LIL FOR L-STATISTICS WITHOUT VARIANCE

by

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Abstract

Let $\{X, X_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function $F(x)$. For each positive integer n, let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of $X_1, X_2, \cdots, X_n$. Let $H(\cdot)$ be a real Borel-measurable function defined on $\mathcal{R}$ such that $\mathbb{E}|H(X)| < \infty$ and let $J(\cdot)$ be a Lipschitz function of order one defined on $[0, 1]$. Write $\mu = \mu(F, J, H) = \mathbb{E}(J(U)H(F^{\leftarrow}(U)))$ and $\mathbb{L}_n(F, J, H) = \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{i}{n}\right)$ $H(X_{i:n}),\ \ n \geq 1$, where U is a random variable with uniform $(0, 1)$ distribution and $F^{\leftarrow}(t) = \inf\{x;\ F(x) \geq t\},\ 0 < t < 1$. In this thesis, the Chung-Smirnov LIL for empirical processes and the Einmahl-Li LIL for partial sums of i.i.d. random variables without variance are used to establish necessary and sufficient conditions for having with probability 1: $0 \leq \limsup_{n\to\infty} \sqrt{n/\varphi(n)}\,|\mathbb{L}_n(F, J, H) - \mu| < \infty$, where $\varphi(\cdot)$ is from a suitable subclass of the positive, nondecreasing, and slowly varying functions defined on $[0,\ \infty)$. The almost sure value of the limsup is identified under suitable conditions. Specializing our result to $\varphi(x) = 2(\log\log x)^p, p > 1$ and to $\varphi(x) = 2(\log x)^r, r > 0$, we obtain an analog of the Hartman-Wintner-Strassen LIL for L-statistics in the infinite variance case. A stability result for L-statistics in the infinite variance case is also obtained.

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Contents

Chapter 1

Historical Review

In this thesis we study the strong limit theorems for linear combinations of order statistics (in short, $L-$statistics). The asymptotic theory of order statistics is concerned with the distribution of $X_{r:n}$, suitably standardized, as n approaches ∞. We usually assume that $X_{r:n}$ is the rth order statistic in a random sample of n from some population with cdf (cumulative distribution function) $F(x)$. If $r/n \to p$ as $n \to \infty$, fundamentally different results are obtained according as (a) $0 < p < 1$ (central or quantile case) or (b) r or $n - r$ is held fixed (extreme case), or (c) $p = 0$ or 1, with r or $n - r$ being a function of n (intermediate case).

The class of $L-$statistics appears to have been first studied in 1920 and such statistics have received growing attention since the 1960s since some of the robust estimators of location (e.g., the trimmed and Winsorized means) are examples of such statistics. Another $L-$statistic, placing more weight on the extremes, is the Gini (1912) mean difference.

The asymptotic normality of $\mathbb{L}_n = \frac{1}{n}\sum_{i=1}^{n} c_{i,n} H(X_{i:n}), \quad n \geq 1$, requires suitable conditions on both the $c_{i,n}$ and the form of the cdf $F(x)$. Some authors discussed the $c_{i,n}$ and weak on $F(x)$, others the reverse. Lots of the notable contributions in this area appeared from the 1960s till mid-1980s. such as Chernoff et al. (1967), van Zwet (1983), Chernoff and Savage (1958), Chemoff et al. (1967), Shorack (1969), Stigler (1969, 1974), Hájek's (1968), Sen (1978), Boos (1979), Serfling (1980), Shorack and Wellner (1986) and Helmers (1977, 1980, 1982).

Some authors have established rates of convergence to normality. Beginning with Rosenkrantz and O'Reilly (1972), Bjerve (1977) obtained a Berry-Esséen type bound of order $n^{1/2}$ for trimmed linear combinations of order statistics using the representation of Chernoff et al. (1967). Helmers (1977, 1982) provided a Berry-Esséen-type bound of order $n^{1/2}$ between $\Phi(x)$, the standard normal cdf, and the cdf of L_n^*,L_n^0, and $\hat{L_n^*}$. Similar results for unbounded weight functions was obtained by Helmers and Hušková (1984). Singh (1981) obtained some nonuniform rates of convergence to normality that are helpful in the study of moment convergence. Helmers et al. (1990) established Berry-Esséen-type results for $L-$statistics based on generalized order statistics, as defined by Choudhury and Serfling (1988). Putt and Chinchilli (1999) corrected their expression for the variance of the limiting normal distribution. See also Putt and Chinchilli (2002).

Weiss (1969b), Shorack (1973), Stigler (1974), and Ruymgaart and van Zuijlen (1978) discussed the asymptotic normality of $L-$statistics obtained from independent nonidentically distributed rv's, and Xiang (1994) obtained Berry-Esséen bounds. Mehra and Rao (1975), Gastwirth and Rubin (1975), Sotres and Ghosh (1979), and Singh (1983) established the asymptotic normality under weak dependence. Puri and Ruymgaart (1993) established asymptotic normality for a large class of time series data that includes dependent nonidentically distributed sequences of variates. Shao (1994) obtained several limit results for $L-$statistics and sample quantiles for the survey data arising from a stratified multistage sampling design.

The asymptotic joint distribution of $L-$statistics from multivariate populations has been studied by Siddiqui and Butler (1969). Numerous references on $L-$statistics were listed in H.A. David et al. (2003).

Chapter 2

Purpose of This Thesis

Let $\{X, X_n;\ n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) real random variables with distribution function $F(x) = P(X \leq x)$, $x \in \mathcal{R} = (-\infty, \infty)$ and let $\{U, U_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables with the uniform $(0, 1)$ distribution. For each positive integer n, let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of $X_1, X_2, \cdots, X_n$. Let $H(\cdot)$ be a real-valued Borel-measurable function defined on $\mathcal{R}$. A linear combination of order statistics (in short, an L-statistic) is a statistic of the form

$$\mathbb{L}_n = \frac{1}{n} \sum_{i=1}^{n} c_{i,n} H(X_{i:n})$$

where the weights $c_{i,n}, 1 \leq i \leq n$ are real numbers and $n \geq 1$. Define $Lt = \log_e \max\{e, t\}$ and $LLt = L(Lt)$ for $t \in \mathcal{R}$. The classical Hartman-Wintner-Strassen law of the iterated logarithm (LIL) states that

$$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \frac{\sum_{i=1}^{n} H(X_i)}{\sqrt{2nLLn}} \overset{+}{=} (-)\, \sigma \quad \text{almost surely (a.s.)} \tag{2.1}$$

if and only if

$$\mathbb{E}H(X) = 0 \quad \text{and} \quad \sigma^2 = \mathbb{E}H^2(X) < \infty. \tag{2.2}$$

Moreover, if (2.2) holds, then

$$C\left(\left\{\sum_{i=1}^{n} H(X_i)/\sqrt{2nLLn};\ n \geq 1\right\}\right) = [-\sigma, \sigma] \ \text{ a.s.}, \tag{2.3}$$

where $C(\{x_n;\ n \geq 1\})$ stands for the cluster set (i.e., the set of limit points) of the numerical sequence $\{x_n;\ n \geq 1\}$. See Hartman and Wintner (1941) for the "if" part and Strassen (1966) for the converse. The conclusion (2.3) is due to Strassen (1964).

Alternative proofs of the Hartman-Wintner (1941) LIL were discovered by Strassen (1964), Heyde (1969), Egorov (1971), Teicher (1974), Csörgő and Révész (1981, p. 119), and de Acosta (1983). Substantially simpler proofs of Strassen's (1966) converse were obtained by Feller (1968), Heyde (1968), and Steiger and Zaremba (1972). Martikainen (1980), Rosalsky (1980), and Pruitt (1981) simultaneously and independently obtained a "one-sided" converse to the Hartman-Wintner (1941) LIL. Specifically, they proved that each part of (2.1) *individually* implies (2.2).

Many authors, including Helmers (1977), Helmers, Janssen, and Serfling (1988), Li, Rao, and Tomkins (2001), Mason (1982), Sen (1978), van Zwet (1980), and Wellner (1977a,b), have investigated the strong limiting behavior for a class of L-statistics of the form

$$\mathbb{L}_n(F, J, H) \equiv \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(X_{i:n}), \quad n \geq 1 \tag{2.4}$$

where $J(\cdot)$ is a real-valued function, often called a *score function*, defined on $[0, 1]$. Helmers (1977), Mason (1982), Sen (1978), van Zwet (1980), and Wellner (1977a) have studied the strong law of large numbers (SLLN) for $\mathbb{L}_n$, $n \geq 1$ and have shown that under a variety of conditions on $J(\cdot)$ and $H(\cdot)$

$$\lim_{n\to\infty} \mathbb{L}_n(F_U, J, H) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(U_{i:n}) = \int_0^1 J(t)H(t)dt \text{ (finite) a.s.,}$$

where F_U is the distribution function of the random variable U and $U_{i:n}$, $1 \leq i \leq n$, are the order statistics of the U_i, $1 \leq i \leq n$, $n \geq 1$.

If $J(\cdot)$ is a Lipschitz function of order one defined on $[0, 1]$ and

$$\mathbb{E}|H(X)| < \infty, \tag{2.5}$$

let us write

$$\begin{cases} Z = J(U)H(F^{\leftarrow}(U)), \\ Y = -Z + \mu - \int_0^1 \left(I(U \leq t) - t\right) J'(t)H(F^{\leftarrow}(t))dt, \end{cases} \tag{2.6}$$

where $F^{\leftarrow}(t)$ is the quantile function

$$F^{\leftarrow}(t) = \inf\left\{s;\ F(s) \geq t\right\}, \quad 0 < t < 1,$$

and

$$\mu = \mu(F, J, H) = \mathbb{E}Z = \mathbb{E}\left(J(U)H(F^{\leftarrow}(U))\right). \tag{2.7}$$

Then μ exists and is finite and Y and Z are both well-defined random variables under (2.5). Moreover,

$$\sigma_Y^2 = \mathrm{Var}(Y) = \mathbb{E}Y^2. \tag{2.8}$$

To see this, note that (2.8) is equivalent to

$$\mathbb{E}Y = -\mathbb{E}\left(\int_0^1 \left(I(U \le t) - t\right) J'(t)H(F^{\leftarrow}(t))dt\right) = 0. \tag{2.9}$$

Since $\mathbb{E}\left(I(U \le t) - t\right) = 0$, (2.9) follows from an application of Fubini's theorem, subject to the existence of the integral

$$I = \int_0^1 J'(t)H(F^{\leftarrow}(t))dt = \int_0^1 H(F^{\leftarrow}(t))dJ(t) = \mathbb{E}\left(J'(U)H(F^{\leftarrow}(U))\right).$$

But the score function $J(\cdot)$ has an almost everywhere (with respect to Lebesgue measure) bounded derivative $J'(\cdot)$. From this fact and the equality $\mathbb{E}|H(X)| = \mathbb{E}|H(G(U))| < \infty$, it follows that I exists and is finite; clearly, $\sigma_Y^2 < \infty$ if and only if

$$\mathbb{E}Z^2 < \infty. \tag{2.10}$$

Recall that a sequence of random variables $\{\xi_n;\ n \ge 1\}$ is said to be *bounded in probability* if

$$\lim_{x\to\infty} \sup_{n\ge1} P\left(|\xi_n| \ge x\right) = 0.$$

Combining the Chung-Smirnov LIL (see Chung (1949) and Smirnov (1944)) and the Finkelstein functional LIL (see Finkelstein (1971)) for empirical processes, Li, Rao, and Tomkins (2001, Theorem 2.1) proved that the following three statements are equivalent:

$$(2.10) \ \text{holds;}$$

$$\limsup_{n\to\infty} \sqrt{n/(2LLn)}\, |\mathbb{L}_n(F,J,H)-\mu| < \infty \quad \text{a.s.;}$$

$$\left\{\sqrt{n}\,(\mathbb{L}_n(F,J,H)-\mu)\,;\quad n\geq 1\right\} \quad \text{is bounded in probability.}$$

Moreover, if any of the three statements above holds, then

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\sqrt{n/(2LLn)}\,(\mathbb{L}_n(F,J,H)-\mu) = \overset{+}{(-)}\,\sigma_Y \qquad \text{a.s.,} \tag{2.11}$$

$$C\left(\left\{\sqrt{n/(2LLn)}\,(\mathbb{L}_n(F,J,H)-\mu)\,;\ n\geq 1\right\}\right) = [-\sigma_Y,\ \sigma_Y] \ \text{ a.s.,} \tag{2.12}$$

and

$$\sqrt{n}\,(\mathbb{L}_n(F,J,H)-\mu) \xrightarrow{d} N(0,\ \sigma_Y^2), \tag{2.13}$$

where "$\xrightarrow{d}$" denotes convergence in distribution. This powerful result contains many previous results obtained under more restrictive conditions, although it is still not the last word as the authors mention in an open problem (to weaken the conditions on $J(\cdot)$). The authors illustrate with examples that their result can handle some cases that previous results could not; for example, the Gini mean-difference statistic.

The main purpose of the thesis is to find necessary and sufficient conditions for

$$0 < \limsup_{n\to\infty} \sqrt{n/\varphi(n)}\, |\mathbb{L}_n(F,J,H)-\mu| < \infty \ \text{ a.s.,}$$

where $\varphi(\cdot)$ is from a suitable subclass of the positive, nondecreasing, and slowly varying functions. But we also treat the case where the limit is 0 a.s. We emphasize that we are not assuming that $\mathbb{E}Z^2 < \infty$ where Z is as in (2.6).

The plan of this thesis is as follows. Our main results, Theorems 3.1 and 3.2 and their proof and corollaries are presented in Chapter 3. The proof is obtained via a nice application of the Chung-Smirnov LIL (see Chung (1949) and Smirnov (1944)) for empirical processes and the Einmahl-Li LIL (see Einmahl and Li (2005)) for partial sums of i.i.d. random variables without variance. In Chapter 4, we provide an interesting example to illustrate our results.

Chapter 3

Main Results

Let $\mathcal{H}$ be the set of continuous, nondecreasing functions $\varphi(\cdot) : [0, \infty) \to (0, \infty)$, which are slowly varying at infinity. By monotonicity, the slow variation of $\varphi(\cdot)$ is equivalent to $\lim_{t\to\infty} \varphi(et)/\varphi(t) = 1$. Very often one can even show that $\lim_{t\to\infty} \varphi(tf(t))/\varphi(t) = 1$, where $f(\cdot)$ is a nondecreasing function such that $\lim_{t\to\infty} f(t) = \infty$. Set $f_\tau(t) = \exp((Lt)^\tau), 0 \leq \tau < 1$. Given $0 \leq q < 1$, let $\mathcal{H}_q \subset \mathcal{H}$ the class of functions $\varphi(\cdot)$ such that

$$\lim_{t\to\infty} \frac{\varphi\left(tf_\tau(t)\right)}{\varphi(t)} = 1, \quad 0 \leq \tau < 1 - q$$

and set $\mathcal{H}_1 = \mathcal{H}$.

We consider q to be a measure for how slow is the slow variation. So functions in $\mathcal{H}_0$ are the "slowest" and it will turn out that this class is particulary interesting for LIL type results (see Theorem 3.2 below). Examples of functions in $\mathcal{H}_0$ are $\varphi(t) = (Lt)^r$, $r \geq 0$ and $\varphi(t) = (LLt)^p$, $p \geq 0$.

The following Theorem 3.1 gives LIL type results when $\lambda > 0$ and stability results when $\lambda = 0$ with respect to a large class of normalizing sequences, without assuming that $\mathbb{E}Z^2 < \infty$, where Z is defined in (2.6).

THEOREM 3.1. *Let $\{X, X_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables with distribution function $F(x) = P(X \leq x),\ x \in \mathcal{R}$ and let $H(\cdot)$ be a real-valued Borel-measurable function defined on $\mathcal{R}$ satisfying (2.5). Let $J(\cdot)$ be a Lipschitz function of order one defined on $[0,1]$ and let $\mathbb{L}_n(F,J,H),\ n \geq 1,\ Z,$ and μ be defined by (2.4), (2.6), and (2.7), respectively. Given a function $\varphi(\cdot) \in \mathcal{H}_q$ where $0 \leq q \leq 1$, set $\Psi(x) = \sqrt{x\varphi(x)}, x \in \mathcal{R}$. If*

$$\lim_{x\to\infty} \frac{\log\log x}{\varphi(x)} = \infty \tag{3.1}$$

and

$$\mathbb{E}\Psi^{-1}(|Z|) < \infty \quad \text{and } \lambda = \sqrt{2 \limsup_{x\to\infty} \frac{\Psi^{-1}(xLLx)}{x^2 LLx} \mathbb{E}\left(Z^2 I\{|Z| \leq x\}\right)}, \tag{3.2}$$

then when $\lambda < \infty$ we have

$$(1-q)^{1/2}\lambda \leq \limsup_{n\to\infty} \sqrt{n/\varphi(n)}\,|\mathbb{L}_n(F,J,H) - \mu| \leq \lambda \quad a.s. \tag{3.3}$$

Conversely, if $q < 1$, then the relation

$$\limsup_{n\to\infty} \sqrt{n/\varphi(n)}\,|\mathbb{L}_n(F,J,H) - \mu| < \infty \quad a.s. \tag{3.4}$$

implies that (3.2) holds with $\lambda < \infty$.

Moreover, the limsup in (3.3) is positive and finite if and only if (3.2) holds with $0 < \lambda < \infty$.

For slowly varying functions $\varphi(\cdot) \in \mathcal{H}_0$ and λ as in (3.2) with $0 < \lambda < \infty$, we obtain for L-statistics $\{\mathbb{L}_n(F, J, H),\ n \geq 1\}$ the following complete analogue of the Hartman-Wintner-Strassen LIL. Of course, (3.5) follows from (3.6) but nevertheless it is worthwhile to label them separately.

THEOREM 3.2. *Assume that* $H(\cdot)$ *is a real-valued Borel-measurable function defined on* $\mathcal{R}$ *satisfying (2.5) and that* $\varphi(\cdot) \in \mathcal{H}_0$ *satisfies (3.1). Assume that* $\{X, X_n;\ n \geq 1\}$, $J(\cdot)$, $\mathbb{L}_n(F, J, H)$, Z, μ, *and* $\Psi(\cdot)$ *are as in Theorem 3.1. Let* $0 \leq \lambda < \infty$. *Then*

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\sqrt{n/\varphi(n)}\,(\mathbb{L}_n(F, J, H) - \mu) \overset{+}{=}(-)\ \lambda \quad a.s. \tag{3.5}$$

and

$$C\left(\left\{\sqrt{n/\varphi(n)}\,(\mathbb{L}_n(F, J, H) - \mu);\ n \geq 1\right\}\right) = [-\lambda,\ \lambda] \quad a.s. \tag{3.6}$$

if and only if condition (3.2) holds.

REMARK 3.1. Due to condition (3.1), Theorems 3.1 and 3.2 do not include as a special case the classical Hartman-Wintner-Strassen LIL for L-statistics obtained by Li, Rao, and Tomkins (2001, Theorem 2.1). It is interesting to note that, under the condition (3.1), the limiting behavior in Theorems 3.1 and 3.2 is determined by the distribution of

Z, whereas, by contrast, the three conclusions (2.11), (2.12), and (2.13) depend on the distribution of Y, where Y is defined in (2.6).

REMARK 3.2. In general, it is not easy to find $F^{\leftarrow}(t), 0 < t < 1$. However, if the distribution function $F(\cdot)$ of the random variable X is continuous, then

$$Z = J(U)H(F^{\leftarrow}(U)) \stackrel{d}{=} J(F(X))H(X)$$

where "$\stackrel{d}{=}$" denotes "equal in distribution".

REMARK 3.3. We conjecture that Theorems 3.1 and 3.2 are still true without condition (2.5).

REMARK 3.4. We note that if $\mathbb{E}Z^2 < \infty$, then the constant λ in Theorems 3.1 and 3.2 is simply

$$\lambda = \sqrt{2(\mathbb{E}Z^2) \limsup_{x\to\infty} \frac{\Psi^{-1}(xLLx)}{x^2 LLx}}.$$

We shall illustrate Theorem 3.2 by considering the following two special cases:

Case I. Choose $\varphi(x) = 2(LLx)^p$ where $p > 1$. Then one can check that

$$\lim_{x\to\infty} \frac{\Psi^{-1}(xLLx)/(x^2LLx)}{1/(2(LLx)^{p-1})} = 1.$$

Case II. Take $\varphi(x) = 2(Lx)^r$ where $r > 0$. Then one also easily sees that

$$\lim_{x\to\infty} \frac{\Psi^{-1}(xLLx)/(x^2LLx)}{LLx/(Lx)^r} = 1.$$

Thus, Theorem 3.2 yields the following two results.

COROLLARY 3.1. *Assume that $H(\cdot)$ is a real-valued Borel-measurable function defined on $\mathcal{R}$ satisfying (2.5). Let $p > 1$. For any constant $0 \le \lambda < \infty$, we have:*

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\sqrt{\frac{n}{2(LLn)^p}}\left(\mathbb{L}_n(F,J,H)-\mu\right) \overset{+}{=}(-)\,\lambda \quad a.s.$$

and

$$C\left(\left\{\sqrt{\frac{n}{2(LLn)^p}}\left(\mathbb{L}_n(F,J,H)-\mu\right);\ n\ge 1\right\}\right)=[-\lambda,\ \lambda] \quad a.s.$$

if and only if

$$\mathbb{E}\left(\frac{Z^2}{(LL|Z|)^p}\right)<\infty \quad and \quad \lambda=\sqrt{\limsup_{x\to\infty}(LLx)^{1-p}\mathbb{E}\left(Z^2 I\{|Z|\le x\}\right)}.$$

COROLLARY 3.2. *Assume that $H(\cdot)$ is a real-valued Borel-measurable function defined on $\mathcal{R}$ satisfying (2.5). Let $r > 0$. For any constant $0 \le \lambda < \infty$, we have:*

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\sqrt{\frac{n}{2(Ln)^r}}\left(\mathbb{L}_n(F,J,H)-\mu\right) \overset{+}{=}(-)\,\lambda \quad a.s.$$

and

$$C\left(\left\{\sqrt{\frac{n}{2(Ln)^r}}\left(\mathbb{L}_n(F,J,H)-\mu\right);\ n\ge 1\right\}\right)=[-\lambda,\ \lambda] \quad a.s.$$

if and only if

$$\mathbb{E}\left(\frac{Z^2}{(L|Z|)^r}\right)<\infty \quad and \quad \lambda=\sqrt{2^{-r}\limsup_{x\to\infty}\frac{LLx}{(Lx)^r}\mathbb{E}\left(Z^2 I\{|Z|\le x\}\right)}.$$

If condition (3.2) is satisfied with $\lambda = 0$, we obtain the following stability result for L-statistics.

COROLLARY 3.3. *Assume that $H(\cdot)$ is a real-valued Borel-measurable function defined on $\mathcal{R}$ satisfying (2.5). Let $\varphi(\cdot) \in \mathcal{H}$ and let $\Psi(\cdot)$ be as in Theorem 3.1. If*

$$\mathbb{E}\Psi^{-1}(|Z|) < \infty \quad \textit{and} \quad \lim_{x\to\infty} \frac{\Psi^{-1}(xLLx)}{x^2 LLx}\mathbb{E}\left(Z^2 I\{|Z| \le x\}\right) = 0, \quad (3.7)$$

then

$$\lim_{n\to\infty} \sqrt{n/\varphi(n)}\,(\mathbb{L}_n(F,J,H) - \mu) = 0 \quad a.s. \quad (3.8)$$

Moreover, if $\varphi(\cdot) \in \mathcal{H}_q$ for some $0 \le q < 1$, then condition (3.7) is necessary and sufficient for (3.8) to hold.

PROOF OF THEOREMS 3.1 AND 3.2. Let $\{U,\ U_n;\ n \ge 1\}$ represent a sequence of i.i.d. random variables with the uniform $(0,1)$ distribution. Then it is well known that

$$\{X,\ X_n;\ n \ge 1\} \stackrel{d}{=} \{F^{\leftarrow}(U),\ F^{\leftarrow}(U_n);\ n \ge 1\}.$$

It now follows that

$$\{X_{i:n};\ 1 \le i \le n, n \ge 1\} \stackrel{d}{=} \{F^{\leftarrow}(U_{i:n});\ 1 \le i \le n, n \ge 1\},$$

where $U_{i:n}$, $1 \le i \le n$, are the order statistics of U_i, $1 \le i \le n$. Thus, one may set without loss of generality $X_n = F^{\leftarrow}(U_n)$ and $X_{i:n} =$

$F^{\leftarrow}(U_{i:n})$ for $1 \le i \le n$ and $n \ge 1$. Note that

$$P\left(U_i \neq U_j \text{ for all } 1 \le i < j < \infty\right) = 1.$$

So we have that

$$\begin{aligned}
&\sum_{i=1}^{n} J(\frac{i}{n}) H\left(X_{i:n}\right) \\
&\quad = \sum_{i=1}^{n} J(\frac{i}{n}) H\left(F^{\leftarrow}(U_{i:n})\right) \\
&\quad = \sum_{i=1}^{n} J(U_{i:n}) H\left(F^{\leftarrow}(U_{i:n})\right) \\
&\qquad + \sum_{i=1}^{n} \left(J(\frac{i}{n}) - J(U_{i:n})\right) H\left(F^{\leftarrow}(U_{i:n})\right) \qquad (3.9) \\
&\quad \overset{\text{a.s.}}{=} \sum_{i=1}^{n} J(U_i) H\left(F^{\leftarrow}(U_i)\right) \\
&\qquad + \sum_{i=1}^{n} \left(J(\mathbb{D}_n(U_{i:n})) - J(U_{i:n})\right) H\left(F^{\leftarrow}(U_{i:n})\right) \\
&\quad = \sum_{i=1}^{n} J(U_i) H\left(F^{\leftarrow}(U_i)\right) + R_n \quad \text{(say)}, \ n \ge 1
\end{aligned}$$

where $\mathbb{D}_n(t) \equiv n^{-1} \sum_{i=1}^{n} I\{U_i \le t\}$ is the empirical distribution function of $U_1, U_2, ..., U_n$. Since $J(\cdot)$ is a Lipschitz function of order one defined on $[0,\ 1]$, there exists a constant $0 \le C < \infty$, depending on $J(\cdot)$ only, such that $|J(t_1) - J(t_2)| \le C|t_1 - t_2|$ uniformly for $t_1,\ t_2 \in [0,\ 1]$. Hence

$$\begin{aligned} |R_n| &\le C \max_{1\le i\le n} |\mathbb{D}_n(U_{i:n}) - U_{i:n}| \sum_{i=1}^{n} |H(F^{\leftarrow}(U_{i:n}))| \\ &\le C \sup_{0\le t\le 1} |\mathbb{D}_n(t) - t| \sum_{i=1}^{n} |H(F^{\leftarrow}(U_i))|, \quad n \ge 1. \end{aligned}$$

Since (2.5) holds, i.e., $\mathbb{E}|H(F^{\leftarrow}(U))| = \mathbb{E}|H(X)| < \infty$, by the Kolmogorov SLLN, we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} |H(F^{\leftarrow}(U_i))|}{n} = \mathbb{E}|H(X)| \ \text{ a.s.}$$

Note that the Chung-Smirnov LIL for empirical processes (see Chung (1949) and Smirnov (1944)) states that

$$\limsup_{n\to\infty} \sqrt{\frac{n}{2LLn}} \sup_{0\le t\le 1} |\mathbb{D}_n(t) - t| = \frac{1}{2} \ \text{ a.s.}$$

Thus, for any given $\varphi(\cdot) \in \mathcal{H}$ satisfying (3.1), since

$$\frac{|R_n|}{\sqrt{n\varphi(n)}} \le C\sqrt{\frac{n}{2LLn}} \sup_{0\le t\le 1} |\mathbb{D}_n(t) - t| \times \sqrt{\frac{2LLn}{\varphi(n)}} \frac{\sum_{i=1}^{n} |H(F^{\leftarrow}(U_i))|}{n}, \quad n \ge 1,$$

we have

$$\lim_{n\to\infty} \frac{R_n}{\sqrt{n\varphi(n)}} = 0 \ \text{ a.s.} \tag{3.10}$$

It then follows from (3.9) and (3.10) that, for any given $\varphi(\cdot) \in \mathcal{H}$ satisfying (3.1),

$$\lim_{n\to\infty} \sqrt{n/\varphi(n)} \left(\mathbb{L}_n(F, J, H) - n^{-1} \sum_{i=1}^{n} J(U_i) H\left(F^{\leftarrow}(U_i)\right) \right) = 0 \ \text{ a.s.} \tag{3.11}$$

It is easy to see that $\varphi(\cdot) \in \mathcal{H}_q$ and (3.1) imply that $\frac{\Psi^{-1}(xLLx)}{x^2LLx}$ is slowly varying at infinity with

$$\lim_{x\to\infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} = 0.$$

Thus (3.2) with $\lambda < \infty$ is equivalent to

$$\mathbb{E}\Psi^{-1}(|V|) < \infty \quad \text{and} \quad \limsup_{x\to\infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx}\mathbb{E}\left(V^2 I\{|Z| \le x\}\right) = \frac{\lambda^2}{2} \tag{3.12}$$

where $V = J(U)H(F^{\leftarrow}(U)) - \mu = Z - \mu$. Then, by Theorem 1 of Einmahl and Li (2005), (3.12) implies that

$$(1-q)^{1/2}\lambda \le \limsup_{n\to\infty} \sqrt{n/\varphi(n)}\left|\frac{\sum_{i=1}^n J(U_i)H(F^{\leftarrow}(U_i))}{n} - \mu\right| \le \lambda \text{ a.s.} \tag{3.13}$$

and (3.3) follows from (3.11) and (3.13).

Conversely, by Theorem 1 of Einmahl and Li (2005), if $0 \le q < 1$, then the relation

$$\limsup_{n\to\infty} \sqrt{n/\varphi(n)}\left|\frac{\sum_{i=1}^n J(U_i)H(F^{\leftarrow}(U_i))}{n} - \mu\right| < \infty \text{ a.s.} \tag{3.14}$$

implies that (3.12) holds with $\lambda < \infty$ and, moreover, the lim sup in (3.14) is positive if and only if (3.12) holds with $0 < \lambda < \infty$. Thus combining (3.4) and (3.11) yields (3.14) and hence (3.12) holds with $\lambda < \infty$. As was noted above, (3.2) with $\lambda < \infty$ is equivalent to (3.12), and the last assertion in Theorem 3.1 is now immediate.

Similarly, if $\varphi(\cdot) \in \mathcal{H}_0$ and (3.1) holds then by combining Theorem 2 of Einmahl and Li (2005) and (3.11), the proof of Theorem 3.2 follows. □

Chapter 4

An Interesting Example

In this chapter, we shall provide an example to illustrate our results.

EXAMPLE. Take $J(t) = 4t - 2,\ 0 \le t \le 1$, and $H(x) = x,\ x \in \mathcal{R}$. Then the L-statistic

$$\mathbb{L}_n(F, J, H) = \frac{1}{n} \sum_{i=1}^{n} \left(4 \cdot \frac{i}{n} - 2\right) X_{i:n}$$

is related to Gini's mean difference,

$$\frac{2}{n(n-1)} \sum_{1 \le i < j \le n} |X_i - X_j| = \frac{1}{n} \sum_{i=1}^{n} \left(4 \cdot \frac{i-1}{n-1} - 2\right) X_{i:n},$$

which is a well-known U-statistic for unbiased estimation of the dispersion parameter

$$\theta = E(|X_1 - X_2|);$$

see, e.g., Serfling (1980, p. 263) or Shorack and Wellner (1986, p. 676). Li, Rao, and Tomkins (2001, Theorem 3.3) established analogues of the classical SLLN, LIL, and central limit theorem for Gini's mean difference. Given a function $\varphi(\cdot) \in \mathcal{H}_q$ satisfying (3.1) where $0 \le q \le 1$, let $\Psi(\cdot)$ be as in Theorem 3.1. Then it is easy to check that, for any constant $0 \le \lambda < \infty$, (3.2) holding with $Z = (4U-2)F^{\leftarrow}(U)$ is equivalent to

$$\mathbb{E}\Psi^{-1}(|X|) < \infty \quad \text{and} \quad \limsup_{x\to\infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx}\mathbb{E}\left(Z^2 I\{|Z| \le x\}\right) = \frac{\lambda^2}{2}. \tag{4.1}$$

Note that $\mu = \mathbb{E}Z = \mathbb{E}|X_1 - X_2| = \theta$ and that

$$\limsup_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\le i<j\le n} |X_i - X_j| < \infty \qquad \text{a.s.}$$

if and only if

$$\mathbb{E}|X| < \infty;$$

see Li, Rao, and Tomkins (2001, Theorem 3.3(i)). So, under (4.1), we have

$$(1-q)^{1/2}\lambda \le \limsup_{n\to\infty} \sqrt{\frac{n}{\varphi(n)}} \left| \frac{2}{n(n-1)} \sum_{1\le i<j\le n} |X_i - X_j| - \theta \right| \le \lambda \ \text{ a.s..}$$

Conversely, if $q < 1$, then the relation

$$\limsup_{n\to\infty} \sqrt{\frac{n}{\varphi(n)}} \left| \frac{2}{n(n-1)} \sum_{1\le i<j\le n} |X_i - X_j| - \theta \right| < \infty \ \text{ a.s.} \tag{4.2}$$

implies that (4.1) holds with $\lambda < \infty$. Moreover, the $\limsup$ in (4.2) is positive if and only if (4.1) holds with $0 < \lambda < \infty$. If $q = 0$, then for any constant $0 \le \lambda < \infty$,

$$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \sqrt{\frac{n}{\varphi(n)}} \left(\frac{2}{n(n-1)} \sum_{1\le i<j\le n} |X_i - X_j| - \theta \right) \overset{+}{=} (-)\, \lambda \quad \text{a.s.}$$

and

$$C\left(\left\{ \sqrt{\frac{n}{\varphi(n)}} \left(\frac{2}{n(n-1)} \sum_{1\le i<j\le n} |X_i - X_j| - \theta \right);\ n \ge 1 \right\}\right) = [-\lambda,\ \lambda] \text{ a.s.}$$

if and only if condition (4.1) holds.

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