1}{2}}(\theta_k)$
1	2	$\frac{2i}{\sin\beta_1\sin^2\frac{1}{2}\beta_2}$
2	1	$-\frac{2i}{\sin\beta_1\sin^2\frac{1}{2}\beta_2}$
2	3	$\frac{i\tan\frac{\beta_1}{2}}{\sin^2\frac{1}{2}\beta_2}$
3	2	$-\frac{i\tan\frac{\beta_1}{2}}{\sin^2\frac{1}{2}\beta_2}$
3	4	$\frac{2i}{\sin\beta_2}$
4	3	$-\frac{2i}{\sin\beta_2}$

(5.4.26)

with all other entries 0. These coefficients differ from those in the expression of the Poisson bracket of equation (5.4.21) by an overall factor of $-2i$.

Table (5.9) contains some Moyal brackets calculated via equation (5.4.25).

Table 5.6: Moyal bracket between the Wigner symbol of some generators and the operator $\hat{B}$, where $\lambda = 15$.

αJ	$\hat{B}$	$\{W_{\hat{T}^{15}_{1;\alpha J}}, W_{\hat{B}}\}_{\mathcal{M}}$
$(201)\frac{1}{2}$	$\hat{T}^{15}_{1;(111)0}$	$\frac{1}{136\sqrt{5}}\exp(i\alpha_2)\cos(\frac{\beta_1}{2})\sin\beta_2$
$(012)\frac{1}{2}$	$(\hat{T}^{15}_{1;(201)\frac{1}{2}})^2$	$\sqrt{\frac{133}{51}}\frac{1}{4080}\exp(-i\alpha_2)\sin(\beta_2)\cos(\frac{\beta_1}{2})$ $\times(\cos(\beta_1)-(\cos(\beta_1)+3)\cos(\beta_2)-1)$
$(111)0$	$\hat{T}^{15}_{1;(210)\frac{1}{2}}$	$\frac{1}{136\sqrt{5}}\exp(i(\alpha_1+\alpha_2))\sin(\frac{\beta_1}{2})\sin(\beta_2)$
$(111)1$	$\hat{T}^{15}_{2;(024)1}$	$-\frac{1}{816}\sqrt{\frac{133}{5}}\exp(2i(\alpha_1+\alpha_2))\sin^2(\frac{\beta_1}{2})\sin^2(\beta_2)$

5.5 Correspondence Rules for the Generators and Their Products

In order to proceed with the $\star$-product we need to go back to equations (5.3.15) and (5.3.19) and observe that the sums will contain terms with $\nu_1 = 1$ and $I = 0, 1$ for which we do not yet have a differential action. These terms were not present in the commutation relations between a generator $\hat{T}^{\lambda}_{1;\alpha J}$ and the quantization kernel $\hat{w}_{\lambda}(\Omega)$ of equation (5.3.22) because the difference between $a^{L}_{\tau;\nu_1 I}(\lambda;\bar{\nu}_1, I)$ and $a^{R}_{\tau;\nu_1 I}(\lambda;\bar{\nu}_1, I)$ in equation (5.3.21) is zero for $\nu = 1$ and $I = 0, 1$. It is possible to find a second order differential operator $\mathbb{S}^{(2)}_{(1\nu_2\nu_3)I}$ which acts on the $D^{(\sigma,\sigma)}_{\mu j;(\sigma\sigma\sigma)0}(\Omega)$ functions to produce $D^{(\sigma,\sigma)}_{\mu j;(1\nu_2\nu_3)1}(\Omega)$.

5.5.1 Expression for the $\star$-product in Terms of D-functions

If we start with equation (5.3.6) and write $\hat{A} = \hat{T}^{\lambda}_{1;\alpha J}$, we can then use equation (5.3.19) to obtain

$$W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega) \star W_{\hat{B}}(\Omega) = \sum_{\nu I} D^{(1,1)}_{\nu I;\alpha J}(\Omega^{-1}) \sum_{\tau\nu' I'} F^{\tau}_{\lambda}\, a^{R}_{\tau;\nu_1 I}(\lambda;\bar{\nu}_1, I) D^{(\tau,\tau)}_{\nu' I';\bar{\nu} I}(\Omega) \mathrm{Tr}\left(\hat{T}^{\lambda}_{\tau;\nu' I'}\hat{B}\right). \tag{5.5.1}$$

Specializing to $\hat{B} = \hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}$ and using

$$\mathrm{Tr}\left(\hat{T}^{\lambda}_{\tau;\nu' I'}\hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}\right) = (-1)^{\tau+\nu'_2}\mathrm{Tr}\left((\hat{T}^{\lambda}_{\tau;\nu'^{*} I'})^{\dagger}\hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}\right) = \delta_{\sigma\tau}\delta_{\bar{\mu}\nu'^{*}}\delta_{\bar{J}I'} \tag{5.5.2}$$

so that

$$\begin{aligned} W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega) \star W_{\hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}}(\Omega) = &\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1}) F^{\sigma}_{\lambda}(-1)^{\sigma-\bar{\mu}_2} a^{R}_{\tau;\nu_1\frac{1}{2}}(\lambda;\bar{\nu}_1,\frac{1}{2}) D^{(\sigma,\sigma)}_{\bar{\mu}^{*}J;\bar{\nu}\frac{1}{2}}(\Omega) \\ &+ \sum_{\nu_2\nu_3} D^{(1,1)}_{(1\nu_2\nu_3)1;\alpha J}(\Omega^{-1}) F^{\sigma}_{\lambda}(-1)^{\sigma-\bar{\mu}_2} a^{R}_{\tau;11}(\lambda;\tau,1) D^{(\sigma,\sigma)}_{\bar{\mu}^{*}J;(1\nu_2\nu_3)1}(\Omega) \\ &+ D^{(1,1)}_{(111)0;\alpha J}(\Omega^{-1}) F^{\sigma}_{\lambda}(-1)^{\sigma-\bar{\mu}_2} a^{R}_{\tau;10}(\lambda;\tau,0) D^{(\sigma,\sigma)}_{\bar{\mu}^{*}J;(111)0}(\Omega)\,. \end{aligned} \tag{5.5.3}$$

We can use equation (5.4.12) to rewrite $D^{(\sigma,\sigma)}_{\bar{\mu}^* J;\bar{\nu}\frac{1}{2}}(\Omega)$ as a differential operator $\hat{\mathbb{S}}_{\bar{\nu};\frac{1}{2}}$ acting on a function $D^{(\sigma,\sigma)}_{\bar{\mu}^* J;(\sigma\sigma\sigma)0}(\Omega)$. The last term in the sum of equation (5.5.3) is already a function of the right form. There remains to express the middle part in the form

$$D^{(\sigma,\sigma)}_{\bar{\mu}^* J;(1\nu_2\nu_3)1}(\Omega) \sim \hat{\mathbb{S}}^{(2)}_{(1\nu_2\nu_3)1} D^{(\sigma,\sigma)}_{\bar{\mu}^* J;(111)0}(\Omega) \tag{5.5.4}$$

It turns out there is *no first order operator* that will satisfy equation (5.5.4), but we can find a *second order operator* that will produce what we want. This is a major difference between the $su(3)$ algebra and the other two cases of $su(2)$ and hw algebra.

5.5.2 Products of differential operators

In order to obtain a differential action like the one in equation (5.5.4), we consider

$$\begin{aligned}
\hat{\mathbb{S}}_\alpha \hat{\mathbb{S}}_\beta D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) &= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\left(\hat{\mathcal{C}}_\alpha + \sum_{ak} d_\alpha(\theta_k)c_a(\theta_k)\hat{\mathcal{C}}_a\right)\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,, &(5.5.5)\\
&= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_\alpha\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad + \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\sum_{ak} d_\beta(\theta_k)c_a(\theta_k)\hat{\mathcal{C}}_a\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,. &(5.5.6)
\end{aligned}$$

Now, since $\hat{\mathcal{C}}_\beta$ is an element of the $u(1)\oplus su(2)$ subalgebra, we have

$$\begin{aligned}
\hat{\mathcal{C}}_a\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle &= [\hat{\mathcal{C}}_a,\hat{\mathcal{C}}_\beta]|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle + \hat{\mathcal{C}}_\beta\hat{\mathcal{C}}_a|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&= [\hat{\mathcal{C}}_a,\hat{\mathcal{C}}_\beta]|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&= \sum_\gamma g^\gamma_{a\beta}\hat{\mathcal{C}}_\gamma|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle &(5.5.7)
\end{aligned}$$

As we have for $\hat{\mathcal{C}}_\gamma$

$$\hat{\mathbb{S}}_\gamma D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) = \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_\gamma|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,, \tag{5.5.8}$$

we find that

$$\begin{aligned}
\hat{\mathbb{S}}_\alpha \hat{\mathbb{S}}_\beta D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) &= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\left(\hat{\mathcal{C}}_\alpha + \sum_{ak} d_\alpha(\theta_k)c_a(\theta_k)\hat{\mathcal{C}}_a\right)\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&= \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_\alpha\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad + \sum_{ak\gamma} d_\alpha(\theta_k)c_a(\theta_k)g^\gamma_{a\beta}\hat{\mathbb{S}}_\gamma D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) &(5.5.9)
\end{aligned}$$

or

$$\left(\hat{\mathbb{S}}_\alpha\hat{\mathbb{S}}_\beta - \sum_{a\gamma} f_{\alpha a} g^\gamma_{a\beta}\hat{\mathbb{S}}_\gamma\right) D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega) = \langle(\sigma,\sigma)\mu J|\hat{R}(\Omega)\hat{\mathcal{C}}_\alpha\hat{\mathcal{C}}_\beta|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\,, \tag{5.5.10}$$

where, for economy, we denote

$$f_{\alpha a} := \sum_k d_\alpha(\theta_k) c_a(\theta_k)\,. \tag{5.5.11}$$

These coefficients are given in Table 5.7.

Table 5.7: The sums $f_{\beta a} = \sum_k d_\beta(\theta_k) c_a(\theta_k)$.

β	$a = 120;1$	$111;1$
$012;\frac{1}{2}$	$-2e^{-i(2\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)$ $\times\sin(\beta_1)\sin^4\left(\frac{\beta_2}{4}\right)\csc(\beta_2)$	$e^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{\sqrt{2}}\left(\sin^2\left(\frac{\beta_2}{4}\right)\cos(\beta_1)\sec\left(\frac{\beta_2}{2}\right)-1\right)$
$021;\frac{1}{2}$	$-\frac{1}{4}e^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\left(\sec\left(\frac{\beta_2}{2}\right)\left(2\sin^2\left(\frac{\beta_2}{4}\right)\cos(\beta_1)+1\right)+3\right)$	$\frac{1}{\sqrt{2}}e^{-i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\left(\sin^2\left(\frac{\beta_2}{4}\right)\cos(\beta_1)\sec\left(\frac{\beta_2}{2}\right)+1\right)$
$201;\frac{1}{2}$	$e^{-i\alpha_1}\sin\alpha_2(\cot\alpha_2+i)$ $\times\sin^2\beta_1\csc\left(\frac{\beta_1}{2}\right)\sin^4\left(\frac{\beta_2}{4}\right)\csc\beta_2$	$-\frac{1}{\sqrt{2}}e^{i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\left(\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1\sec\left(\frac{\beta_2}{2}\right)+1\right)$
$210;\frac{1}{2}$	$e^{i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{4}\left(\sec\left(\frac{\beta_2}{2}\right)\left(2\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1-1\right)-3\right)$	$e^{i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{\sqrt{2}}\left(\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1\sec\left(\frac{\beta_2}{2}\right)-1\right)$
β	$a = 102;1$	$111;0$
$012;\frac{1}{2}$	$e^{-i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{4}\left(\sec\left(\frac{\beta_2}{2}\right)\left(1-2\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1\right)+3\right)$	$\frac{1}{2}\sqrt{\frac{3}{2}}e^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{2}\right)$
$021;\frac{1}{2}$	$-e^{-i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\frac{1}{4}\left(\sec\left(\frac{\beta_2}{2}\right)\left(2\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1+1\right)+3\right)$	$\frac{1}{2}\sqrt{\frac{3}{2}}e^{-i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{2}\right)$
$201;\frac{1}{2}$	$e^{-i\alpha_1}\sin\alpha_2\csc\left(\frac{\beta_1}{2}\right)\csc\beta_2$ $\times\sin^2(\beta_1)\sin^4\left(\frac{\beta_2}{4}\right)(\cot(\alpha_2)+i)$	$-\frac{1}{2}\sqrt{\frac{3}{2}}e^{i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{2}\right)$
$210;\frac{1}{2}$	$\frac{1}{4}e^{i\alpha_2}\cos\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{4}\right)$ $\times\left(\sec\left(\frac{\beta_2}{2}\right)\left(2\sin^2\left(\frac{\beta_2}{4}\right)\cos\beta_1-1\right)-3\right)$	$\frac{1}{2}\sqrt{\frac{3}{2}}e^{i(\alpha_1+\alpha_2)}\sin\left(\frac{\beta_1}{2}\right)\tan\left(\frac{\beta_2}{2}\right)$

5.5.3 The Second Order Operator $\hat{\mathbb{S}}^{(2)}_{(1\nu_2\nu_3)I}$

In order to obtain the second order operator, which shifts the I label by one, I had to work with products of the operators $\hat{\mathbb{S}}_{\nu\frac{1}{2}}$.

If we consider

$$\hat{\mathbb{S}}_{(021)\frac{1}{2}}\hat{\mathbb{S}}_{(210)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma,\sigma,\sigma)0}(\Omega) = \langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{(021)\frac{1}{2}}\hat{\mathcal{C}}_{(210)\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle + \sum_a f_{(021)\frac{1}{2};a}\,\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\Big[\hat{\mathcal{C}}_a,\hat{\mathcal{C}}_{(210)\frac{1}{2}}\Big]\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle \quad , \tag{5.5.12}$$

where the $f_{\alpha a}$ coefficients that appear in the expression above are given in equation (5.5.11). Using

$$\begin{aligned}
&\sum_a f_{(021)\frac{1}{2};a}\,\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\Big[\hat{\mathcal{C}}_a,\hat{\mathcal{C}}_{(210)\frac{1}{2}}\Big]\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad=\Big(-f_{(021)\frac{1}{2};(102)\frac{1}{2}}\,\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{(201)\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\qquad+\Big(-\tfrac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)1}+\sqrt{\tfrac{3}{2}}f_{(021)\frac{1}{2};(111)0}\Big)\,\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{(210)\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\Big)\\
&\quad=\Big(-f_{(021)\frac{1}{2};(102)\frac{1}{2}}\hat{\mathbb{S}}_{(201)\frac{1}{2}}+\Big(-\tfrac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)1}+\sqrt{\tfrac{3}{2}}f_{(021)\frac{1}{2};(111)0}\Big)\hat{\mathbb{S}}_{(210)\frac{1}{2}}\Big)\,D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)
\end{aligned}\tag{5.5.13}$$

and

$$\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{(021)\frac{1}{2}}\hat{\mathcal{C}}_{(210)\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle = -\frac{\sigma(\sigma+2)}{\sqrt{6}}D^{(\sigma,\sigma)}_{\mu J;(\sigma,\sigma+1,\sigma-1)1}(\Omega) \tag{5.5.14}$$

we obtain the expression

$$\begin{aligned}
&-\frac{\sigma(\sigma+2)}{\sqrt{6}}D^{(\sigma,\sigma)}_{\mu J;(\sigma,\sigma+1,\sigma-1)1}(\Omega)\\
&\quad=\hat{\mathbb{S}}_{(021)\frac{1}{2}}\hat{\mathbb{S}}_{(210)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}\\
&\qquad+\Big(f_{(021)\frac{1}{2};(102)\frac{1}{2}}\hat{\mathbb{S}}_{(201)\frac{1}{2}}+\tfrac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)1}\hat{\mathbb{S}}_{(210)\frac{1}{2}}-\sqrt{\tfrac{3}{2}}f_{(021)\frac{1}{2};(111)0}\hat{\mathbb{S}}_{(210)\frac{1}{2}}\Big)\,D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\\
&\quad:=\hat{\mathbb{S}}^{(2)}_{(120);1}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}\quad.(\Omega)
\end{aligned}\tag{5.5.15}$$

Similarly, starting with

$$\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{(012)\frac{1}{2}}\hat{\mathcal{C}}_{(201)\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle = -\frac{\sigma(\sigma+2)}{\sqrt{6}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma,\sigma-1,\sigma+1)1}(\Omega) \quad , \tag{5.5.16}$$

we easily reach

$$\begin{aligned}
&-\frac{\sigma(\sigma+2)}{\sqrt{6}}D^{(\sigma,\sigma)}_{\mu J;(\sigma,\sigma-1,\sigma+1)1}(\Omega)\\
&\quad=\hat{\mathbb{S}}_{(012)\frac{1}{2}}\hat{\mathbb{S}}_{(201)\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}\\
&\qquad-\Big(f_{(012)\frac{1}{2};(120)1}\hat{\mathbb{S}}_{(210)\frac{1}{2}}+\Big(\tfrac{\sqrt{2}}{2}f_{(012)\frac{1}{2};(111)1}+\sqrt{\tfrac{3}{2}}f_{(012)\frac{1}{2};(111)0}\Big)\hat{\mathbb{S}}_{(201)\frac{1}{2}}\Big)D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\\
&\quad:=\hat{\mathbb{S}}^{(2)}_{(102);1}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\quad.
\end{aligned}\tag{5.5.17}$$

Finally, we consider the action

$$\begin{aligned}
\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\Big(\hat{\mathcal{C}}_{021;\frac{1}{2}}\hat{\mathcal{C}}_{201;\frac{1}{2}}+\hat{\mathcal{C}}_{012;\frac{1}{2}}\hat{\mathcal{C}}_{210;\frac{1}{2}}\Big)\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle &= -\frac{\sigma(\sigma+2)}{\sqrt{3}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)1}(\Omega)\\
&= -\frac{1}{\sqrt{3}}\hat{\mathbb{S}}^{(2)}_{(111);1}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\,.
\end{aligned}\tag{5.5.18}$$

We can then verify that

$$
\begin{aligned}
&\hat{\mathbb{S}}_{021;\frac{1}{2}}\hat{\mathbb{S}}_{201;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)}(\Omega)\\
&\quad=\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{021;\frac{1}{2}}\hat{\mathcal{C}}_{201;\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle+f_{(021)\frac{1}{2};(120)1}\,\langle(\sigma,\sigma)\mu J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{210;\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\qquad+\left(\sqrt{\tfrac{3}{2}}f_{(021)\frac{1}{2};(111)1}+\tfrac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)0}\right)\langle(\sigma,\sigma)\mu J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{201;\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad=\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{021;\frac{1}{2}}\hat{\mathcal{C}}_{201;\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\qquad+\left(f_{(021)\frac{1}{2};(120)1}\hat{\mathbb{S}}_{210;\frac{1}{2}}+\left(\sqrt{\tfrac{3}{2}}f_{(021)\frac{1}{2};(111)1}+\tfrac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{201;\frac{1}{2}}\right)D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\quad,
\end{aligned}
\tag{5.5.19}
$$

where equation (5.4.13) has been used.

Duplicating the same steps, this time for $\hat{\mathbb{S}}_{012;\frac{1}{2}}\hat{\mathbb{S}}_{210;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)$, yields

$$
\begin{aligned}
&\hat{\mathbb{S}}_{012;\frac{1}{2}}\hat{\mathbb{S}}_{210;\frac{1}{2}}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\\
&\quad=\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{012;\frac{1}{2}}\hat{\mathcal{C}}_{210;\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle-f_{(012)\frac{1}{2};(102)1}\,\langle(\sigma,\sigma)\mu J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{201;\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\qquad+\left(\sqrt{\tfrac{3}{2}}f_{(012)\frac{1}{2};(111)1}-\tfrac{\sqrt{2}}{2}f_{(012)\frac{1}{2};(111)0}\right)\langle(\sigma,\sigma)\mu J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{210;\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\quad=\langle(\sigma,\sigma)\mu;J|\,\hat{R}(\Omega)\hat{\mathcal{C}}_{021;\frac{1}{2}}\hat{\mathcal{C}}_{201;\frac{1}{2}}\,|(\sigma,\sigma)\sigma\sigma\sigma;0\rangle\\
&\qquad+\left(f_{(012)\frac{1}{2};(102)1}\hat{\mathbb{S}}_{201;\frac{1}{2}}+\left(\sqrt{\tfrac{3}{2}}f_{(012)\frac{1}{2};(111)1}-\tfrac{\sqrt{2}}{2}f_{(012)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{210;\frac{1}{2}}\right)D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\,.
\end{aligned}
\tag{5.5.20}
$$

Hence:

$$
\begin{aligned}
&-\frac{1}{\sqrt{3}}\hat{\mathbb{S}}^{(2)}_{(111);1}D^{(\sigma,\sigma)}_{\mu,J;(\sigma\sigma\sigma)0}(\Omega)\\
&=\left(\hat{\mathbb{S}}_{021;\frac{1}{2}}\hat{\mathbb{S}}_{201;\frac{1}{2}}-f_{(021)\frac{1}{2};(120)1}\hat{\mathbb{S}}_{210;\frac{1}{2}}-\left(\sqrt{\tfrac{3}{2}}f_{(021)\frac{1}{2};(111)1}+\tfrac{\sqrt{2}}{2}f_{(021)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{201;\frac{1}{2}}\right)D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\\
&+\left(\hat{\mathbb{S}}_{012;\frac{1}{2}}\hat{\mathbb{S}}_{210;\frac{1}{2}}-f_{(012)\frac{1}{2};(102)1}\hat{\mathbb{S}}_{201;\frac{1}{2}}-\left(\sqrt{\tfrac{3}{2}}f_{(012)\frac{1}{2};(111)1}-\tfrac{\sqrt{2}}{2}f_{(012)\frac{1}{2};(111)0}\right)\hat{\mathbb{S}}_{210;\frac{1}{2}}\right)D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)\,.
\end{aligned}
\tag{5.5.21}
$$

We can summarize equations (5.5.15), (5.5.17) and (5.5.21) as

$$
\hat{\mathbb{S}}^{(2)}_{1\nu_2\nu_3;1}D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}(\Omega)=-\sigma(\sigma+2)\sqrt{\frac{(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}{6}}D^{(\sigma,\sigma)}_{\mu J;(1\nu_2\nu_3)1}(\Omega)\,,
\tag{5.5.22}
$$

The operators that were constructed via equation (5.5.22) together with the ones that shift the label I by $\frac{1}{2}$ form a set of differential operators that act on the functions $D^{(\sigma,\sigma)}_{\mu J;(\sigma\sigma\sigma)0}$, changing the index $(\sigma\sigma\sigma)0$.

5.5.4 The $\star$-product in terms of differential operators

We can now write the $\star$-product of equation (5.5.3) in terms of the differential operators $\hat{\mathbb{S}}_{\nu\frac{1}{2}}$ and $\hat{\mathbb{S}}^{(2)}_{\nu 1}$ acting on functions $D^{(\sigma,\sigma)}_{\mu^* J;(\sigma\sigma\sigma)0}(\Omega)$. Using equations (5.4.12) and (5.5.22), we obtain

$$
\begin{aligned}
&W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\star W_{\hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}}(\Omega)\\
&\quad=\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\, a^R_{\sigma;\nu\frac{1}{2}}(\lambda;\bar{\nu},\frac{1}{2})(-1)^{\nu_1/2}\sqrt{\frac{2}{\sigma(\sigma+2)}}\hat{\mathbb{S}}_{\nu\frac{1}{2}}\left(F^{\sigma}_{\lambda}(-1)^{\sigma-\bar{\mu}_2}D^{(\sigma,\sigma)}_{\bar{\mu}^* J;(\sigma\sigma\sigma)0}(\Omega)\right)\\
&\qquad-\sum_{\nu_2\nu_3} D^{(1,1)}_{(1\nu_2\nu_3)1;\alpha J}(\Omega^{-1})\, a^R_{\sigma;11}(\lambda;\sigma,1)\left(\sqrt{\frac{6}{1+\delta_{\nu_2 1}\delta_{\nu_3 1}}}\frac{1}{\sigma(\sigma+2)}\right)\hat{\mathbb{S}}^{(2)}_{(1\nu_2\nu_3);1}\left(F^{\sigma}_{\lambda}(-1)^{\sigma-\bar{\mu}_2}D^{(\sigma,\sigma)}_{\bar{\mu}^* J;(\sigma\sigma\sigma)0}\right)\\
&\qquad+D^{(1,1)}_{(111)0;\alpha J}(\Omega^{-1})\, a^R_{\sigma;10}(\lambda;\sigma,0)\left(F^{\sigma}_{\lambda}(-1)^{\sigma-\bar{\mu}_2}D^{(\sigma,\sigma)}_{\bar{\mu}^* J;(111)0}(\Omega)\right)\,.
\end{aligned}
\tag{5.5.23}
$$

Finally, using equation (5.3.3), we can write

$$
\begin{aligned}
\left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^* &= (-1)^{\frac{\nu_1}{2}}\sqrt{\frac{2}{3}}\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}D^{(1,1)}_{\alpha J;(111)0}(\Omega)\right)^* = (-1)^{\frac{\nu_1}{2}}\sqrt{\frac{(\lambda+1)(\lambda+2)}{24}}\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\right)^* W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\\
\left(D^{(1,1)}_{\alpha J;\nu 1}(\Omega)\right)^* &= -\sqrt{\frac{2}{3(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}}\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}D^{(1,1)}_{\alpha J;(111)0}(\Omega)\right)^*\\
&= -\sqrt{\frac{(\lambda+1)(\lambda+2)}{24(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}}\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}\right)^* W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\quad .
\end{aligned}
\tag{5.5.24}
$$

With this we can bring equation (5.5.23) into a more symmetrical form:

$$
\begin{aligned}
&\sqrt{\frac{24}{(\lambda+1)(\lambda+2)}}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\star W_{\hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}}(\Omega)\\
&\quad=\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3}\left(\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\right)^* W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right) a^R_{\sigma;\nu_1\frac{1}{2}}(\lambda;\bar{\nu}_1,\frac{1}{2})\sqrt{\frac{2}{\sigma(\sigma+2)}}\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}W_{\hat{T}^{\lambda}_{\sigma;\mu J}}(\Omega)\right)\\
&\qquad+\sum_{\nu_2\nu_3}\left(\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}\right)^* W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right) a^R_{\sigma;11}(\lambda;\sigma,1)\frac{\sqrt{6}}{\sigma(\sigma+2)(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}W_{\hat{T}^{\lambda}_{\sigma;\mu J}}(\Omega)\right)\\
&\qquad+\sqrt{\frac{3}{2}}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\, a^R_{\sigma;10}(\lambda;\sigma,0)W_{\hat{T}^{\lambda}_{\sigma;\mu J}}(\Omega)
\end{aligned}
\tag{5.5.25}
$$

We can compare this form to that of the $su(2)$ $\star$-product presented in equation (3.2.25) from chapter 2 of this thesis and found by Klimov and Espinoza in [10]. The factor $(\lambda+1)(\lambda+2)$ is basically the size of the irrep $(\lambda,0)$ of the Hilbert space of quantum states. The operators $\hat{\mathbb{S}}_{\nu 1/2}$ and $\hat{\mathbb{S}}^{(2)}_{\nu 1}$ play the role of $\hat{\mathbb{S}}^{\pm(j)}$ of Klimov and Espinoza.

The full correspondence rules are obtained by one final last step: to express the coefficients $a^R_{\sigma;\nu_1 I}(\lambda;\bar{\nu}_1,I)$ of equation (5.3.18) as functions of the $su(3)$ Casimir invariant $\hat{\mathcal{C}}^{(2)}$. Here, it is enough to know that this operator commutes with any $SU(3)$ transformation and any $su(3)$ generator. Acting on any state $|(\tau,\tau)\nu I\rangle$, we have

$$\hat{\mathcal{C}}^{(2)}\,|(\tau,\tau)\nu I\rangle = 2\tau(\tau+2)\,|(\tau,\tau)\nu I\rangle\ . \tag{5.5.26}$$

This operator is given by

$$\hat{\mathcal{C}}^{(2)} = \sum_{i\neq j}\hat{C}_{ij}\hat{C}_{ji} + \hat{H}_1^2 + \hat{H}_0^2\,, \tag{5.5.27}$$

with

$$\hat{H}_1 = -\frac{1}{\sqrt{2}}(\hat{C}_{22}-\hat{C}_{33})\,, \qquad \hat{H}_0 = \frac{1}{\sqrt{6}}(2\hat{C}_{11}-\hat{C}_{22}-\hat{C}_{33})\,. \tag{5.5.28}$$

A function of λ and τ, such as $\sqrt{\lambda-\tau+1}$, can then be formally expanded as

$$\sqrt{\lambda-\tau+1} = \sqrt{\lambda+1}\sqrt{1-\frac{\tau}{\lambda+1}} = \sqrt{\lambda+1}\left(\frac{\tau^2}{8(\lambda+1)^2}+\frac{\tau}{2(\lambda+1)}+\dots\right)\,. \tag{5.5.29}$$

Using again the formal identify valid for any state in (τ,τ):

$$\tau = -1 + \frac{\sqrt{2+\hat{\mathcal{C}}^{(2)}}}{\sqrt{2}} \tag{5.5.30}$$

one can recompose the series of equation (5.5.29) and express $\sqrt{\lambda-\tau+1}$ as a formal function of $\hat{\mathcal{C}}^{(2)}$. Expressing the coefficients $a^R_{\tau;2\frac{1}{2}}(\lambda;\tau+1,\frac{1}{2})$ this way, and using the differential operators $\hat{\mathbb{S}}_{\nu I}$ and $\hat{\mathbb{S}}^{(2)}_{\nu I}$, gives the complete expression for the correspondence rules.

5.5.5 Asymptotic form of the $\star$-product

Because the exact functions $a^R_{\tau;2\frac{1}{2}}(\lambda;\tau+1,\frac{1}{2})$ are quite complicated functions of $\hat{\mathcal{C}}^{(2)}$, and because Wigner functions provide a bridge to the semi-classical limit reached when $\lambda\to\infty$, the correspondence rules and the $\star$-product are used in practical situation in this classical limit. Therefore, one should consider that in the limit of large λ we have

$$a^L_{\tau;1,1}(\infty;\tau,1) = a^R_{\tau;1,1}(\infty;\tau,1) \sim -\frac{\tau(\tau+2)}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\frac{(2\lambda-3)}{4\lambda^2}\,, \tag{5.5.31}$$

$$a^L_{\tau;1,0}(\infty;\tau,0) = a^R_{\tau;1,0}(\infty;\tau,0) \sim \frac{1}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\left(4\lambda+6-\frac{3(2\lambda-3)(\tau(\tau+2)+3)}{4\lambda^2}\right)\,, \tag{5.5.32}$$

$$a^L_{\tau;0,\frac{1}{2}}(\infty;\tau-1,\frac{1}{2}) = a^R_{\tau;2,\frac{1}{2}}(\infty;\tau+1,\frac{1}{2}) \sim \sqrt{\frac{3\tau(\tau+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\left(1+\frac{3}{4\lambda}-\frac{9}{8\lambda^2}\right)\quad, \tag{5.5.33}$$

$$a^L_{\tau;2,\frac{1}{2}}(\infty;\tau+1,\frac{1}{2}) = a^R_{\tau;0,\frac{1}{2}}(\infty;\tau-1,\frac{1}{2}) \sim \sqrt{\frac{3\tau(\tau+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\left(-1+\frac{3}{4\lambda}-\frac{9}{8\lambda^2}\right)\quad. \tag{5.5.34}$$

In this section we discuss the simplifications that occur in the semi-classical limit of $\lambda \to \infty$. We first start with the terms

$$a^R_{\tau;\nu_1,\frac{1}{2}}(\infty;\bar{\nu}_1,\frac{1}{2})\sqrt{\frac{2}{\tau(\tau+2)}} = \sqrt{\frac{6}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\left(-(-1)^{\nu_1/2}+\frac{3}{4\lambda}-\frac{9}{8\lambda^2}\right), \tag{5.5.35}$$

$$a^R_{\tau;\nu_1,1}(\infty;\tau,1)\frac{\sqrt{6}}{\tau(\tau+2)(1+\delta_{\nu_2 1}\delta_{\nu_3 1})} = -\frac{\sqrt{6}}{\tau(\tau+2)(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}\frac{\tau(\tau+2)}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\frac{(2\lambda-3)}{4\lambda^2}, \tag{5.5.36}$$

$$= -\sqrt{\frac{6}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\frac{1}{1+\delta_{\nu_2 1}\delta_{\nu_3 1}}\left(\frac{1}{2\lambda}-\frac{3}{4\lambda^2}\right), \tag{5.5.37}$$

$$a^R_{\tau;1,0}(\infty;\tau,0) = \frac{1}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\left(4\lambda+6-\frac{3(2\lambda-3)(\tau(\tau+2)+3)}{4\lambda^2}\right). \tag{5.5.38}$$

We note for later discussion that these coefficients scale differently with λ:

$$\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\, a^R_{\tau;1,0}(\infty;\tau,0) \sim \lambda\,, \tag{5.5.39}$$

$$\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\, a^R_{\tau;\nu_1,\frac{1}{2}}(\infty;\bar{\nu}_1,\frac{1}{2}) \sim 1\,, \tag{5.5.40}$$

$$\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\, a^R_{\tau;\nu_1,1}(\infty;\tau,1) \sim \lambda^{-1}\,. \tag{5.5.41}$$

In the asymptotic limit we therefore have

$$\begin{aligned}
&\hat{w}_\lambda(\Omega)\hat{T}^\lambda_{1;\alpha J} := \hat{\mathcal{A}}^R_{\alpha J}\hat{w}_\lambda(\Omega) \\
&= \frac{1}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\Bigg[\sum_{\nu_1=0,2}\left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^*\left(-(-1)^{\nu_1/2}+\frac{3}{4\lambda}-\frac{9}{8\lambda^2}\right)\sqrt{6}\,(-1)^{\nu_1/2}\,\hat{\$}_{\nu;\frac{1}{2}} \\
&\qquad - \sum_{\nu_2\nu_3}\left(D^{(1,1)}_{\alpha J;(1\nu_2\nu_3)1}(\Omega)\right)^*\sqrt{\frac{6}{(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}}\left(\frac{1}{2\lambda}-\frac{3}{4\lambda^2}\right)\hat{\$}^{(2)}_{\nu;1} \\
&\qquad + \left(D^{(1,1)}_{\alpha J;(111)0}(\Omega)\right)^*\left(4\lambda+6-\frac{3(2\lambda-3)(\mathcal{C}^2+3)}{4\lambda^2}\right)\Bigg]\hat{w}_\lambda(\Omega)\,,
\end{aligned} \tag{5.5.42}$$

where $\hat{\mathcal{C}}^{(2)}$ is the quadratic Casimir operator:

$$\hat{\mathcal{C}}^{(2)}D^{(\tau,\tau)}_{\nu J;(\tau\tau\tau)0}(\Omega) = \tau(\tau+2)D^{(\tau,\tau)}_{\nu J;(\tau\tau\tau)0}(\Omega)\quad . \tag{5.5.43}$$

In the same limit, the $\star$-product of equation (5.5.25) simplifies to

$$
\begin{aligned}
&2\sqrt{\lambda(\lambda+3)}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\star W_{\hat{T}^{\lambda}_{\sigma;\bar{\mu}\bar{J}}}(\Omega)\\
&\quad=\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3}\left(\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\right)^{*}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(-(-1)^{\nu_1/2}+\frac{3}{4\lambda}-\frac{9}{8\lambda^2}\right)\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}W_{\hat{T}^{\lambda}_{\sigma;\mu J}}(\Omega)\right)\\
&\qquad-\sum_{\nu_2\nu_3}\left(\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}\right)^{*}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(\frac{1}{1+\delta_{\nu_2 1}\delta_{\nu_3 1}}\left(\frac{1}{2\lambda}-\frac{3}{4\lambda^2}\right)\right)\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}W_{\hat{B}}(\Omega)\right)\\
&\qquad+\left(\mathbb{1}\,W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\frac{1}{2}\left(2(2\lambda+3)-\frac{3(2\lambda-3)(\hat{\mathcal{C}}^{(2)}+3)}{4\lambda^2}\right)\left(\mathbb{1}\,W_{\hat{T}^{\lambda}_{\sigma;\mu J}}(\Omega)\right)\quad.
\end{aligned}
\tag{5.5.44}
$$

As there is no σ-dependence in any of these expressions, we have quite generally

$$
\begin{aligned}
&2\sqrt{\lambda(\lambda+3)}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\star W_{\hat{B}}(\Omega)\\
&\quad\sim\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3}\left(\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\right)^{*}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(-(-1)^{\nu_1/2}+\frac{3}{4\lambda}-\frac{9}{8\lambda^2}\right)\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}W_{\hat{B}}(\Omega)\right)\\
&\qquad-\sum_{\nu_2\nu_3}\left(\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}\right)^{*}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(\frac{1}{1+\delta_{\nu_2 1}\delta_{\nu_3 1}}\left(\frac{1}{2\lambda}-\frac{3}{4\lambda^2}\right)\right)\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}W_{\hat{T}^{\lambda}_{\sigma;\mu J}}(\Omega)\right)\\
&\qquad+\left(\mathbb{1}\,W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\frac{1}{2}\left(2(2\lambda+3)-\frac{3(2\lambda-3)(\hat{\mathcal{C}}^{(2)}+3)}{4\lambda^2}\right)\left(\mathbb{1}\,W_{\hat{B}}(\Omega)\right)
\end{aligned}
\tag{5.5.45}
$$

which is valid in the $\lambda\to\infty$ for any operator $\hat{B}$. An equivalent but slightly more physically meaningful expression is obtained when expressing equation (5.5.45) in terms of the semi-classical parameter ϵ of equation (5.4.23) rather than λ:

$$
\begin{aligned}
&\epsilon W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\star W_{\hat{B}}(\Omega)\\
&\quad\sim\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3}\left(\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\right)^{*}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(-(-1)^{\nu_1/2}+\frac{3}{2\epsilon}\right)\left(\hat{\mathbb{S}}_{\nu;\frac{1}{2}}W_{\hat{B}}(\Omega)\right)\\
&\qquad-\sum_{\nu_2\nu_3}\left(\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}\right)^{*}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\frac{1}{\epsilon}\,\frac{1}{1+\delta_{\nu_2 1}\delta_{\nu_3 1}}\left(\hat{\mathbb{S}}^{(2)}_{\nu;1}W_{\hat{B}}(\Omega)\right)\\
&\qquad+\left(\mathbb{1}\,W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(\epsilon-\frac{3\hat{\mathcal{C}}^{(2)}}{2\epsilon}\right)\left(\mathbb{1}\,W_{\hat{B}}(\Omega)\right)\quad.
\end{aligned}
\tag{5.5.46}
$$

The calculation of $\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_\lambda(\Omega)$ is simple since only the first term of the $I = 1/2$ changes sign so that

$$
\begin{aligned}
&\hat{T}^{\lambda}_{1;\alpha J}\hat{w}_\lambda(\Omega) := \hat{\mathcal{A}}^{L}_{\alpha J}\hat{w}_\lambda(\Omega) \\
&= \frac{1}{\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\Bigg[\sum_{\nu_1=0,2}\left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^*\left((-1)^{\nu_1/2}+\frac{3}{4\lambda}-\frac{9}{8\lambda^2}\right)\sqrt{6}\,(-1)^{\nu_1/2}\,\hat{\$}_{\nu;\frac{1}{2}} \\
&\qquad -\sum_{\nu_2\nu_3}\left(D^{(1,1)}_{\alpha J;(1\nu_2\nu_3)1}(\Omega)\right)^*\sqrt{\frac{6}{(1+\delta_{\nu_2 1}\delta_{\nu_3 1})}}\left(\frac{1}{2\lambda}-\frac{3}{4\lambda^2}\right)\hat{\$}^{(2)}_{\nu;1} \\
&\qquad +\left(D^{(1,1)}_{\alpha J;(111)0}(\Omega)\right)^*\left(4\lambda+6-\frac{3(2\lambda-3)(\mathcal{C}^2+3)}{4\lambda^2}\right)\Bigg]\hat{w}_\lambda(\Omega)\,, \qquad (5.5.47)
\end{aligned}
$$

so that

$$
\begin{aligned}
\epsilon W_{\hat{B}}(\Omega)\star W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega) \sim &\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3}\left(\left(\hat{\$}_{\nu;\frac{1}{2}}\right)^* W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left((-1)^{\nu_1/2}+\frac{3}{2\epsilon}\right)\left(\hat{\$}_{\nu;\frac{1}{2}}W_{\hat{B}}(\Omega)\right) \\
&-\sum_{\nu_2\nu_3}\left(\left(\hat{\$}^{(2)}_{\nu;1}\right)^* W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\frac{1}{\epsilon}\frac{1}{1+\delta_{\nu_2 1}\delta_{\nu_3 1}}\left(\hat{\$}^{(2)}_{\nu;1}W_{\hat{B}}(\Omega)\right) \\
&+\left(\mathbb{1}\,W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(\epsilon-\frac{3\hat{\mathcal{C}}^{(2)}}{2\epsilon}\right)\left(\mathbb{1}\,W_{\hat{B}}(\Omega)\right) \quad . \qquad (5.5.48)
\end{aligned}
$$

Finally, we get

$$
\{W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega), W_{\hat{B}}(\Omega)\}_{\mathcal{M}} \sim -\frac{2}{\epsilon}\sum_{\nu_1=0,2}\sum_{\nu_2\nu_3}(-1)^{\frac{\nu_1}{2}}\left(\left(\hat{\$}_{\nu\frac{1}{2}}\right)^* W_{\hat{T}^{\lambda}_{1;\alpha J}}\right)\left(\hat{\$}_{\nu\frac{1}{2}}W_{\hat{B}}(\Omega)\right) \qquad (5.5.49)
$$

which has been brought to the form of a Poisson bracket between the Wigner symbol of one of the generators $\hat{T}^{\lambda}_{1;\alpha J}$ and an arbitrary operator $\hat{B}$.

5.6 Application to $(\hat{T}^{\lambda}_{1;\alpha J})^2$-type operators

Hamiltonians which contain powers of generators have been considered by many authors in multiple contexts, such as the $su(2)$-LMG model described earlier in this thesis. Also, Dinani *et al.* in [13] have considered Hamiltonians containing terms $\hat{H}_0^2$ and $\hat{H}_1^2$ in their study of $su(3)$ squeezing. We explore the $\star$-product and evolution equations for such Hamiltonians.

Consider first

$$
\begin{aligned}
\{W_{(\hat{T}^{\lambda}_{1;\alpha J})^2}, W_{\hat{B}}\}_{\mathcal{M}} &= \mathrm{Tr}\Big(\hat{w}_\lambda(\Omega)(\hat{T}^{\lambda}_{1;\alpha J})^2\hat{B}\Big) - \mathrm{Tr}\Big(\hat{w}_\lambda(\Omega)\hat{B}(\hat{T}^{\lambda}_{1;\alpha J})^2\Big) \\
&= \mathrm{Tr}\Big(\Big[\hat{w}_\lambda(\Omega),(\hat{T}^{\lambda}_{1;\alpha J})^2\Big]\hat{B}\Big) \qquad (5.6.1)
\end{aligned}
$$

and if we recall that

$$
\Big[\hat{A},\hat{B}\hat{C}\Big] = \Big[\hat{A},\hat{B}\Big]\hat{C}+\hat{B}\Big[\hat{A},\hat{C}\Big] \qquad (5.6.2)
$$

we can write equation (5.6.1) as

$$\{W_{(\hat{T}^{\lambda}_{1;\alpha J})^2}, W_{\hat{B}}\}_{\mathcal{M}} = \mathrm{Tr}\Big(\Big[\hat{w}_{\lambda}(\Omega), \hat{T}^{\lambda}_{1;\alpha J}\Big]\hat{T}^{\lambda}_{1;\alpha J}\hat{B}\Big) + \mathrm{Tr}\Big(\hat{T}^{\lambda}_{1;\alpha J}\Big[\hat{w}_{\lambda}(\Omega), \hat{T}^{\lambda}_{1;\alpha J}\Big]\hat{B}\Big). \tag{5.6.3}$$

We can now use equation (5.4.14) to obtain

$$\begin{aligned}\{W_{(\hat{T}^{\lambda}_{1;\alpha J})^2}, W_{\hat{B}}\}_{\mathcal{M}} = &-\sqrt{\frac{24}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\\ &\times\sum_{\nu} D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\left(W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\star W_{\hat{B}}(\Omega) + W_{\hat{B}}(\Omega)\star W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right) \quad .\end{aligned} \tag{5.6.4}$$

Since

$$D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1}) = (-1)^{\nu_1/2}\sqrt{\frac{(\lambda+1)(\lambda+2)}{24}}\left[\hat{\mathbb{S}}^*_{\nu;\frac{1}{2}}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right] \tag{5.6.5}$$

we can use this together with equations (5.5.45) and (5.5.48) to rewrite equation (5.6.4) in the limit of large λ, as

$$\begin{aligned}\{W_{(\hat{T}^{\lambda}_{1;\alpha J})^2}, W_{\hat{B}}\}_{\mathcal{M}} = &-\frac{2}{\epsilon^2}\sum_{\nu}\left[(-1)^{\nu_1/2}\hat{\mathbb{S}}^*_{\nu;\frac{1}{2}}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right]\\ &\times\left[\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\epsilon\left(W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\star W_{\hat{B}}(\Omega) + W_{\hat{B}}(\Omega)\star W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\right] , \end{aligned}\tag{5.6.6}$$

$$\begin{aligned}= &-\frac{4}{\epsilon^2}\sum_{\nu}\left[(-1)^{\nu_1/2}\hat{\mathbb{S}}^*_{\nu;\frac{1}{2}}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right]\\ &\times\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\left[\sum_{\nu'_1=0,2}\sum_{\nu'_2\nu'_3}\left(\left(\hat{\mathbb{S}}_{\nu';\frac{1}{2}}\right)^*W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\frac{3}{2\epsilon}\left(\hat{\mathbb{S}}_{\nu';\frac{1}{2}}W_{\hat{B}}(\Omega)\right)\right.\\ &\qquad -\sum_{\nu'_2\nu'_3}\left(\left(\hat{\mathbb{S}}^{(2)}_{\nu';1}\right)^*W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\frac{1}{\epsilon}\,\frac{1}{1+\delta_{\nu'_2 1}\delta_{\nu'_3 1}}\left(\hat{\mathbb{S}}^{(2)}_{\nu';1}W_{\hat{B}}(\Omega)\right)\\ &\qquad \left.+\left(\mathbb{1}\,W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(\epsilon-\frac{3\hat{\mathcal{C}}^{(2)}}{2\epsilon}\right)\left(\mathbb{1}\,W_{\hat{B}}(\Omega)\right)\right] ,\end{aligned}\tag{5.6.7}$$

where the expansion has been done in terms of the more physically meaningful semi-classical parameter ϵ. This can be brought into a more insightful form by writing

$$\begin{aligned}\{W_{(\hat{T}^{\lambda}_{1;\alpha J})^2}, W_{\hat{B}}\}_{\mathcal{M}} = &-\frac{4}{\epsilon^2}\sum_{\nu}\left[(-1)^{\nu_1/2}\hat{\mathbb{S}}^*_{\nu;\frac{1}{2}}W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right]\\ &\times\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\left[\sum_{\nu'_1=0,2}\sum_{\nu'_2\nu'_3}\left(\left(\hat{\mathbb{S}}_{\nu';\frac{1}{2}}\right)^*W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\frac{3}{2\epsilon}\left(\hat{\mathbb{S}}_{\nu';\frac{1}{2}}W_{\hat{B}}(\Omega)\right)\right.\\ &\qquad\left.-\sum_{\nu'_2\nu'_3}\left(\left(\hat{\mathbb{S}}^{(2)}_{\nu';1}\right)^*W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\frac{1}{\epsilon}\,\frac{1}{1+\delta_{\nu'_2 1}\delta_{\nu'_3 1}}\left(\hat{\mathbb{S}}^{(2)}_{\nu';1}W_{\hat{T}^{\lambda}_{\sigma;\alpha J}}(\Omega)\right)\right]\\ &-\frac{4}{\epsilon^2}\sum_{\nu}\left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^*\sqrt{\frac{24}{(\lambda+1)(\lambda+2)}}\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\left[\left(W_{\hat{T}^{\lambda}_{1;\alpha J}}(\Omega)\right)\left(\epsilon-\frac{3\hat{\mathcal{C}}^{(2)}}{2\epsilon}\right)\left(W_{\hat{B}}(\Omega)\right)\right] .\end{aligned}\tag{5.6.8}$$

Now, $W_{\hat{T}^{(1,1)}_{\alpha J}}(\Omega) \propto \left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^*$ so the last term contains an expression of the form

$$\sum_\nu \left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^* \hat{\$}_{\nu;\frac{1}{2}} \left(D^{(1,1)}_{\alpha J;(111)0}(\Omega)\right)^* . \tag{5.6.9}$$

It is not hard to see, using orthogonality of $D^{(1,1)}_{\alpha J;(111)0}(\Omega)$ functions or by direct calculation that

$$\sum_\nu D^{(1,1)}_{\nu\frac{1}{2};\alpha J}(\Omega^{-1})\hat{\$}_{\nu;\frac{1}{2}} \left(D^{(1,1)}_{\alpha J;(111)0}(\Omega)\right)^* = 0 . \tag{5.6.10}$$

In other words, the last term is nothing but

$$\begin{aligned}
&-\frac{4}{\epsilon^2}\sum_\nu \left(D^{(1,1)}_{\alpha J;\nu\frac{1}{2}}(\Omega)\right)^* \sqrt{\frac{24}{(\lambda+1)(\lambda+2)}}\hat{\$}_{\nu;\frac{1}{2}}\left[\left(W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right)\left(\epsilon-\frac{3\hat{\mathcal{C}}^{(2)}}{2\epsilon}\right)\left(W_{\hat{B}}(\Omega)\right)\right] \\
&\quad = -\frac{4}{\epsilon}\, W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\sum_\nu \left[(-1)^{\nu_1/2}\hat{\$}^*_{\nu;\frac{1}{2}} W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right]\left[\hat{\$}_{\nu;\frac{1}{2}} W_{\hat{B}}(\Omega)\right] \\
&\qquad + \frac{6}{\epsilon^3}\, W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\sum_\nu \left[(-1)^{\nu_1/2}\hat{\$}^*_{\nu;\frac{1}{2}} W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right]\left[\mathcal{C}^{(2)}\hat{\$}_{\nu;\frac{1}{2}} W_{\hat{B}}(\Omega)\right] , \qquad (5.6.11)\\
&\quad = 2W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\{W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega), W_{\hat{B}}(\Omega)\}_{\mathcal{M}} \\
&\qquad + \frac{6}{\epsilon^3}\, W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\sum_\nu \left[(-1)^{\nu_1/2}\hat{\$}^*_{\nu;\frac{1}{2}} W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right]\left[\mathcal{C}^{(2)}\hat{\$}_{\nu;\frac{1}{2}} W_{\hat{B}}(\Omega)\right] , \qquad (5.6.12)\\
&\quad = -\frac{2i}{\epsilon}W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\{W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega), W_{\hat{B}}(\Omega)\}_P \\
&\qquad + \frac{6}{\epsilon^3}\, W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\sum_\nu \left[(-1)^{\nu_1/2}\hat{\$}^*_{\nu;\frac{1}{2}} W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right]\left[\mathcal{C}^{(2)}\hat{\$}_{\nu;\frac{1}{2}} W_{\hat{B}}(\Omega)\right] , \qquad (5.6.13)\\
&\quad = -\frac{i}{\epsilon}\{(W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega))^2, W_{\hat{B}}(\Omega)\}_P \\
&\qquad + \frac{6}{\epsilon^3}\, W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\sum_\nu \left[(-1)^{\nu_1/2}\hat{\$}^*_{\nu;\frac{1}{2}} W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right]\left[\mathcal{C}^{(2)}\hat{\$}_{\nu;\frac{1}{2}} W_{\hat{B}}(\Omega)\right] , \qquad (5.6.14)
\end{aligned}$$

where the derivative property

$$\{f, g^2\}_{\mathcal{P}} = -\{f,g\}_{\mathcal{P}}\, g + g\{f,g\}_{\mathcal{P}} = 2g\{f,g\}_{\mathcal{P}} \tag{5.6.15}$$

of the Poisson bracket has been used.

The Moyal bracket can thus be written in the form

$$\{W_{(\hat{T}^\lambda_{1;\alpha J})^2}, W_{\hat{B}}\}_{\mathcal{M}} = -\frac{i}{\epsilon}\{(W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega))^2, W_{\hat{B}}(\Omega)\}_{\mathcal{P}} + \text{correction terms} \tag{5.6.16}$$

where the correction terms

$$
\begin{aligned}
&-\frac{4}{\epsilon^2}\sum_{\nu}\left[(-1)^{\nu_1/2}\hat{\mathbb{S}}^*_{\nu;\frac{1}{2}}W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right]\hat{\mathbb{S}}_{\nu;\frac{1}{2}}\\
&\quad\times\left[\sum_{\nu'_1=0,2}\sum_{\nu'_2\nu'_3}\left(\left(\hat{\mathbb{S}}_{\nu';\frac{1}{2}}\right)^* W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right)\frac{3}{2\epsilon}\left(\hat{\mathbb{S}}_{\nu';\frac{1}{2}}W_{\hat{B}}(\Omega)\right)\right.\\
&\qquad\left.-\sum_{\nu'_2\nu'_3}\left(\left(\hat{\mathbb{S}}^{(2)}_{\nu';1}\right)^* W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right)\frac{1}{\epsilon}\frac{1}{1+\delta_{\nu'_2 1}\delta_{\nu'_3 1}}\left(\hat{\mathbb{S}}^{(2)}_{\nu';1}W_{\hat{B}}(\Omega)\right)\right]\\
&\quad+\frac{6}{\epsilon^3}W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\sum_{\nu}\left[(-1)^{\nu_1/2}\hat{\mathbb{S}}^*_{\nu;\frac{1}{2}}W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\right]\left[\mathcal{C}^{(2)}\hat{\mathbb{S}}_{\nu;\frac{1}{2}}W_{\hat{B}}(\Omega)\right]
\end{aligned}
\tag{5.6.17}
$$

are two powers of the semi-classical parameter ϵ smaller than the leading term, which is nothing but the classical Poisson bracket. Note that, in general,

$$
(W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega))^2 = W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega))W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega) \neq W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega)\star W_{\hat{T}^\lambda_{1;\alpha J}}(\Omega) = W_{(\hat{T}^\lambda_{1;\alpha J})^2}(\Omega) \quad . \tag{5.6.18}
$$

Hence, to leading order, the quantum evolution equation obtained from the Moyal bracket agrees with the equations of motion obtained from the classical Poisson bracket.

Examples of nonlinear Hamiltonians

We are now interested in explicit expressions for the leading term and the correction terms of the Moyal bracket of equation (5.6.16) for systems that possess Hamiltonians of the type

$$
\hat{H} = \left(\hat{T}^\lambda_{1;\alpha\frac{1}{2}}\right)^2 \tag{5.6.19}
$$

As mentioned previously, the leading term of this Moyal bracket is the Poisson bracket

$$
-\frac{i}{\epsilon}\{(W_{\hat{T}^\lambda_{1;\alpha J}})^2(\Omega), W_{\hat{B}}(\Omega)\}_{\mathcal{P}}
$$

and the correction terms are given in equation (5.6.17). A summary of the leading term of the Moyal brackets for various Wigner symbols of the operator $\hat{B}$ and their correction terms can be found in Tables 5.8 and 5.9, respectively.

Table 5.8: The leading term and correction term of some Moyal brackets

$\hat{T}^{\lambda}_{1;\alpha\frac{1}{2}}$	$\hat{B}$	$-\frac{i}{2\sqrt{\lambda(\lambda+3)}}\{W_{\left(\hat{T}^{\lambda}_{1;\alpha\frac{1}{2}}\right)^2}, W_{\hat{B}}\}_{\mathcal{P}}$
$\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}$	$\hat{T}^{\lambda}_{1;(111)1}$	$\frac{24\sqrt{3}e^{2i(\alpha_1+\alpha_2)}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2(\beta_2)}{((\lambda+1)(\lambda+2))^{3/2}\sqrt{\lambda(\lambda+3)}}$
$\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}$	$\hat{T}^{\lambda}_{2;(042)1}$	$\frac{36\sqrt{5}e^{2i\alpha_1}\sin^2(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right)\sin^2(\beta_2)}{((\lambda+1)(\lambda+2))^{3/2}\sqrt{\lambda(\lambda+3)}}$
$\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}$	$\hat{T}^{\lambda}_{2;(222)0}$	$-\frac{24\sqrt{6}e^{2i(\alpha_1+\alpha_2)}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2(\beta_2)(5\cos(\beta_2)+1)}{((\lambda+1)(\lambda+2))^{3/2}\sqrt{\lambda(\lambda+3)}}$
$\hat{T}^{\lambda}_{1;(012)\frac{1}{2}}$	$\hat{T}^{\lambda}_{1;(120)1}$	$-\frac{12\sqrt{6}e^{-i(\alpha_1+2\alpha_2)}\sin(\beta_1)\sin^2(\beta_2)}{((\lambda+1)(\lambda+2))^{3/2}\sqrt{\lambda(\lambda+3)}}$
$\hat{T}^{\lambda}_{1;(012)\frac{1}{2}}$	$\hat{T}^{\lambda}_{1;(240)2}$	$-\frac{36\sqrt{5}e^{-2i\alpha_2}\sin^2(\beta_1)\sin^2\left(\frac{\beta_2}{2}\right)\sin^2(\beta_2)}{((\lambda+1)(\lambda+2))^{3/2}\sqrt{\lambda(\lambda+3)}}$
$\hat{T}^{\lambda}_{1;(012)\frac{1}{2}}$	$\hat{T}^{\lambda}_{1;(222)0}$	$\frac{24\sqrt{6}e^{-2i(\alpha_1+\alpha_2)}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2(\beta_2)(5\cos(\beta_2)+1)}{((\lambda+1)(\lambda+2))^{3/2}\sqrt{\lambda(\lambda+3)}}$

Table 5.9: The correction terms of some Moyal brackets

$\hat{T}^{\lambda}_{1;\alpha\frac{1}{2}}$	$\hat{B}$	Correction Terms
$\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}$	$\hat{T}^{\lambda}_{1;(111)1}$	$-\frac{3072\sqrt{3}}{\epsilon^6}e^{2i(\alpha_1+\alpha_2)}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2\left(\beta_2\right)$
$\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}$	$\hat{T}^{\lambda}_{2;(042)1}$	$\frac{1152\sqrt{5}}{\epsilon^6}e^{2i\alpha_1}\sin^2\left(\beta_1\right)\sin^4\left(\frac{\beta_2}{2}\right)\left(20\cos\left(\beta_2\right)+13\right)$
$\hat{T}^{\lambda}_{1;(210)\frac{1}{2}}$	$\hat{T}^{\lambda}_{2;(222)0}$	$\frac{384\sqrt{6}}{\epsilon^6}e^{2i(\alpha_1+\alpha_2)}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2\left(\beta_2\right)\left(100\cos\left(\beta_2\right)+13\right)$
$\hat{T}^{\lambda}_{1;(012)\frac{1}{2}}$	$\hat{T}^{\lambda}_{1;(120)1}$	$\frac{1536\sqrt{6}}{\epsilon^6}e^{-i(\alpha_1+2\alpha_2)}\sin\left(\beta_1\right)\sin^2\left(\beta_2\right)$
$\hat{T}^{\lambda}_{1;(012)\frac{1}{2}}$	$\hat{T}^{\lambda}_{2;(240)2}$	$\frac{1152\sqrt{5}}{\epsilon^6}e^{-2i\alpha_2}\sin^2\left(\beta_2\right)\cos^2\left(\frac{\beta_1}{2}\right)\left(10\cos\left(\beta_1\right)\left(\cos\left(\beta_2\right)-1\right)-10\cos\left(\beta_2\right)+3\right)$
$\hat{T}^{\lambda}_{1;(012)\frac{1}{2}}$	$\hat{T}^{\lambda}_{2;(222)0}$	$-\frac{384\sqrt{6}}{\epsilon^6}e^{-2i(\alpha_1+\alpha_2)}\sin^2\left(\frac{\beta_1}{2}\right)\sin^2\left(\beta_2\right)\left(100\cos\left(\beta_2\right)+13\right)$

Chapter 6

Conclusion and Final Remarks

In this work I have obtained some correspondence rules for $SU(3)$ systems. These correspondence rules allow one to write the $\star$-product in differential form, and I was able to do this explicitly and simplify the expression to a more convenient form in the limit case of $\lambda \to \infty$. In principle, one can do the same for any value of λ, but this involves expressing $\mathfrak{a}^{R}_{\tau;\nu,I}(\lambda;\bar{\nu},I)$ coefficients in terms of complicated functions of the Casimir operator $\hat{\mathcal{C}}^{(2)}$. Even when given in terms of the (complicated) $\mathfrak{a}^{R}_{\tau;\nu,I}(\lambda;\bar{\nu},I)$ coefficients and differential operators, the differential form is quite convenient and can be advantageously compared with the integral form found, for instance, in recent work by Rundle [22].

The physics of the $\lambda \to \infty$ limit of $SU(3)$ systems is similar to the limit of large S in angular momentum systems. Large S values are obtained by combining many spin-$1/2$ particles, and large λ values are obtained for instance, in BEC of many interacting neutral atoms trapped in a symmetric triple well in a three-mode approximation, as studied in [29].

For these systems, the methods presented in this thesis capture the dominant part of the quantum dynamics described by the semiclassical correspondence

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = \left[\hat{H},\hat{\rho}\right] \to i\frac{\partial}{\partial t}W_{\rho}(\Omega) = \{W_H(\Omega), W_{\rho}(\Omega)\}_{\mathcal{M}}$$

and also give the leading order correction to the classical approximation. It is remarkable that, despite the considerable machinery deployed to obtain various coefficients, the dominant term in the evolution equation of a quantum state described by the density matrix ρ is easily identifiable as the classical Poisson bracket of the corresponding phase space symbols. This highlights in the $SU(3)$ context the deep idea first put forward by Niels Bohr that quantum systems ought to behave, in some limit, as classical systems. The expansion of the exact quantum evolution equation makes it possible to identify the leading order correction to this classical dynamics. Thus, in principle, one could go beyond the semiclassical analysis of squeezing done by Dinani in [13], which was limited to the Poisson bracket term in the so-called truncated Wigner approximation

One strength of this approach is that, in an $SU(3)$ system, the size of matrices for a system containing λ particles evolving in a Hilbert space that carries the irrep $(\lambda, 0)$ of $SU(3)$ is $\frac{1}{2}(\lambda+1)(\lambda+2) \times \frac{1}{2}(\lambda+1)(\lambda+2)$. Thus, even for a moderate number of atoms - say 15 or 30 - the matrices are of dimension 136×136 and 496×496 respectively, but the number of phase space parameters - i.e. the angles $\Omega = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ that enter as arguments of the phase space symbols - remains unchanged. The correction term for a Hamiltonian containing the square of a generator, i.e. a Hamiltonian such as $\hat{H}_1^2$ as studied by Dinani previously, is of size $\epsilon^{-2} \sim 1/1020$ and $\sim 1/3840$ for $\lambda = 15$ and 30 respectively, illustrating how the semiclassical limit is quickly reached and why one should believe that including the first correction ought to be sufficient to obtain the fundamental information of the dynamics of these kinds of systems. Experimentally, the number of particles in BEC systems is typically of the size $N \sim 10^3 - 10^5$ as Corre *et al* points out in [4].

We note that, for $SU(n)$ systems, the size of the matrices now grows like (number of particles)$^{n-1}$ but the number of parameters grows like $2(n-1)$. Thus, the $SU(4)$ version of the BEC problem would require matrices of size 816×816 and 5456×5456 for 15 and 30 atoms respectively, but only 6 angles.

The drawbacks of this approach are also apparent in the work presented here: the ingredients required to obtain the correspondence rules - the Clebsch-Gordan coefficients, recoupling coefficients and group functions - require a considerable amount of technical calculations. Yet, the results of this thesis show how some important recognizable features become tractable in the large λ limit and could be reproduced in higher symmetries than $SU(3)$.

First, the algorithm used to find the $SU(3)$ Clebsch-Gordan coefficients in chapter 4 was similar to the one used to encounter these coefficients in the $SU(2)$ case in chapter 2. That is I constructed the $SU(3)$ highest weight states and acted on them with a raising operator of the $su(3)$ algebra, leading to a recursion relation which allowed me to calculate the $SU(3)$ highest weight CGs. By the action of the lowering operator I was able to find the remaining CGs. Although it may look straightforward to find the $SU(3)$ CGs, finding the CGs that I was interested in was not so trivial. I focused on the Clebsch-Gordan coefficients of the coupling $(1,1) \otimes (\sigma,\sigma)$ and it turned out that this coupling had decomposition

$$(1,1) \otimes (\sigma,\sigma) = (\sigma-1,\sigma-1) \oplus (\sigma+1,\sigma+1) \oplus 2(\sigma,\sigma). \tag{6.0.1}$$

The $SU(3)$ highest weight CGs for the resulting irrep (σ,σ) introduced a labeling issue into my derivations. This is because the irrep (σ,σ) occurs twice in the decomposition of the direct product of the irreps $(1,1)$ and (σ,σ) as showed in equation (6.0.1). In order to solve this problem, I had to use the label ρ for the CGs of the irrep (σ,σ). The copy labeled $\rho = 1$ was chosen using the usual convention that the $SU(3)$ CGs agree with the Wigner-Eckart theorem when the generators are considered as $SU(3)$ tensors transforming by the $(1,1)$ representation. The copy labeled $\rho = 2$ was chosen to be orthogonal to the $\rho = 1$ copy. This major difference is also a characteristic of higher symmetries and, therefore, the methods used throughout this thesis can also be used for these higher symmetries.

As presented in chapter 5, these $SU(3)$ Clebsch-Gordan coefficients are fundamental pieces in the construction of the $SU(3)$ tensor operators $\hat{T}^{\lambda}_{\sigma;\gamma I_\gamma}$. Moreover, the quantization kernel $\hat{w}_\lambda(\Omega)$, which plays an essential role in the quantum mechanics phase space formalism by mapping operators of a Hilbert space into c-valued functions of phase space, was constructed as a linear expansion of tensor operators of the type $\hat{T}^{\lambda}_{\sigma;\nu J}$ as well as the functions $D^{(\sigma,\sigma)}_{\nu J;(\sigma\sigma\sigma)0}(\Omega)$ as it was given in equation (5.1.13). In order to find the correspondence rules in $SU(3)$, I followed the scheme given by Klimov and Espinoza in [10], which finds the correspondence rules for the generators of the $su(2)$ algebra, and extended it to the $su(3)$ algebra. The basic idea of this scheme, which can be expanded to any $SU(n)$ symmetry, consisted of replacing a $\star$-product by a differential operator $\hat{\mathbb{S}}^{(j)}_{\gamma I_\gamma}$

$$W_{\hat{T}^{\lambda}_{1;\gamma I_\gamma}}(\Omega) \star W_{\hat{B}}(\Omega) := \hat{\mathbb{S}}^{(j)}_{\gamma I_\gamma} W_{\hat{B}}(\Omega) \tag{6.0.2}$$

where $j = 1, 2$ and the operator $\hat{\mathbb{S}}^{(j)}_{\gamma I_\gamma}$ only depends on the generators . In the case of $SU(2)$, the differential operators equivalent to $\hat{\mathbb{S}}^{(j)}_{\gamma I_\gamma}$ are of first order only, however, my research showed that, for the $SU(3)$ case, some of the differential operators of equation (6.0.2) have second order dependence. The first and second order differential operators were represented by $\hat{\mathbb{S}}_{\gamma\frac{1}{2}}$ and $\hat{\mathbb{S}}^{(2)}_{\gamma 1}$, respectively. This was a very surprising and new result in comparison to the $SU(2)$ results found by Klimov and Espinoza on the correspondence rules of the generators.

As mentioned above, the exact evolution of a quantum particle in phase space depends on the evaluation of a Moyal bracket between the symbols of the Hamiltonian $W_{\hat{H}}$ and density operator $W_{\hat{\rho}}$, and the semiclassical limit of this bracket is the well known Poisson bracket as it was stressed in chapter 1. First, I was able to evaluate the Moyal brackets between operators that are linear in the generators and an arbitrary operator $W_{\hat{B}}$. Although, the $\star$-product of equation (6.0.2) has a second order differential dependence that occurs because of the non-vanishing coefficients $a_{1,1}(\lambda;\tau)$ and $a_{1,0}(\lambda;\tau)$, the evaluation of the Moyal bracket eliminates this second order differential dependence and gives us an expression that can be expanded in terms of the semiclassical parameter, leading to the recognition of the Poisson bracket with extra correction terms.

One of the most interesting features of my derivations was the structure of the asymptotic form of the $\star$-product found in chapter 5. In the case of the $SU(2)$ semiclassical approximation, the structure of the $\star$-product has only first order differential operators [10]. However, when we go to a higher symmetry, for example $SU(3)$, second order differential operators do not vanish and this semiclassical expansion yields equation (3.2.32). In fact, my approach to obtain these correspondence rules is quite systematic and could be used to find the rules of any $SU(n)$ system. I strongly believe that this differential structure, with second order derivatives, will be also found in problems with higher symmetry due to the multiplicity that do not occur in $SU(2)$.

My results can also be compared to the findings of Dinani on the Truncated Wigner Approximation of $SU(3)$ squeezing [26]. This type of approximation is obtained when one truncates equation (1.1.6) leaving behind only the Poisson bracket. If we consider the TWA, one finds that the evolution under Hamiltonians that are linear in the generators is exact. Therefore, in order to compare my results with those presented by Dinani, I had to develop a general and explicit expression for the semiclassical limit of Moyal brackets of the type

$$\{W_{\left(\hat{T}^{\lambda}_{1;\gamma I_\gamma}\right)^2}(\Omega), W_{\hat{B}}(\Omega)\}_{\mathcal{M}}.$$

This expression can be found in section (5.6). Therefore, one can choose the operator $\hat{B}$ to be the density operator $\hat{\rho}$ and calculate the semiclassical evolution of quantum particles in phase space. This is a very advantageous approach because I was able to recover the dependence of the Moyal bracket on the Poisson bracket plus the first correction term of this expansion. In addition, I presented some analytical forms of the leading term and the correction terms for specific cases of the Moyal bracket as shown in tables (5.8) and (5.9), respectively. These tables demonstrate that the leading term (Poisson bracket) happens with order ϵ^{-4} while the first correction term happens with order ϵ^{-6}. The reason for this ϵ^{-6} dependence of the correction term comes from the fact that every Wigner symbol of equation (5.6.17) has a factor ϵ^{-1}.

In the future, I would like to investigate the evolution of quantum systems using the tools developed in this thesis, *i.e.*, I would like to include the correction term of equation (5.6.17) into the Truncated Wigner Approximation and compare the results of Dinani on $SU(3)$ squeezing. One would expect a more accurate evolution when the correction term is added to TWA.

Appendix A

Density Matrix Theory

A.1 Pure Spin States

Following Blum [2], if it is possible to find an orientation of the Stern-Gerlach apparatus for which a given beam is completely transmitted, then we will say that the beam is in a pure spin state. For instance, consider a beam of spin-1/2 particles which passes through a Stern-Gerlach magnet which has its field gradient aligned along the z direction with respect to a fixed coordinate system (x, y, z). The beam will be transmitted through the magnet and the emerging particles will be in a state which corresponds to the eigenvalue $m = +\frac{1}{2}$ (or $m = -\frac{1}{2}$) of the z component of the spin operator $\hat{S}$. figure A.1 represents an electron beam that is completely transmitted to $+z$.

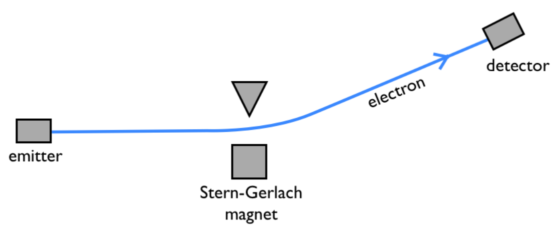

Figure A.1: Stern-Gerlach apparatus. Retrieved from `http://i.stack.imgur.com/SKqat.png`

The representation of the two possible eigenstates of the operator $\hat{S}_z$ are written mathematically by the standard notation of the bra-ket, which describes quantum mechanical systems. In this notation, the possible outcomes for the spin of an electron are written as

$$\begin{aligned} \left|+\tfrac{1}{2}\right\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{Spin "Up"} \quad , \\ \left|-\tfrac{1}{2}\right\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{Spin "Down"} \quad . \end{aligned} \tag{A.1.1}$$

These are the ket states of the operator $\hat{S}_z$. One may notice that the states of equation (A.1.1) are normalized. The adjoint, or complex transpose, of these states are called the bra states and they are given by

$$\left\langle +\frac{1}{2}\right| = \begin{pmatrix}1 & 0\end{pmatrix} \quad \text{Spin "Up"}$$
$$\left\langle -\frac{1}{2}\right| = \begin{pmatrix}0 & 1\end{pmatrix} \quad \text{Spin "Down"} \tag{A.1.2}$$

A general state will be written as it follows

$$|\chi\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 |+1/2\rangle + a_2 |-1/2\rangle \tag{A.1.3}$$

The normalization of this is state is given by

$$\langle\chi|\chi\rangle = |a_1|^2 + |a_2|^2 = 1 \tag{A.1.4}$$

A Stern-Gerlach apparatus works as a filter, because irrespective of the state of the beam sent through it, the emerging beam is in a definite spin state which is defined by the orientation of the magnet. Passing a beam through the filter can therefore be regarded as a method of preparing a beam of particles in a pure state.

A.2 The Polarization Vector

The components of the polarization vector are given by

$$|P_i| = \langle\hat{\sigma}_i\rangle \quad , \tag{A.2.1}$$

where $i = x, y, z$. Also, σ_i are the Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{A.2.2}$$

The states $\left|+\frac{1}{2}\right\rangle$ and $\left|-\frac{1}{2}\right\rangle$ are characterized by polarization vectors of unit magnitude pointing in the $+z$ and $-z$ directions, respectively. Also, these states are said to be states of opposite polarization [2].

A general pure state like the one of equation (A.1.3) is parametrized as

$$|\chi\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\delta}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \quad , \tag{A.2.3}$$

where θ and δ are the polar angle and azimuth angles. Using equations (A.2.1) and (A.2.3), we can find

$$|P_x| = \sin\theta\cos\delta \quad |P_y| = \sin\theta\sin\delta \quad |P_z| = \cos\theta \quad . \tag{A.2.4}$$

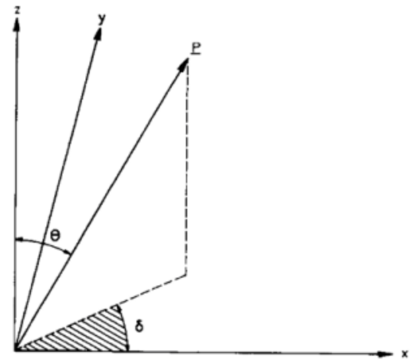

Figure A.2: Polarization vector in the three dimensional space. This image was extracted from [2]

The magnitude of the polarization vector is

$$|P| = \sqrt{|P_x|^2 + |P_y|^2 + |P_z|^2} = 1 \tag{A.2.5}$$

It is possible to choose a different frame of reference, say x', y', z', such that two components of the polarization vector, say P'_x and P'_y, yield zero and the remaining one, say P'_z, yields 1. It means that all particles have spin up with respect to the z' component. If a beam is sent through a Stern-Gerlach filter oriented parallel to $\vec{P}$, the whole beam will pass through the filter.

A.3 Mixed Spin States

The quantum state made out of a mixture of two or more different beam states will be called a mixed spin state. Suppose that the first beam is prepared with N_1 particles in the state $\left|+\frac{1}{2}\right\rangle$, and the second with N_2 in the state $\left|-\frac{1}{2}\right\rangle$. It is important to say that the second beam is prepared independently of the first beam because in this way there is no definite phase relation between the two beams, which makes it impossible to construct states of the form of equation (A.1.3). The total number of particles in this system is $N = N_1 + N_2$. If this mixture of beam states is led to a Stern-Gerlach apparatus oriented in the z-axes, it will be noticed that N_1 particles will be found being spin up. Similarly, there will be N_2 spin down particles [2].

The probability of finding a particle in the state $\left|+\frac{1}{2}\right\rangle$ will be $W_1 = \frac{N_1}{N}$, likewise $W_2 = \frac{N_2}{N}$ is the probability for finding a particle in the state $\left|-\frac{1}{2}\right\rangle$. Recalling equation (A.1.3), it is possible to identify W_1 and W_2 as being equal to $|a_1|^2$ and $|a_2|^2$, respectively. For a mixture of N_1 particles in the state $\left|+\frac{1}{2}\right\rangle$ and N_2 particles in the state $\left|-\frac{1}{2}\right\rangle$ prepared independently, the components of the polarization vector are written as

$$|P_i| = W_1 \left\langle +\frac{1}{2} \middle| \hat{\sigma}_i \middle| +\frac{1}{2} \right\rangle + W_2 \left\langle -\frac{1}{2} \middle| \hat{\sigma}_i \middle| -\frac{1}{2} \right\rangle \quad , \tag{A.3.1}$$

or separately

$$|P_x| = 0, \quad |P_y| = 0, \quad |P_z| = \frac{N_1 - N_2}{N} = W_1 - W_2 \quad . \tag{A.3.2}$$

Therefore, the magnitude of the polarization vector can be smaller than 1. However, this is not the most generalized form of a mixed state. Suppose a beam is prepared independently with N_a particles in the state $|\chi_a\rangle$ and N_b in the state $|\chi_b\rangle$. So, the components of the polarization vector are given by

$$|P_i| = W_a \langle\chi_a| \hat{\sigma}_i |\chi_a\rangle + W_b \langle\chi_b| \hat{\sigma}_i |\chi_b\rangle = W_a P_i^{(a)} + W_b P_i^{(b)} \quad , \tag{A.3.3}$$

and the polarization vector is simply

$$\vec{P} = W_a \vec{P}^{(a)} + W_b \vec{P}^{(b)} \tag{A.3.4}$$

where $\vec{P}^{(a)}$ and $\vec{P}^{(b)}$ are the polarization vectors associated with the constituent beams. However, as mentioned before, the polarization vector of the constituent beams have magnitude 1. Then,

$$\vec{P}^2 = \left(W_a \vec{P}^{(a)} + W_b \vec{P}^{(b)}\right)^2 = W_a^2 + W_b^2 + 2W_a W_b \vec{P}^{(a)} \cdot \vec{P}^{(b)} \tag{A.3.5}$$

but

$$W_a^2 + W_b^2 + 2W_a W_b \vec{P}^{(a)} \cdot \vec{P}^{(b)} \leq W_a^2 + W_b^2 + 2W_a W_b = (W_a + W_b)^2 = 1 \quad . \tag{A.3.6}$$

Therefore,

$$0 \leq |\vec{P}| \leq 1 \tag{A.3.7}$$

the maximum polarization ($|\vec{P}| = 1$) is obtained if (and only if) the two beams under consideration are in the same pure state, whereas mixtures necessarily have a polarization magnitude that is less than 1.

A.4 More on Mixed States

In classical mechanics, the information of a system is given by the positions of the particles and their momenta. This information describes the system completely. However, in quantum mechanics, position and momentum are characterized as operators and a simultaneous measurement of these operators will bring uncertainty because position and momentum operators do not commute. In general, a simultaneous measurement of two operators is only possible if these operators commute. Increasing the number of commuting operators will give more information about the quantum system. The eigenvalues $q_1, q_2, q_3 \ldots$ give a more precise classification of the system [2]. If we introduce a new operator and this operator does not commute with the other commuting ones, the new operator will introduce uncertainty into the system.

To deduce the eigenvalues of an operator, an experiment has to be performed multiple times. If the operators $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3, \ldots$ commute, a complete experiment will give the respective set of eigenvalues $q_1, q_2, q_3, \ldots$ and they will be used to label a single common ket state $|q_1, q_2, q_3, \ldots\rangle$.

Let us assume two sets of operators $\{\hat{Q}_i; i = 1, 2, \ldots\}$ and $\{\hat{Q}'_i; i = 1, 2, \ldots\}$ with eigenstates $|\psi\rangle = |q_1, q_2, \ldots\rangle$ and $|\phi\rangle = |q'_1, q'_2, \ldots\rangle$, respectively, where at least one of the operators $\hat{Q}'_i$ does not commute with the first set. It is possible to describe $|\psi\rangle$ as a linear combination of the orthonormal basis that represent the operators $\hat{Q}'_1, \hat{Q}'_2, \ldots$ as it follows

$$|\psi\rangle = \sum_n a_n |\phi_n\rangle \quad a_n = \langle\phi_n|\psi\rangle \quad , \tag{A.4.1}$$

where the index n represents different eigenstates and a_n are the coefficients of the expansion. In fact, $|a_n|^2$ is the probability of finding the particle in the state $|\phi_n\rangle$. Assuming for simplicity that the basis is orthonormal, we can write

$$\langle\phi_n|\phi_m\rangle = \delta_{nm} \quad . \tag{A.4.2}$$

But we also can write the expansion of unity

$$\sum_n |\phi_n\rangle\langle\phi_n| = \mathbb{1} \rightarrow \quad \langle\phi_m|\phi_m\rangle = \langle\phi_m|\mathbb{1}|\phi_m\rangle = \sum_n |\langle\phi_n|\phi_m\rangle|^2 = 1 \quad . \tag{A.4.3}$$

In general, an experiment cannot control all possible variables precisely. For instance, one cannot control the polarization of the photons that are emitted by an incandescent bulb. As a result, there is no orientation of a polarizer that will allow 100% of the photons to pass or 100% of the photons to be blocked; the photons are not in an eigenstate of the polarization operator for if they were, one could pass them with 100% probability by aligning the polarizer correctly, and block them with 100% probability by aligning the polarizer perpendicular to this polarization. The state of a photon cannot be expanded in terms of a polarization basis like $|\updownarrow\rangle$ or $|\leftrightarrow\rangle$. Instead, one reproduces the experimental result by thinking of the collection of photons as a *statistical mixture* of vertically or horizontally polarized states. The essential difference between a statistical mixture and a linear combination of the type given in equation (A.4.1) is that terms in equation (A.4.1) can interfere whereas different parts of a statistical mixture cannot. The terms in a statistical mixture come multiplied with a non-negative statistical weight which gives the probability of getting this specific component of the mixture during the preparation.

Usually, the ensembles treated in classical or quantum mechanics have a large number of particles. In this sense, the best approach to quantify operators would be calculating their averages. As an illustration, let us suppose that we have an ensemble of particles in the pure state $|\psi\rangle$, but this state is not an eigenstate of the operator $\hat{Q}$. Then, measurements made on the ensemble of particles, which are in the state $|\psi\rangle$, will produce all the eigenvalues of the operator $\hat{Q}$. The average of these eigenvalues is the expectation value of the operator, i.e.

$$\langle\hat{Q}\rangle = \langle\psi|\,\hat{Q}\,|\psi\rangle \quad . \tag{A.4.4}$$

For a mixture of states $\{|\psi_1\rangle, |\psi_2\rangle, \dots\}$, the expectation value of an operator $\hat{Q}$ is given by

$$\langle\hat{Q}\rangle = \sum_n W_n \langle\psi_n|\,\hat{Q}\,|\psi_n\rangle \quad , \tag{A.4.5}$$

where W_n are the weights of every pure state $|\psi_n\rangle$.

Finally, the statistical theory is necessary to describe two aspects of equation (A.4.5). First, the perturbations caused during measurements, since a state that is in a superposition of eigenstates of the operator $\hat{Q}$ will collapse to a single eigenstate [7, 23]. Second, there is a lack of information caused by the several pure states which the system may be in [2]. The introduction of the density matrix accounts for this lack of information, since it will contain all the information of the quantum system.

A.5 The Density Matrix

Why should we introduce a density matrix and density operator? Firstly, statistical methods must be applied because of the uncontrollable perturbation of states by any measuring apparatus. Secondly, when dealing with mixtures, it is only known that the particles can be in any one of several spin states. Then, a statistical description must be applied because of the lack of information available on the system. It was primarily for the purpose of describing this latter case that the density matrix formalism was developed.

Given N_a particles prepared in the state $|\chi_a\rangle$ and N_b in the state $|\chi_b\rangle$, independently, we define the density operator

$$\hat{\rho} = W_a |\chi_a\rangle \langle\chi_a| + W_b |\chi_b\rangle \langle\chi_b| \quad . \tag{A.5.1}$$

This operator describes the preparations which have been performed, and it contains all the information obtained on the beam. For a pure state, the density operator is simply written as

$$\hat{\rho} = |\chi\rangle \langle\chi| \quad , \tag{A.5.2}$$

and we can expand this general state[1] into the basis states $\{\left|+\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\right\rangle\}$:

$$\begin{aligned} |\chi_a\rangle &= a_1^{(a)} \left|+\frac{1}{2}\right\rangle + a_2^{(a)} \left|-\frac{1}{2}\right\rangle \rightarrow \quad |\chi_a\rangle \langle\chi_a| = \begin{pmatrix} a_1^{(a)} a_1^{(a)*} & a_1^{(a)} a_2^{(a)*} \\ a_2^{(a)} a_1^{(a)*} & a_2^{(a)} a_2^{(a)*} \end{pmatrix} \\ |\chi_b\rangle &= a_1^{(b)} \left|+\frac{1}{2}\right\rangle + a_2^{(b)} \left|-\frac{1}{2}\right\rangle \rightarrow \quad |\chi_b\rangle \langle\chi_b| = \begin{pmatrix} a_1^{(b)} a_1^{(b)*} & a_1^{(b)} a_2^{(b)*} \\ a_2^{(b)} a_1^{(b)*} & a_2^{(b)} a_2^{(b)*} \end{pmatrix} \quad . \end{aligned} \tag{A.5.3}$$

Substituting equation (A.5.3) into equation (A.5.1), we find

$$\hat{\rho} = \begin{pmatrix} W_a |a_1^{(a)}|^2 + W_b |a_1^{(b)}|^2 & W_a a_1^{(a)} \left(a_2^{(a)}\right)^* + W_b a_1^{(b)} \left(a_2^{(b)}\right)^* \\ W_a a_2^{(a)} \left(a_1^{(a)}\right)^* + W_b a_2^{(b)} \left(a_1^{(b)}\right)^* & W_a |a_2^{(a)}|^2 + W_b |a_2^{(b)}|^2 \end{pmatrix} \quad . \tag{A.5.4}$$

This is the density matrix in the $\{|\pm 1/2\rangle\}$ representation. Now, if we define $|+1/2\rangle = |\chi_1\rangle$ and $|-1/2\rangle = |\chi_2\rangle$, a matrix element of $\hat{\rho}$ is then written as

$$\langle\chi_i| \hat{\rho} |\chi_j\rangle = W_a a_i^{(a)} \left(a_j^{(a)}\right)^* + W_b a_i^{(b)} \left(a_j^{(b)}\right)^* \quad , \tag{A.5.5}$$

with $i, j = 1, 2$. It is clear that a different basis will lead to a different density matrix than the one given in equation (A.5.5). The trace of equation (A.5.4) is

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}) &= W_a |a_1^{(a)}|^2 + W_b |a_1^{(b)}|^2 + W_a |a_2^{(a)}|^2 + W_b |a_2^{(b)}|^2 \\ &= W_a + W_b = 1 \quad . \end{aligned} \tag{A.5.6}$$

Therefore, the trace of a density matrix is equal to 1 and it is independent of the basis representation.

As an example, suppose a system with N_1 particles in the state $\left|+\frac{1}{2}\right\rangle$ and N_2 particles in the state $\left|-\frac{1}{2}\right\rangle$. So,

$$\begin{aligned} \left|+\frac{1}{2}\right\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \quad a_1^{(a)} = 1, a_2^{(a)} = 0 \\ \left|-\frac{1}{2}\right\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \quad a_1^{(b)} = 0, a_2^{(b)} = 1 \quad , \end{aligned} \tag{A.5.7}$$

which results in the following density matrix

$$\hat{\rho} = \begin{pmatrix} W_a & 0 \\ 0 & W_b \end{pmatrix} = \begin{pmatrix} \frac{N_1}{N} & 0 \\ 0 & \frac{N_2}{N} \end{pmatrix} \quad . \tag{A.5.8}$$

[1] It should always be assumed that the vector states are orthonormal like in equation (A.1.4).

A.6 Significance of the Density Matrix

Considering equation (A.5.5) for $i = j = 1,2$ (or more directly, the diagonal elements of $\hat{\rho}$), one can notice the direct physical meaning of this expression. Since, W_a and $|a_i^{(a)}|^2$ are the probabilities of finding a particle of the mixture in the vector states $|\chi_a\rangle$ and $|\chi_i\rangle$, respectively, the product $W_a|a_i^{(a)}|^2$ is the probability of finding a particle, which was originally prepared in the state $|\chi_a\rangle$, in the vector state $|\chi_i\rangle$. Therefore, the diagonal element ($i = 1,2$) of the density matrix in equation (A.5.4) gives the total probability of finding a particle in the corresponding basis state $|\chi_i\rangle$.

A form of generalizing this result is taking the inner product of equation (A.5.1) with the states $|\chi\rangle$ and $\langle\chi|$

$$\begin{aligned}\langle\chi|\,\rho\,|\chi\rangle &= W_a\,\langle\chi|\chi_a\rangle\,\langle\chi_a|\chi\rangle + W_b\,\langle\chi|\chi_b\rangle\,\langle\chi_b|\chi\rangle \\ &= W_a|a^{(a)}|^2 + W_b|a^{(b)}|^2 \quad .\end{aligned} \tag{A.6.1}$$

This expression means that the probability of finding a particle in the state $|\chi\rangle$ within the mixture is expressed as equation (A.6.1), and the mixture is represented by $\hat{\rho}$.

A.6.1 Basic Properties of the Density Operator

In general, we can write the density operator as

$$\hat{\rho} = \sum_n W_n\,|\psi_n\rangle\,\langle\psi_n| \tag{A.6.2}$$

the states $|\psi_n\rangle$ are not necessarily orthonormal to each other and the sum is over all the states of the mixture.

A matrix form of the density operator $\hat{\rho}$ is found when the states $|\psi_n\rangle$ are expanded into an orthonormal basis $\{|\phi_m\rangle\}$

$$|\psi_n\rangle = \sum_m a_m^{(n)}\,|\phi_m\rangle \quad , \tag{A.6.3}$$

which brings equation (A.6.2) to the form

$$\hat{\rho} = \sum_{n,m,m'} W_n a_m^{(n)} \left(a_{m'}^{(n)}\right)^* |\phi_m\rangle\,\langle\phi_{m'}| \quad . \tag{A.6.4}$$

Therefore, the matrix elements of the density matrix are given by

$$\langle\phi_i|\,\hat{\rho}\,|\phi_j\rangle = \sum_n W_n a_i^{(n)} \left(a_j^{(n)}\right)^* \quad . \tag{A.6.5}$$

The density operator is Hermitian and this means that

$$\hat{\rho} = \hat{\rho}^\dagger \tag{A.6.6}$$

The diagonal matrix elements of the density matrix have a very important meaning in quantum mechanics. Since, the probability of finding a particle in the state $|\psi_n\rangle$ is W_n and the probability that this state will be found in the state $|\phi_m\rangle$ is $|a_m^{(b)}|^2$, then the probability of finding the system in the state $|\phi_m\rangle$ is given by the diagonal element

$$\hat{\rho}_{mm} = \sum_n W_n |a_m^{(n)}|^2 \quad . \tag{A.6.7}$$

Clearly, the diagonal matrix elements of $\hat{\rho}$ must be positive because they represent probabilities. Finally, the probability $W(\psi)$ of finding the system in the state $|\psi\rangle$ after a measurement is

$$W(\psi) = \langle\psi|\,\rho\,|\psi\rangle = \sum_n W_n |\langle\psi_n|\psi\rangle|^2 \quad . \tag{A.6.8}$$

Finally, we calculate the average value of an operator $\hat{Q}$ using the density operator

$$\langle\hat{Q}\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{Q}\right) = \sum_{n,m,m'} W_n a_m^{(n)} \left(a_{m'}^{(n)}\right)^* \langle\phi_{m'}|\,\hat{Q}\,|\phi_m\rangle \quad . \tag{A.6.9}$$

If the states that compose the density matrix are not normalized, the averages are calculated as

$$\langle\hat{Q}\rangle = \frac{\mathrm{Tr}\left(\hat{\rho}\hat{Q}\right)}{\mathrm{Tr}(\hat{\rho})} \tag{A.6.10}$$

As we stated in the beginning of this chapter, the best approach to quantify quantities in quantum mechanics is by calculating expectation values of operators. Therefore, equations (A.6.9) and (A.6.10) will be very important in the development of this thesis, especially when we present the quasi-distribution approach.

Example 1: An Ensemble of Atoms of Spin S

Consider an ensemble of atoms with spin S and magnetic number m characterized by state vectors $|S, m\rangle$. Consider the case where all atoms have the same values of S, but where the ensemble is a mixture with respect to m such that the weights are given by

$$W_m = \frac{1}{2S+1} \quad , \tag{A.6.11}$$

meaning that all the each value of m is equally probable. The density operator is expressed as

$$\hat{\rho} = \sum_m W_m\,|S,m\rangle\,\langle S,m| = \frac{1}{2S+1}\sum_m |S,m\rangle\,\langle S,m| = \frac{\mathbb{1}}{2S+1} \quad , \tag{A.6.12}$$

where $\mathbb{1}$ is the unit matrix of dimension $(2S+1)$ in the $\{|S,m\rangle\}$ basis. This density matrix is in its diagonal form with constant values given by equation (A.6.11).

Example 2: Density Operator as a Linear Combination of Irreducible Tensor Operators

There is another way to define the density operator of equation (A.6.2), that is expressing it as a linear combination of irreducible tensor operators $\hat{T}^S_{L,M}$. These tensors are defined as

$$\hat{T}^S_{L,M} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-S}^{S} \left\langle \begin{matrix} S \\ m \end{matrix} ; \begin{matrix} L \\ M \end{matrix} \middle| \begin{matrix} S \\ m' \end{matrix} \right\rangle |S, m'\rangle \langle S, m| \quad , \tag{A.6.13}$$

and they have an orthogonality condition

$$\mathrm{Tr}\left(\left(\hat{T}^S_{L',M'} \right)^\dagger \hat{T}^S_{L,M} \right) = \delta_{L'L}\delta_{M'M} \quad . \tag{A.6.14}$$

In this manner, we can express the density operator as

$$\hat{\rho} = \sum_{L=0}^{2S} \sum_{M=-L}^{L} \tilde{\rho}^S_{L,M} \hat{T}^S_{L,M} \quad , \tag{A.6.15}$$

where the coefficients $\tilde{\rho}^S_{L,M}$ are found by the orthogonality condition of equation (A.6.14)

$$\tilde{\rho}^S_{L,M} = \mathrm{Tr}\left(\hat{\rho} \left(\hat{T}^S_{L,M} \right)^\dagger \right) \quad . \tag{A.6.16}$$

Therefore, given the density operator of the mixture, we can express this operator as a linear combination of tensor operators. For instance, let us take the previous example and expand

$$\hat{\rho} = \frac{\mathbb{1}}{2S+1} \tag{A.6.17}$$

into a linear combination of irreducible tensor operators in the case of $S = 1$

$$\hat{\rho} = \sum_{L=0}^{2S} \sum_{M=-L}^{L} \tilde{\rho}^S_{L,M} \hat{T}^S_{L,M} = \tilde{\rho}^1_{0,0} \hat{T}^1_{0,0} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \tag{A.6.18}$$

where $\tilde{\rho}^1_{0,0} = \frac{1}{\sqrt{3}}$ and

$$\hat{T}^1_{0,0} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{A.6.19}$$

Appendix B

Derivation of the Wigner Function of a Particle Using the Displacement Operator

Our goal is to derive the Wigner function of equation (1.2.6). This derivation is based on the work of Royer in [21], where he used the displacement operator $\hat{D}(q,p)$ given by equation (1.2.8) and the parity operator to demonstrate that the Wigner function is the expectation value of the quantization kernel of equation (1.2.7).

Let us start with the definitions of the displacement operator

$$\hat{D}(q,p) = \exp\left(\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right) \quad , \tag{B.1.1}$$

parity operator

$$\hat{P} = \int dq \, |-q\rangle \langle q| = \int dp \, |-p\rangle \langle p| \tag{B.1.2}$$

and quantization kernel

$$\hat{w}(q,p) = \hat{D}(q,p)\hat{P}\hat{D}^\dagger(q,p) \quad , \tag{B.1.3}$$

where the integration range is $[-\infty, \infty]$ for both integrals for the parity operator.

For equation (B.1.1), we can use

$$\mathrm{e}^{\hat{A}+\hat{B}} = \mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}}\mathrm{e}^{-\frac{1}{2}[\hat{A},\hat{B}]} \tag{B.1.4}$$

to find

$$\hat{D}(q,p) = \exp\left(\frac{i}{\hbar}p\hat{q}\right)\exp\left(-\frac{i}{\hbar}q\hat{p}\right)\exp\left(\frac{i}{2\hbar}pq\mathbb{1}\right) \quad . \tag{B.1.5}$$

The application of the displacement operator on the parity operator is written as

$$\begin{aligned} \hat{D}(q,p)\hat{P} &= \exp\left(\frac{i}{\hbar}p\hat{q}\right)\exp\left(-\frac{i}{\hbar}q\hat{p}\right)\exp\left(\frac{i}{2\hbar}pq\mathbb{1}\right)\int dp \, |-\bar{p}\rangle \langle \bar{p}| \\ &= \frac{\exp\left(-\frac{i}{2\hbar}pq\right)}{\sqrt{2\pi\hbar}}\int d\bar{p}\exp\left(-\frac{i}{\hbar}q\bar{p}\right)|p+\bar{p}\rangle \langle -\bar{p}| \quad . \end{aligned} \tag{B.1.6}$$

The last piece of this calculation is the evaluation of $\langle -\bar{p}| \hat{D}^{\dagger}(q,p)$ which is found to be

$$\langle -\bar{p}| \hat{D}^{\dagger}(q,p) = \frac{\exp\left(\frac{i}{2\hbar}pq\right)\exp\left(-\frac{i}{\hbar}q\bar{p}\right)}{\sqrt{2\pi\hbar}} \langle p-\bar{p}| \quad . \tag{B.1.7}$$

Combining equations (B.1.6) and (B.1.7) we find the quantization kernel to be

$$\hat{w}(q,p) = \frac{1}{2\pi\hbar}\int d\bar{p}\exp\left(\frac{2iq\bar{p}}{\hbar}\right)|p+\bar{p}\rangle\langle p-\bar{p}| \quad . \tag{B.1.8}$$

Now, we have all the pieces necessary to determine the Wigner function of a density operator $\hat{\rho} = |\psi\rangle\langle\psi|$, and this quasi-distribution is written as

$$\tilde{W}_{\hat{\rho}}(q,p) = \frac{1}{\pi\hbar}\int dy \exp\left(-\frac{2iqy}{\hbar}\right)\psi^{*}(p+y)\psi(p-y) \tag{B.1.9}$$

or by performing a Fourier transform

$$W_{\hat{\rho}}(q,p) = \frac{1}{2\pi\hbar}\int dy \exp\left(\frac{-ipy}{\hbar}\right)\psi^{*}(q-\frac{1}{2}y)\psi(q+\frac{1}{2}y) \tag{B.1.10}$$

Appendix C

The product $\hat{T}^{\lambda}_{1;\nu J} T^{\lambda}_{\sigma;\mu I}$ of SU(3) tensors

This is the explicit calculation of the $c^{(\tau,\tau),\nu I}_{\bar{\nu} I;(\sigma\sigma\sigma)0}$ coefficients which are important in the construction of the $\boldsymbol{a}$ coefficients of equation (5.3.12). The latter coefficients are important in the derivation of the $\hat{\mathbb{S}}_{\nu\frac{1}{2}}$ and $\hat{\mathbb{S}}^{(2)}_{\nu 1}$ operators. Although, the derivation of the c coefficients for the $\boldsymbol{a}^{R}_{\tau;\nu I}(\lambda;\tau_1\tau_2\tau_3,I)$ coefficients is not presented here, they were obtained in the same fashion as the $c^{(\tau,\tau),\nu I}_{\bar{\nu} I;(\sigma\sigma\sigma)0}$ and they can be easily derived by the interested reader.

C.1.1 Case $\nu, I = (111)1$:

$$\hat{T}^{\lambda}_{1;(111)1}\hat{w}_{\lambda}(\Omega)(0) = \sum_{\sigma=0}^{\sigma=\lambda} \hat{T}^{\lambda}_{1;(111)1}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} \tag{C.1.1}$$

$$\hat{T}^{\lambda}_{1;(111);1}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} = \sum_{\tau,\tau_1,\tau_2,\tau_3} c^{(\tau,\tau),(\tau_1,\tau_2,\tau_3);1}_{(111);1;(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{\tau;(\tau_1,\tau_2,\tau_3);1} \tag{C.1.2}$$

$$c^{(\tau,\tau),(\tau_1,\tau_2,\tau_3);1}_{(111);1;(\sigma\sigma\sigma);0} = \sqrt{\frac{16}{(\lambda+1)(\lambda+2)}}\sum_{\rho}\left\langle \begin{smallmatrix}(1,1)\\1;1\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\\sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\tau,\tau)\\\tau_1;1\end{smallmatrix}\right\rangle_{\rho} \tag{C.1.3}$$
$$\times U_{SU(3)}((1,1),(\lambda,0),(\tau,\tau),(0,\lambda);(\lambda,0),(\sigma,\sigma))_{\rho}$$

a) $\tau = \sigma + 1$ we have

$$c^{(\sigma+1,\sigma+1),(\sigma+1,\sigma+1,\sigma+1);1}_{(111);1;(\sigma\sigma\sigma);0} = \sqrt{\frac{16}{(\lambda+1)(\lambda+2)}}\left\langle \begin{smallmatrix}(1,1)\\1;1\end{smallmatrix}; \begin{smallmatrix}(\sigma,\sigma)\\\sigma;0\end{smallmatrix} \middle\| \begin{smallmatrix}(\sigma+1,\sigma+1)\\\sigma+1;1\end{smallmatrix}\right\rangle \tag{C.1.4}$$
$$\times U_{SU(3)}((1,1),(\lambda,0),(\sigma+1,\sigma+1),(0,\lambda);(\lambda,0),(\sigma,\sigma))$$

$$= \sqrt{\frac{16}{(\lambda+1)(\lambda+2)}}\frac{(\sigma+2)(\sigma+3)}{2(\sigma+1)}\sqrt{\frac{1}{3(\sigma+1)(2\sigma+3)}} \tag{C.1.5}$$
$$\times \frac{(\sigma+1)}{2}\sqrt{\frac{3(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+3)(\sigma+2)(2\sigma+3)}}$$

This can be simplified

$$c^{(\sigma+1,\sigma+1),(\sigma+1,\sigma+1,\sigma+1);1}_{(111);1;(\sigma\sigma\sigma);0} = \frac{(\sigma+3)}{2\sigma+3}\sqrt{\frac{(\sigma+2)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\sigma+1)}} \tag{C.1.6}$$

b) $\tau = \sigma$

$$c^{(\sigma,\sigma),(\sigma\sigma\sigma);1}_{(111);1;(\sigma\sigma\sigma);0} = -\frac{2\sigma(\sigma+2)(2\lambda+3)}{(2\sigma+1)(2\sigma+3)\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.7}$$

c) $\tau = \sigma - 1$

$$c^{(\sigma-1,\sigma-1),(\sigma-1,\sigma-1,\sigma-1);1}_{(111);1;(\sigma\sigma\sigma);0} = \frac{(\sigma-1)}{(2\sigma+1)}\sqrt{\frac{\sigma(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\sigma+1)}} \tag{C.1.8}$$

Combining the results of **a**,**b** and **c**, we obtain:

$$\begin{aligned}\hat{T}^{\lambda}_{1;(111);1}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} &= \frac{(\sigma+3)}{2\sigma+3}\sqrt{\frac{(\sigma+2)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\sigma+1)}}T^{\lambda}_{(\sigma+1);(\sigma+1,\sigma+1,\sigma+1);1} \\ &- \frac{2\sigma(\sigma+2)(2\lambda+3)}{(2\sigma+1)(2\sigma+3)\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}T^{\lambda}_{\sigma;(\sigma\sigma\sigma);1} \\ &+ \frac{(\sigma-1)}{(2\sigma+1)}\sqrt{\frac{\sigma(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\sigma+1)}}T^{\lambda}_{(\sigma-1);(\sigma-1,\sigma-1,\sigma-1);1}\end{aligned} \tag{C.1.9}$$

Thus

$$\begin{aligned}\hat{T}^{\lambda}_{1;(111)1}\hat{w}_{\lambda}(\Omega)(0) &= \sum_{\tau}\frac{\sqrt{2(\tau+1)}\tau(\tau+2)}{\sqrt{\lambda(\lambda+3)}(\lambda+1)(\lambda+2)} \\ &\times\left(\frac{\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)}}{(2\tau+1)} - \frac{2(\tau+1)(2\lambda+3)}{(2\tau+1)(2\tau+3)}\right. \\ &\qquad\left. + \sqrt{(\lambda-\tau)(\lambda+\tau+3)}\frac{1}{(2\tau+3)}\right)\hat{T}^{\lambda}_{\tau;(\tau\tau\tau)1}\end{aligned} \tag{C.1.10}$$

C.1.2 Case $\nu, I = (2,0,1); \frac{1}{2}$:

$$\begin{aligned}\hat{T}^{\lambda}_{1;(2,0,1);\frac{1}{2}}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} &= c^{(\sigma+1,\sigma+1),(\sigma+2,\sigma,\sigma+1);\frac{1}{2}}_{(2,0,1);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{(\sigma+1);(\sigma+2,\sigma,\sigma+1);\frac{1}{2}} \\ &+ c^{(\sigma,\sigma),(\sigma+1,\sigma-1,\sigma);\frac{1}{2}}_{(2,0,1);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{\sigma;(\sigma+1,\sigma-1,\sigma);\frac{1}{2}} \\ &+ c^{(\sigma-1,\sigma-1),(\sigma,\sigma-2,\sigma-1);\frac{1}{2}}_{(2,0,1);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{(\sigma-1);(\sigma,\sigma-2,\sigma-1);\frac{1}{2}}\end{aligned} \tag{C.1.11}$$

C.1.2 a): $\tau = \sigma + 1$

$$c^{(\sigma+1,\sigma+1),(\sigma+2,\sigma,\sigma+1);\frac{1}{2}}_{(2,0,1);\frac{1}{2};(\sigma\sigma\sigma);0} = \frac{1}{(2\sigma+3)}\sqrt{\frac{3(\sigma+2)(\sigma+3)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.12}$$

C.1.2 b): $\tau = \sigma$

$$c^{(\sigma,\sigma),(\sigma+1,\sigma-1,\sigma);\frac{1}{2}}_{(2,0,1);\frac{1}{2};(\sigma\sigma\sigma);0} = 2\frac{(\lambda-2\sigma(\sigma+2))}{(2\sigma+1)(2\sigma+3)}\sqrt{\frac{3\sigma(\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.13}$$

C.1.2 c): $\tau = \sigma - 1$

$$c^{(\sigma-1,\sigma-1),(\sigma,\sigma-2,\sigma-1);\frac{1}{2}}_{(2,0,1);\frac{1}{2};(\sigma\sigma\sigma);0} = -\frac{1}{(2\sigma+1)}\sqrt{\frac{3\sigma(\sigma-1)(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.14}$$

Combining these coefficients with the respective tensors:

$$\begin{aligned}\hat{T}^{\lambda}_{1;(201);\frac{1}{2}}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} = &\frac{1}{(2\sigma+3)}\sqrt{\frac{3(\sigma+2)(\sigma+3)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{(\sigma+1);(\sigma+2,\sigma,\sigma+1);\frac{1}{2}} \\ &+2\frac{(\lambda-2\sigma(\sigma+2))}{(2\sigma+1)(2\sigma+3)}\sqrt{\frac{3\sigma(\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{\sigma;(\sigma+1,\sigma-1,\sigma);\frac{1}{2}} \\ &-\frac{1}{(2\sigma+1)}\sqrt{\frac{3\sigma(\sigma-1)(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{(\sigma-1);(\sigma,\sigma-2,\sigma-1);\frac{1}{2}}\end{aligned} \tag{C.1.15}$$

Thus

$$\begin{aligned}\hat{T}^{\lambda}_{1;(201)\frac{1}{2}}\hat{w}_{\lambda}(\Omega)(0) = \sum_{\tau}&\frac{\sqrt{6\tau(\tau+1)(\tau+2)}}{(\lambda+1)(\lambda+2)\sqrt{\lambda(\lambda+3)}}\Big(\frac{\tau}{(2\tau+1)}\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)} \\ &+\frac{2(\tau+1)(\lambda-2\tau(\tau+2))}{(2\tau+1)(2\tau+3)} \\ &-\sqrt{(\lambda-\tau)(\lambda+\tau+3)}\frac{(\tau+2)}{(2\tau+3)}\Big)\hat{T}^{\lambda}_{\tau;(\tau+1,\tau-1,\tau)\frac{1}{2}}\end{aligned} \tag{C.1.16}$$

C.1.3 Case $\nu, I = (0,1,2);\frac{1}{2}$:

$$\begin{aligned}\hat{T}^{\lambda}_{1;(0,1,2);\frac{1}{2}}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} = &c^{(\sigma+1,\sigma+1),(\sigma,\sigma+1,\sigma+2);\frac{1}{2}}_{(0,1,2);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{(\sigma+1);(\sigma,\sigma+1,\sigma+2);\frac{1}{2}} \\ &+c^{(\sigma,\sigma),(\sigma-1,\sigma,\sigma+1);\frac{1}{2}}_{(0,1,2);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{\sigma;(\sigma-1,\sigma,\sigma+1);\frac{1}{2}} \\ &+c^{(\sigma-1,\sigma-1),(\sigma-2,\sigma-1,\sigma);\frac{1}{2}}_{(0,1,2);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{(\sigma-1);(\sigma-2,\sigma-1,\sigma);\frac{1}{2}}\end{aligned} \tag{C.1.17}$$

C.1.3.0 a): $\tau = \sigma + 1$

$$c^{(\sigma+1,\sigma+1),(\sigma,\sigma+1,\sigma+2);\frac{1}{2}}_{(0,1,2);\frac{1}{2};(\sigma\sigma\sigma);0} = \frac{1}{(2\sigma+3)}\sqrt{\frac{3(\sigma+2)(\sigma+3)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.18}$$

C.1.3 b): $\tau = \sigma$

$$c^{(\sigma,\sigma),(\sigma-1,\sigma,\sigma+1);\frac{1}{2}}_{(0,1,2);\frac{1}{2};(\sigma\sigma\sigma);0} = 2\frac{(3+\lambda+2\sigma(\sigma+2))}{(2\sigma+1)(2\sigma+3)}\sqrt{\frac{3\sigma(\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.19}$$

C.1.3 c): $\tau = \sigma - 1$

$$c^{(\sigma-1,\sigma-1),(\sigma-2,\sigma-1,\sigma);\frac{1}{2}}_{(0,1,2);\frac{1}{2};(\sigma\sigma\sigma);0} = -\frac{1}{(2\sigma+1)}\sqrt{\frac{3\sigma(\sigma-1)(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.20}$$

Combining these coefficients with the respective tensors:

$$\begin{aligned}\hat{T}^{\lambda}_{1;(0,1,2);\frac{1}{2}}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} = &\frac{1}{(2\sigma+3)}\sqrt{\frac{3(\sigma+2)(\sigma+3)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{(\sigma+1);(\sigma,\sigma+1,\sigma+2);\frac{1}{2}} \\ &+ 2\frac{(3+\lambda+2\sigma(\sigma+2))}{(2\sigma+1)(2\sigma+3)}\sqrt{\frac{3\sigma(\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{\sigma;(\sigma-1,\sigma,\sigma+1);\frac{1}{2}} \\ &- \frac{1}{(2\sigma+1)}\sqrt{\frac{3\sigma(\sigma-1)(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{(\sigma-1);(\sigma-2,\sigma-1,\sigma);\frac{1}{2}}\,. \end{aligned} \tag{C.1.21}$$

Thus

$$\begin{aligned}\hat{T}^{\lambda}_{1;(012)\frac{1}{2}}\hat{w}_{\lambda}(\Omega)(0) = &\sum_{\tau}\frac{\sqrt{6\tau(\tau+1)(\tau+2)}}{(\lambda+1)(\lambda+2)\sqrt{\lambda(\lambda+3)}}\Big(\frac{\tau}{(2\tau+1)}\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)} \\ &+ \frac{2(\tau+1)(3+\lambda+2\tau(\tau+2))}{(2\tau+1)(2\tau+3)} \\ &- \frac{(\tau+2)}{(2\tau+3)}\sqrt{(\lambda-\tau)(\lambda+\tau+3)}\Big)\hat{T}^{\lambda}_{\tau;(\tau-1,\tau,\tau+1)\frac{1}{2}} \end{aligned} \tag{C.1.22}$$

C.1.4 Case $\nu, I = (0,2,1); \frac{1}{2}$:

$$\begin{aligned}\hat{T}^{\lambda}_{1;(0,2,1);\frac{1}{2}}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} = &c^{(\sigma+1,\sigma+1),(\sigma,\sigma+2,\sigma+1);\frac{1}{2}}_{(0,2,1);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{(\sigma+1);(\sigma,\sigma+2,\sigma+1);\frac{1}{2}} \\ &+ c^{(\sigma,\sigma),(\sigma-1,\sigma+1,\sigma);\frac{1}{2}}_{(0,2,1);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{\sigma;(\sigma-1,\sigma+1,\sigma);\frac{1}{2}} \\ &+ c^{(\sigma-1,\sigma-1),(\sigma-2,\sigma,\sigma-1);\frac{1}{2}}_{(0,2,1);\frac{1}{2};(\sigma\sigma\sigma);0}\hat{T}^{\lambda}_{(\sigma-1);(\sigma-2,\sigma,\sigma-1);\frac{1}{2}} \end{aligned} \tag{C.1.23}$$

These coefficients are given by

C.1.4 a): $\tau = \sigma + 1$

$$c^{(\sigma+1,\sigma+1),(\sigma,\sigma+2,\sigma+1);\frac{1}{2}}_{(0,2,1);\frac{1}{2};(\sigma\sigma\sigma);0} = \frac{1}{(2\sigma+3)}\sqrt{\frac{3(\sigma+2)(\sigma+3)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.24}$$

C.1.4 b): $\tau = \sigma$

$$c^{(\sigma,\sigma),(\sigma-1,\sigma+1,\sigma);\frac{1}{2}}_{(0,2,1);\frac{1}{2};(\sigma\sigma\sigma);0} = 2\frac{(3+\lambda+2\sigma(\sigma+2))}{(2\sigma+1)(2\sigma+3)}\sqrt{\frac{3\sigma(\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.25}$$

C.1.4 c): $\tau = \sigma - 1$

$$c^{(\sigma-1,\sigma-1),(\sigma-2,\sigma,\sigma-1);\frac{1}{2}}_{(0,2,1);\frac{1}{2};(\sigma\sigma\sigma);0} = -\frac{1}{(2\sigma+1)}\sqrt{\frac{3\sigma(\sigma-1)(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.26}$$

Clearly, these coefficients are the same as the ones given in the case $\nu, I = (0,1,2); \frac{1}{2}$. Combining them with the respective tensors:

$$\begin{aligned}\hat{T}^{\lambda}_{1;(0,2,1);\frac{1}{2}}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0} = {}& \frac{1}{(2\sigma+3)}\sqrt{\frac{3(\sigma+2)(\sigma+3)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{(\sigma+1);(\sigma,\sigma+2,\sigma+1);\frac{1}{2}} \\ &+ 2\frac{(3+\lambda+2\sigma(\sigma+2))}{(2\sigma+1)(2\sigma+3)}\sqrt{\frac{3\sigma(\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{\sigma;(\sigma-1,\sigma+1,\sigma);\frac{1}{2}} \\ &- \frac{1}{(2\sigma+1)}\sqrt{\frac{3\sigma(\sigma-1)(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\hat{T}^{\lambda}_{(\sigma-1);(\sigma-2,\sigma,\sigma-1);\frac{1}{2}}\end{aligned} \tag{C.1.27}$$

Finally,

$$\begin{aligned}\hat{T}^{\lambda}_{1;(021)\frac{1}{2}}\hat{w}_{\lambda}(\Omega)(0) = \sum_{\tau} & \frac{\sqrt{6\tau(\tau+1)(\tau+2)}}{(\lambda+1)(\lambda+2)\sqrt{\lambda(\lambda+3)}}\Big(\frac{\tau}{(2\tau+1)}\sqrt{(\lambda-\tau+1)(\lambda+\tau+2)} \\ &+ \frac{2(\tau+1)(3+\lambda+2\tau(\tau+2))}{(2\tau+1)(2\tau+3)} \\ &- \frac{(\tau+2)}{(2\tau+3)}\sqrt{(\lambda-\tau)(\lambda+\tau+3)}\Big)\hat{T}^{\lambda}_{\tau;(\tau-1,\tau+1,\tau)\frac{1}{2}}\end{aligned} \tag{C.1.28}$$

C.1.5 Case $\nu I = (111)0$:

$$\hat{T}^{\lambda}_{1;(111)0}\hat{w}_{\lambda}(\Omega)(0) = \sum_{\sigma} F^{\lambda}_{\sigma}\,\hat{T}^{\lambda}_{1;(111)0}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma);0}\,, \tag{C.1.29}$$

$$\hat{T}^{\lambda}_{1;(111)0}\hat{T}^{\lambda}_{\sigma;(\sigma\sigma\sigma)0} = \sum_{\tau=\sigma-1}^{\sigma+1} c^{(\tau\tau)(\tau\tau\tau)0}_{(111)0;(\sigma\sigma\sigma)0}\hat{T}^{\lambda}_{\tau;(\tau\tau\tau)0}\,, \tag{C.1.30}$$

$$c^{(\tau\tau)(\tau\tau\tau)0}_{(111)0;(\sigma\sigma\sigma)0} = \sqrt{\frac{16}{(\lambda+1)(\lambda+2)}}\sum_{\rho}\left\langle \begin{smallmatrix}(1,1)\\1;0\end{smallmatrix};\begin{smallmatrix}(\sigma\sigma)\\\sigma;0\end{smallmatrix}\,\|\,\begin{smallmatrix}(\tau\tau)\\\tau;0\end{smallmatrix}\right\rangle_{\rho} U[(11)(\lambda 0)(\tau\tau)(0\lambda);(\lambda,0)(\sigma\sigma)]_{\rho} \tag{C.1.31}$$

C.1.5 a): $\tau = \sigma + 1$

We have

$$\left\langle \begin{smallmatrix}(1,1)\\1;0\end{smallmatrix};\begin{smallmatrix}(\sigma,\sigma)\\\sigma;0\end{smallmatrix}\,\|\,\begin{smallmatrix}(\sigma+1,\sigma+1)\\\sigma+1;0\end{smallmatrix}\right\rangle = \frac{\sigma+2}{3}\sqrt{\frac{3}{(2\sigma+3)\sigma+1)}}\,, \tag{C.1.32}$$

$$U[(11)(\lambda 0)(\sigma+1,\sigma+1)(0\lambda);(\lambda 0)(\sigma\sigma)] = \frac{\sigma+1}{2}\sqrt{\frac{3(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+3)(\sigma+2)(\sigma+3)}}\,, \tag{C.1.33}$$

$$c^{(\sigma+1\sigma+1)(\sigma+1,\sigma+1,\sigma+1)0}_{(111)0;(\sigma\sigma\sigma)0} = \frac{3}{\sigma+3}\sqrt{\frac{(\sigma+1)(\sigma+2)(\lambda-\sigma)(\lambda+\sigma+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\,. \tag{C.1.34}$$

C.1.5 b): $\tau = \sigma$

$$c^{(\sigma\sigma)(\sigma\sigma\sigma)0}_{(111)0;(\sigma\sigma\sigma)0} = \frac{4}{\sqrt{(\lambda+1)(\lambda+2)}}\sqrt{\frac{\sigma(\sigma+2)}{(2\sigma+1)(2\sigma+3)}} \times \frac{2\lambda+3}{2}\sqrt{\frac{\sigma(\sigma+2)}{\lambda(\lambda+3)(2\sigma+1)(2\sigma+3)}}\,, \tag{C.1.35}$$

$$= \frac{2(2\lambda+3)\sigma(\sigma+2)}{(2\sigma+1)(2\sigma+3)\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}} \tag{C.1.36}$$

C.1.5 c): $\tau = \sigma = 1$

$$c^{(\sigma-1,\sigma-1)(\sigma-1,\sigma-1,\sigma-1)0}_{(111)0;(\sigma\sigma\sigma)0} = \frac{4}{\sqrt{(\lambda+1)(\lambda+2)}}\frac{\sigma}{2}\sqrt{\frac{3}{2\sigma+1)(\sigma+1)}}\frac{\sigma+1}{2}\sqrt{\frac{3(\lambda-\sigma+1)(\lambda+\sigma+2)}{\lambda(\lambda+3)\sigma(2\sigma+1)}} \tag{C.1.37}$$

Thus,

$$
\begin{aligned}
\hat{T}^{\lambda}_{1;(111)0}\,\hat{w}_{\lambda}(\Omega)(0) &= \sum_{\tau}\sqrt{\frac{2\tau^3}{(\lambda+1)(\lambda+2)}}\frac{3}{2\tau+1}\sqrt{\frac{\tau(\tau+1)(\lambda-\tau+1)(\lambda+\tau+2)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\,\hat{T}^{\lambda}_{\tau;(\tau\tau\tau);0} \\
&+\sum_{\tau}\sqrt{\frac{2(\tau+1)^3}{(\lambda+1)(\lambda+2)}}\frac{2(2\lambda+3)\tau(\tau+2)}{(2\tau+1)(2\tau+3)\sqrt{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\,\hat{T}^{\lambda}_{\tau;(\tau\tau\tau)0} \\
&+\sum_{\tau}\sqrt{\frac{2(\tau+2)^3}{(\lambda+1)(\lambda+2)}}\frac{3}{2\tau+3}\sqrt{\frac{(\tau+1)(\tau+2)(\lambda-\tau)(\lambda+\tau+3)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}}\,\hat{T}^{\lambda}_{\tau;(\tau\tau\tau)0}\,, \qquad \text{(C.1.38)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(\lambda+1)(\lambda+2)}\sqrt{\frac{2}{\lambda(\lambda+3)}} \\
&\times\sum_{\tau}\left[\frac{3\tau^2}{(2\tau+1)}\sqrt{(\tau+1)(\lambda-\tau+1)(\lambda+\tau+2)}+\frac{2(\tau+1)^{3/2}(2\lambda+3)\tau(\tau+2)}{(2\tau+1)(2\tau+3)}\right. \\
&\qquad\left.+\frac{3(\tau+2)^2}{(2\tau+3)}\sqrt{(\tau+1)(\lambda-\tau)(\lambda+\tau+3)}\right]\hat{T}^{\lambda}_{\tau;(\tau\tau\tau)0} \qquad \text{(C.1.39)}
\end{aligned}
$$

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