Neighborhood System In Concept Lattice

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Table of Contents

Table of Contents I

ACKNOWLEDGMENT III

ABSTRACT IV

Chapter 1 INTRODUCTION 1

Chapter 2 LATTICES AND CONCEPT LATTICES AS KNOWLEDGE REPRESENTATION 3

2.1 Introduction 3

2.2 Lattices 3

2.3 Concept lattices and properties 7

2.4 Examples 9

Chapter 3 VARIOUS NEIGHBORHOOD-BASES IN CONCEPTUAL KNOWLEDGE SYSTEMS 13

3.1 Introduction and basic concepts 13

3.2 Three different neighborhood systems in a concept lattice . . 14

3.2.1 Super-sub-relation neighborhood system NS_1 14

3.2.2 Close-relation neighborhood system NS_2 19

3.2.3 Far-relation neighborhood system NS_3 23

3.2.4 Comparison of NS_1, NS_2 & NS_3 27

Chapter 4 NEIGHBORHOOD SYSTEMS IN CONCEPT LATTICES OF MULTI-VALUED CONTEXTS 29

4.1 Introduction 29

4.2 Multi-valued contexts 30

4.3 Neighborhood systems 36

Chapter 5 CONCLUSION 46

REFERENCE 48

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ABSTRACT

In this thesis, we apply and generalize the notion of neighborhood system from topology to study the relation between the concepts in a concept lattice. We classify all concepts in the concept lattice into various classes by seeking similar characters or properties of their attributes. Any element in the concept lattice is associated with a family of subsets of the concept lattice. This family is called a neighborhood system of the element. Each subset in the neighborhood system is called a neighborhood of the element. A concept in some neighborhood of the fixed element in the concept lattice is interpreted to be somewhat near or adjacent to the element. Two concepts in a same neighborhood are considered to be somewhat indiscernible or at least not noticeably distinguishable. We introduce three different neighborhood systems NS_1, NS_2 and NS_3. For the first type NS_1, a concept is said to be in a neighborhood of another concept in the concept lattice if it is a subconcept or a superconcept of the other. For the second type NS_2, a concept is said to be in a neighborhood of another concept if the two concepts have some common attributes. For the third type NS_3, a concept is said to be in the neighborhood of another concept if every object in the concept shares some attribute with some object in the other concept. We prove that $NS_1 \subsetneq NS_2 \subsetneq NS_3$. Examples are given and properties of the neighborhood systems are discussed.

Chapter 1

INTRODUCTION

Concept Lattice is an area of research which is based on a set-theoretic model for concepts and conceptual hierarchies (see [15]). It plays a central role in formal concept analysis. Although there are different understandings of a concept, the philosophical understanding of a concept is as a unit of thoughts consisting of two parts: the extension and the intention; the extension covers all objects belong to the concept while the intention comprises all attributes (or properties) valid for all those objects (see [13]). Since the extension and the intention are described by some subcollections, i.e. sets, a set-theoretic model for these is a natural tool for the formal concept analysis. These "conceptual tools" are considered as a general aid in sciences, economy and administration.

In formal concept analysis, sometimes it is necessary to classify all elements in a concept lattice into various classes by seeking similar characters or properties of the attributes. In this way, the universe, a concept lattice, is divided into different classes of subsets. All elements in the same subset are considered to be indiscernible, or similar. In this situation, it is natural to adopt the notion of neighborhood system from topology (see [9] and [10]), which generalized the concept of indiscernibility into that of neighborhood. In this framework, any element of a universe is associated with a nonempty family of subsets. This family of subsets is called a neighborhood system of the element, and each subset in the family is called a neighborhood of the element. The elements in the same neighborhood of an element can be interpreted to be somewhat indiscernible or at least not noticeably distinguishable. Mathematically, the elements in the same neighborhood are considered to be "close to" or "near to" each other. The main purpose of this thesis is to incorporate the idea of neighborhood systems into a concept lattice for formal concept analysis. We introduce three different neighborhood systems in a concept lattice based on the characteristics of the attributes of the concepts.

This approach allows us to describe and to study the relation between concepts in a concept lattice. We interpret data in terms of the neighborhood systems and study the properties and relations of these neighborhood systems. Since the notion of neighborhood systems come from studies of topological spaces, the mathematical aspect of the neighborhood system will also be discussed.

We organize this thesis as follows. Chapter 2 is intended to be a reference for the terms and notations used throughout this thesis. Basic notions of an abstract lattice and a concept lattice will be given, and some basic properties of them and examples are also be included in this chapter. In chapter 3, we introduce three neighborhood bases in concept lattices. Properties, interpretations, relations between them and examples will be given. In chapter 4, we deal with neighborhood bases in a concept lattice of a multi-valued context.

Chapter 2

LATTICES AND CONCEPT LATTICES AS KNOWLEDGE REPRESENTATION

2.1 Introduction

This chapter is intended to be a reference for the terms and the notations used throughout the thesis. We also include some basic properties of lattices and concept lattices, which will be needed in the thesis.

2.2 Lattices

In this section, we include the definitions and properties of posets and lattices. Since these are algebraic concepts, we can find them in any standard abstract algebra book (see [8]). The notion of lattices is a generalization of the order relation $\leq$ in usual number systems and set-theoretic inclusion $\subseteq$ among subsets of a universal set.

Definition 2.2.1 If S is a set, then any subset of $S \times S$ is called a relation on S. A relation T on S is called a partial order provided that the subset T is

- Reflexive: $(a, a) \in T$ for every $a \in S$.
- Antisymmetric: If $(a, b) \in T$ and $(b, a) \in T$, then $a = b$.
- Transitive: If $(a, b) \in T$ and $(b, c) \in T$, then $(a, c) \in T$.

A set equipped with a partial-order relation is called a partially ordered set (or poset).

The symbol $\leq$ is usually used to denote an arbitrary partial order T:

$$a \leq b \quad \text{means} \quad (a, b) \in T.$$

In this notation, the conditions defining a partial order become

- Reflexive: $a \leq a$ for every $a \in S$.
- Antisymmetric: If $a \leq b$ and $b \leq a$, then $a = b$.
- Transitive: If $a \leq b$ and $b \leq c$, then $a \leq c$.

When such a notation is used, a partial order on S is usually defined without explicit reference to a subset of $S \times S$. We shall also adopt the usual notation:

$$b \geq a \quad \text{means} \quad a \leq b.$$

Example 2.2.2 Let S be the set of all subsets of {x, y, z} and define $A \leq B$ to mean A is a subset of B. The relation $\subseteq$ is reflexive, antisymmetric, and transitive. So S is a partially ordered set. The ordering can be schematically displayed by Figure 2.2.1, in which a line connecting two sets means that the lower of the two is a subset of the higher:

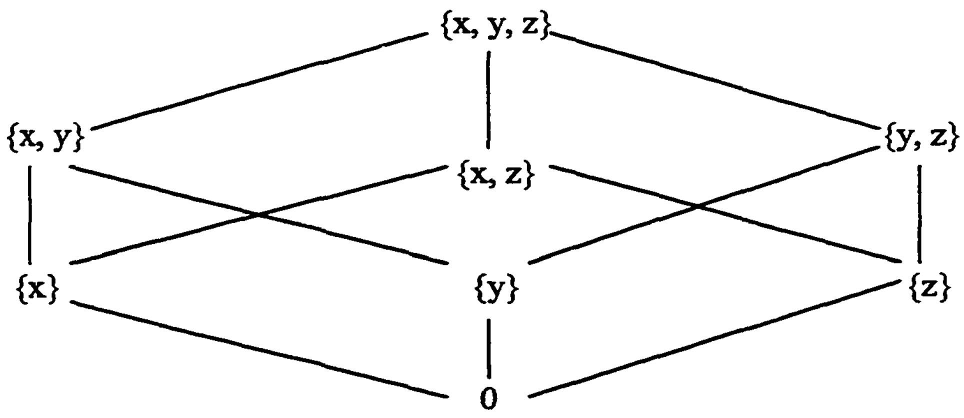

Figure 2.2.1

Example 2.2.3 The set S = {r, s, t, u, v, w, x} is a partially ordered set whose partial order is given by Figure 2.2.2, in which $a \leq b$ means that either $a = b$ or a lies below b and there is a path of line segments from a to b that never moves downward.

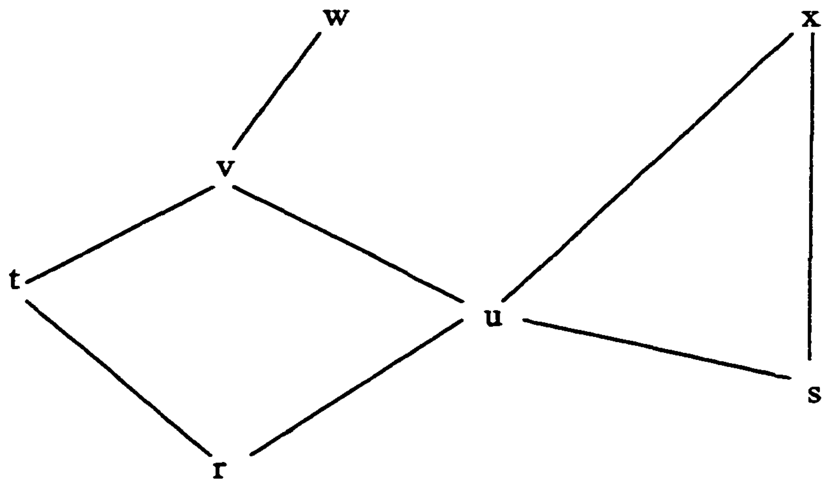

Figure 2.2.2

Thus r ≤ u and r ≤ w, but it is not true that r ≤ s. Similarly, a ≤ w for every a ∈ S except x.

Definition 2.2.4 Let B be a subset of a partially ordered set S. An element u of S is said to be an upper bound of B if b ≤ u for every b ∈ B. The set B may have many upper bounds, some of which are not in B itself, or B may have no upper bounds.

Example 2.2.5 In Example 2.2.3, the only upper bounds of the subset B = {t, u} are v and w. The subset {r, u, s} has four upper bounds (u, v, w, x). In the set Z of integers with the usual ordering, the subset of even integers has no upper bound.

If u is an upper bound of B such that u ≤ v for every other upper bound v of B, then u is the least upper bound (or l.u.b.) of B. Let B be a subset of a partially ordered set A. An element w of A is said to be a lower bound of B if w ≤ b for every b ∈ B. If w is a lower bound of B such that v ≤ w for every other lower bound w of B, then w is the greatest lower bound (or g.l.b.) of B.

Theorem 2.2.6 Let B be a nonempty subset of a partially ordered set S. If B has a least upper bound, then this l.u.b. is unique. If B has a greatest lower bound, then this g.l.b. is unique.

Definition2.2.7 A lattice is a partially ordered set L in which every pair of elements has both a least upper bound and a greatest lower bound. If a, b $\in$ L, then their least upper bound is denoted by $a \vee b$ and called the join of a and b. The greatest lower bound of a and b is denoted by $a \wedge b$ and called meet.

Example 2.2.8 (Rings) If R is a ring, then the set S of all ideals of R, partially ordered by set-theoretic inclusion ($\subseteq$), is a lattice. The g.l.b. of ideals I and J is the ideal $I \cap J$. The union of two ideals may not be an ideal, so $I \cup J$ is not the least upper bound of I and J in this lattice. The l.u.b. of I and J is the ideal I + J.

Example 2.2.9 (Groups) If G is a group, then the set S of all subgroups of G, partially ordered by set-theoretic inclusion, is a lattice. The g.l.b. of subroups H and K is the subgroup $H \cap K$. The set $H \cup K$ may not be subgroup; the l.u.b. of H and K is the subgroup generated by the set $H \cup K$.

Theorem 2.2.10 If L is a lattice, then the binary operations $\vee$ and $\wedge$ satisfy these conditions for all a, b, c $\in$ L:

1. Commutative Laws:

   $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

2. Associative Laws:

   $a \vee (b \vee c) = (a \vee b) \vee c$ and

   $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

3. Absorption Laws:

   $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

4. Idempotent Laws:

   $a \vee a = a$ and $a \wedge a = a$

Theorem 2.2.11 Let L be a nonempty set equipped with two binary operations, $\vee$ and $\wedge$, that obey the commutative, associative, absorption, and idempotent laws. Define a relation $\leq$ on L by: $a \leq b$ if and only if $a \vee b = b$. Then L is a lattice with respect to $\leq$ such that for all $a, b \in L$:

$$\text{l.u.b. } \{a, b\} = a \vee b \quad \text{and} \quad \text{g.l.b. } \{a, b\} = a \wedge b.$$

2.3 Concept lattices and properties

Formal concept analysis has been developed during the last twenty years by many researchers (see [3], [4], [6], [13] and [15]). It is based on the understanding of a concept as a unit of thoughts consisting of two parts, the extension and intension. The extension covers all objects belonging to the concept while the intension comprises all attributes (or properties) valid for all those objects. Naturally, a set theory can be used. This approach to data analysis is a method for formal representation of conceptual knowledge. Formal concept analysis starts with the notion of a context defined as follows.

Definition 2.3.1 A (formal) context which is defined as a triple (G, M, I) where G and M are sets while I is a binary relation between G and M, i.e., $I \subseteq G \times M$; the elements of G and M are called objects and attributes, respectively, and gIm i.e., $(g, m) \in I$, is read: the object g has the attribute m. Frequently used are the following derivation operators represented by "prime":

$$X \rightarrow X' = \{ m \in M \mid gIm \text{ for all } g \in X \},$$
$$Y \rightarrow Y' = \{ g \in G \mid gIm \text{ for all } m \in Y \}.$$

These operators form a so-called Galois connection between the power sets of G and M which can be expressed by the following conditions indicating a natural "duality" between objects and attributes (see [8], pp 122-125):

$$X_1 \subseteq X_2 \text{ implies } X_2' \subseteq X_1' \text{ for } X_1, X_2 \subseteq G;$$
$$Y_1 \subseteq Y_2 \text{ implies } Y_2' \subseteq Y_1' \text{ for } Y_1, Y_2 \subseteq M;$$

$$X \subseteq X'' \text{ and } X' = X''' \text{ for } X \subseteq G;$$

$$Y \subseteq Y'' \text{ and } Y' = Y''' \text{ for } Y \subseteq M;$$

$$(\bigcup_{t\in T} X_t)' = \bigcap_{t\in T} X_t' \text{ for } X_t \subseteq G\ (t \in T)$$

$$(\bigcup_{t\in T} Y_t)' = \bigcap_{t\in T} Y_t' \text{ for } Y_t \subseteq M\ (t \in T)$$

In the frame of a formal context (G, M, I), the philosophical view of a concept as a unit of thoughts constituted by its extension and its intension can be formalized by the following definition.

Definition 2.3.2 A pair (A , B) is said to be a formal concept of the context (G, M, I) if $A \subseteq G$, $B \subseteq M$, $A = B'$ and $B = A'$; A and B are called the extent and the intent of the concept (A , B). The set of all concepts of (G, M, I) is denoted by B(G, M, I).

The most important structure on B(G, M, I) is given by the subconcept-superconcept-relation which is defined as follows.

Definition 2.3.3 The concept (A_1, B_1) is a subconcept of the concept (A_2, B_2) if $A_1 \subseteq A_2$ which is equivalent to $B_2 \subseteq B_1$, (A_2, B_2) is then a superconcept of (A_1, B_1).

A subset of D of a complete lattices L is called infimum-dense (supremum-dense) if each element of L is the infimum (supremum) of some subset of D. An element a of a lattice L is said to be $\wedge$-irreducible ($\vee$-irreducible) if $a = b \wedge c$ ($a = b \vee c$) always implies $a = b$ or $a = c$; the set of all $\wedge$-irreducible ($\vee$-irreducible) elements of L is denoted by J(L) (M(L)).

Theorem 2.3.4 (see [16]) Let (G, M, I) be a context. Then B(G, M, I) is a complete lattice, called the concept lattice of (G, M, I), for which infimum and supremum can be described as follows:

$$\vee(A_t, B_t) = ((\bigcup_{t\in T} A_t)'', \bigcap_{t\in T} B_t)$$

$$\wedge(A_t, B_t) = (\bigcap_{t\in T} A_t, (\bigcup_{t\in T} B_t)'')$$

Definition 2.3.5 This lattice B(G, M, I) is called a concept lattice of the formal context (G, M, I).

In general, a complete lattice L is isomorphic to B(G, M, I) if and only if there exist mappings γ: G → L and μ: M → L such that γG is supremum-dense in L, μM is infimum-dense in L, and gIm is equivalent to $\gamma G \leq \mu M$; in particular, $L \cong B(L, L, \leq)$ and, if L has finite length, $L \cong B(J(L), M(L), \leq)$. To illustrate the definitions, we include the examples in the following section.

2.4 Examples

Contexts are usually described by cross-tables while concept lattices are effectively visualized by labeled line diagrams.

Example 2.4.1 Let G be the set of all students at Lakehead University and let M be all the courses offered in 1999 ~ 2000 school year. For $g \in G$ and $m \in M$, we define gIm if the student g takes the course m in 1999 ~ 2000 school year. Then (G, M, I) is a formal context.

Let $A \subseteq G$ and $B \subseteq M$. Then (A, B) is a concept if A is the set of students who take all courses in B and B is the set of all courses taken by all students in A, i.e. $A = B'$ and $B = A'$. Let $(A_1, B_1), (A_2, B_2) \in B(G, M, I)$. $(A_1, B_1) \leq (A_2, B_2)$ if $A_1 \subseteq A_2$.

Example 2.4.2 Table 2.4.1 can be understood as a description of a formal context: its objects are the eleven persons whose name are heading the rows and its attributes are the twelve cities which are represented by the columns; the crosses indicate when an object has an attribute, i.e., which person has been in that city.

	C1	[illegible]	C3	[illegible]	C5	[illegible]	C7	[illegible]	C9	[illegible]	C11	[illegible]
[illegible]	X	[illegible]	X	[illegible]	X	[illegible]		[illegible]		[illegible]		[illegible]
[illegible]	X	[illegible]	X	[illegible]		[illegible]		[illegible]		[illegible]		[illegible]
[illegible]		[illegible]	X	[illegible]	X	[illegible]		[illegible]		[illegible]		[illegible]
[illegible]	X	[illegible]		[illegible]		[illegible]		[illegible]		[illegible]		[illegible]
[illegible]		[illegible]		[illegible]	X	[illegible]		[illegible]		[illegible]		[illegible]
[illegible]		[illegible]		[illegible]		[illegible]		[illegible]	X	[illegible]	X	[illegible]
[illegible]		[illegible]		[illegible]		[illegible]		[illegible]	X	[illegible]	X	[illegible]
[illegible]		[illegible]		[illegible]		[illegible]		[illegible]		[illegible]		[illegible]
[illegible]		[illegible]		[illegible]		[illegible]	X	[illegible]		[illegible]		[illegible]
[illegible]		[illegible]		[illegible]		[illegible]	X	[illegible]		[illegible]		[illegible]
[illegible]		[illegible]		[illegible]		[illegible]	X	[illegible]		[illegible]		[illegible]

Table 2.4.1. Cross-tables

The concept lattice of a given context (G , M , I) is determined as follows:

First, by the formulas $X' = \bigcap_{t\in T} \{g\}'$ or $Y' = \bigcap_{m\in Y} \{m\}'$, then we form (X″, X′) or (Y′, Y″).

Thus, one can start with the special intents $\{g\}'$ ($g\in G$) or the special extents $\{m\}'$($m\in M$) to form all the concepts, since each intent is the intersection of some special extents $\{g\}'$and each extent is the intersection of some special extents $\{m\}'$.

There are 19 concepts of the context:

1. ({a} , {C_1, C_2, C_3, C_5})
2. ({b} , {C1, C3, C4})
3. ({a, c} , {C_2, C_3, C_5})
4. ({b, d} , {C_1, C_4})
5. ({e} , {C_2, C_5, C_6})
6. ({f} , {C_8, C_9, C_{11}})

7. ({g} , {C_9, C_{10}, C_{11}, C_{12}})
8. ({g, h} , {C_{10}, C_{12}})
9. ({i} , {C_6, C_7, C_8})
10. ({i, j} , {C_7, C_8})
11. ({i, k} , {C_6, C_7})
12. ({a, b} , {C_1, C_3})
13. ({a, b, d} , {C_1})
14. ({a, c, e} , {C_2, C_5})
15. ({a, b, c} , {C_3})
16. ({e, i, k} , {C_6})
17. ({i, j, k} , {C_7})
18. ({f, i, j} , {C_8})
19. ({f, g} , {C_9, C_{11}})

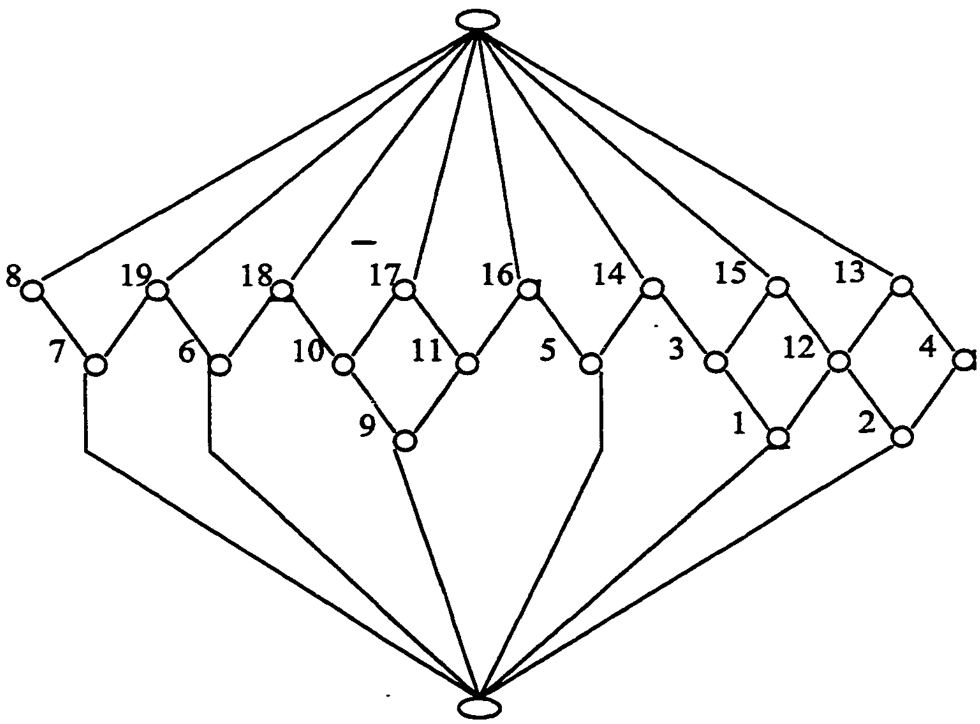

Figure 2.4.2. Concept lattice of the formal context in Figure 2.4.1.

The labeled line diagram is shown in Figure 2.4.2, the little circles represent the 19 concepts of the context (the number there is concepts, not the code of cities) and the ascending paths of the line segments represent the subconcept-superconcept-relation.

A concept lattice can be viewed as a hierarchical conceptual clustering of the objects (via its extents). The concept lattice in Figure 2.4.1, for instance, shows that the conceptual hierarchy classifies the people in mainly three groups with the similar cities they have been. A concept lattice can be understood as a representation of all implications between the attributes (via its intents). An implication of a context (G, M, I) is a pair of subsets of M, denoted by $Y \rightarrow Z$, for which $Y' \subseteq Z'$,i.e., each object from G having all attributes of Y has also all attributes of Z.

Formal contexts and their concept lattices are substantial tools for formal representation of conceptual knowledge. These tools activate the rich source of mathematical developments in order and lattice theory for knowledge representation. In particular, the representation by labeled line diagrams is a powerful instrument if it is combined with the structure theory of concept lattices. Then these diagrams can make transparent the different meanings of concept lattices as, for instance, the hierarchical classification of objects or the logic of attribute implications.

Chapter 3

VARIOUS NEIGHBORHOOD-BASES IN CONCEPTUAL KNOWLEDGE SYSTEMS

3.1 Introduction and basic concepts

In this section, we define three different types of neighborhood systems in a concept lattice: super-sub-relation neighborhood system, close-relation neighborhood system and far-relation neighborhood system. Examples are given to interpret these neighborhood systems. The properties and their relations are discussed extensively.

The notion of neighborhood systems originated from studies of topological space (see [14]) and its generalization called Frechet (V) Space (see [12]). Let X be a topological space and $x \in X$. The neighborhood system U_x of x is defined to be a collection of subsets of X satisfying the following axioms:

- Na) If $u \in U_x$, then $x \in u$.
- Nb) If u, $v \in U_x$, then $u \cap v \in U_x$.
- Nc) If $u \in U_x$, then there is a $v \in U_x$, such that $u \in U_y$ for every $y \in v$.
- Nd) If $u \in U_x$ and $u \subseteq v$, then $v \in U_x$.

In our case, we loose the axioms for neighborhood system substantially. Let X be a nonempty finite set. For any $x \in X$, a neighborhood of x is defined as a subset of X, denoted by n(x). It may or may not contain x itself. A nonempty family of neighborhoods of x, denoted by NS(x), is called a neighborhood system of x. A neighborhood system of X, denoted by NS(X), is the collection of NS(x) for all $x \in X$. NS(X) defines a Frechet

Space, or briefly (V) space, written (X, NS(X)). A neighborhood system can be defined by an operator from X to 2^{2^X}.

Example 3.1.1 Let X = {a, b, c, d, e} be the universe. The following is a neighborhood system of X and NS(X) is the collection:

$$NS(a) = \{\{a\}, \{a, b\}, \{a, d\}\},$$
$$NS(b) = \{\{b\}, \{b, c\}\},$$
$$NS(c) = \{\{c\}, \{c, d\}, \{a, b, c\}\},$$
$$NS(d) = \{\{d\}, \{a, b, d\}, \{c, d\}\}.$$

Example 3.1.2 Let R be the set of real numbers. For $a \in R$, let U_a be the family of all open intervals containing a (an open interval, denoted by (x_1, x_2) is the set of all real numbers x such that $x_1 < x < x_2$). Then U_a is a neighborhood system. Note that this neighborhood system satisfies Na), Nb) and Nc), but not Nd).

3.2 Three different neighborhood systems in a concept lattice

We define three different neighborhood systems by the characteristics of the attributes in the ways that have not only mathematical foundation, but also simulate the relation between events in the real world.

3.2.1 Super-sub-relation neighborhood system NS_1

This neighborhood system is based on the order of the concept lattice. In the real world, it is the measurement of the inclusion of the objects.

Definition 3.2.1.1 Let B(G, M, I) be a concept lattice of a context (G, M, I) and let (A_O, B_O) be a concept in B(G, M, I). We say that a concept (A, B) is in a super-sub-relation neighborhood of (A_O, B_O) if (A, B) is the subconcept of (A_O, B_O) or the superconcept of (A_O, B_O). As a convention, we define (A_O, B_O) to be in every neighborhood of itself. Any subset of B(G, M, I) consisting of (A_0, B_0) and elements in some neighborhood of (A_0, B_0) is called a neighborhood of (A_0, B_0). Let $NS_1(A_0, B_0)$ denote the set of all the neighborhoods of (A_0, B_0). This neighborhood system, denoted by NS_1, is called the super-sub-relation neighborhood system.

Remark: It is always true that $\{(A_0, B_0)\}$ is a neighborhood of (A_0, B_0) by our definition. We do not assume the axiom Na) - Nd) for the neighborhood in topology. It follows from the definition of subconcept and superconcept that the concept (A, B) is in a neighborhood of a concept (A_0, B_0), if and only if $A \subseteq A_0$ or $A_0 \subseteq A$.

Example 3.2.1.2 In example 2.4.2,

$$NS_1(3) = \{\{3\}, \{1, 3\}, \{3, 14\}, \{3, 15\}, \{3, 1, 14\}, \{3, 1, 15\}, \{3, 14, 15\}, \{1, 3, 14, 15\}\}.$$

A concept is in a neighborhood of 3:=({a, c}, $\{C_2, C_3, C_5\}$), if it is a concept immediately above, (i.e. with more objects) or immediately below ({a, c}, $\{C_2, C_3, C_5\}$) (i.e. with less objects).

$$NS_1(5) = \{\{5\}, \{5, 16\}, \{5, 14\}, \{5, 14, 16\}\},$$

$$NS_1(7) = \{\{7\}, \{7, 8\}, \{7, 19\}, \{7, 8, 19\}\},$$

$$NS_1(10) = \{\{10\}, \{10, 9\}, \{10, 17\}, \{10, 18\}, \{10, 9, 17\}, \{10, 9, 18\}, \{10, 17, 18\}, \{10, 9, 17, 18\}\}.$$

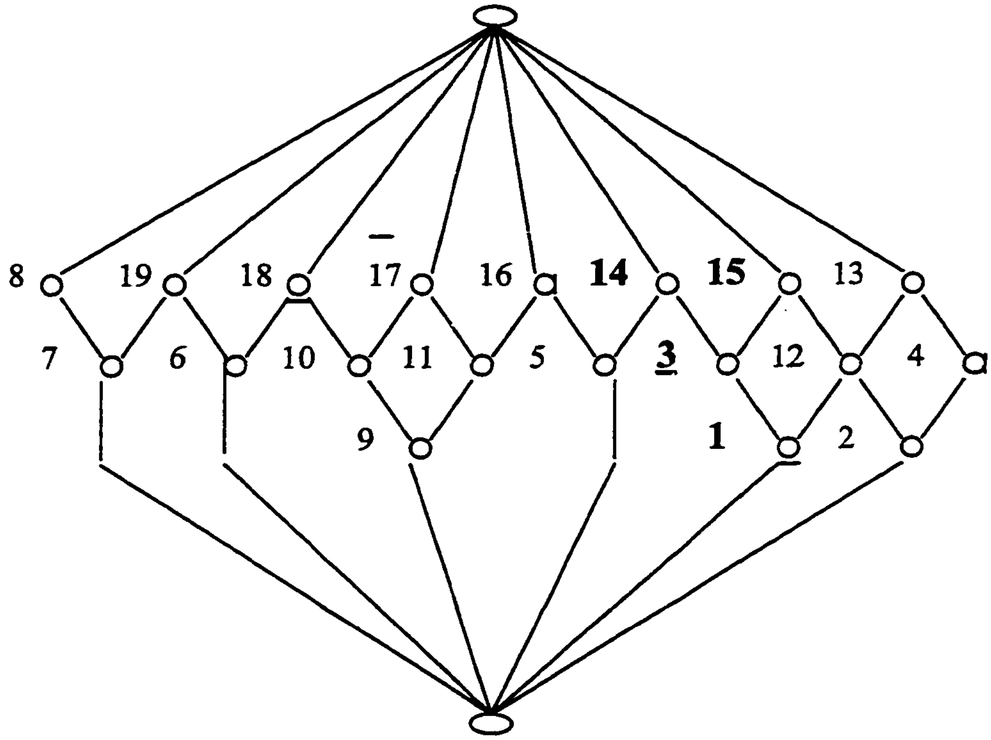

Figure 3.2.1.1 Super-sub relation neighborhood of concept 3.

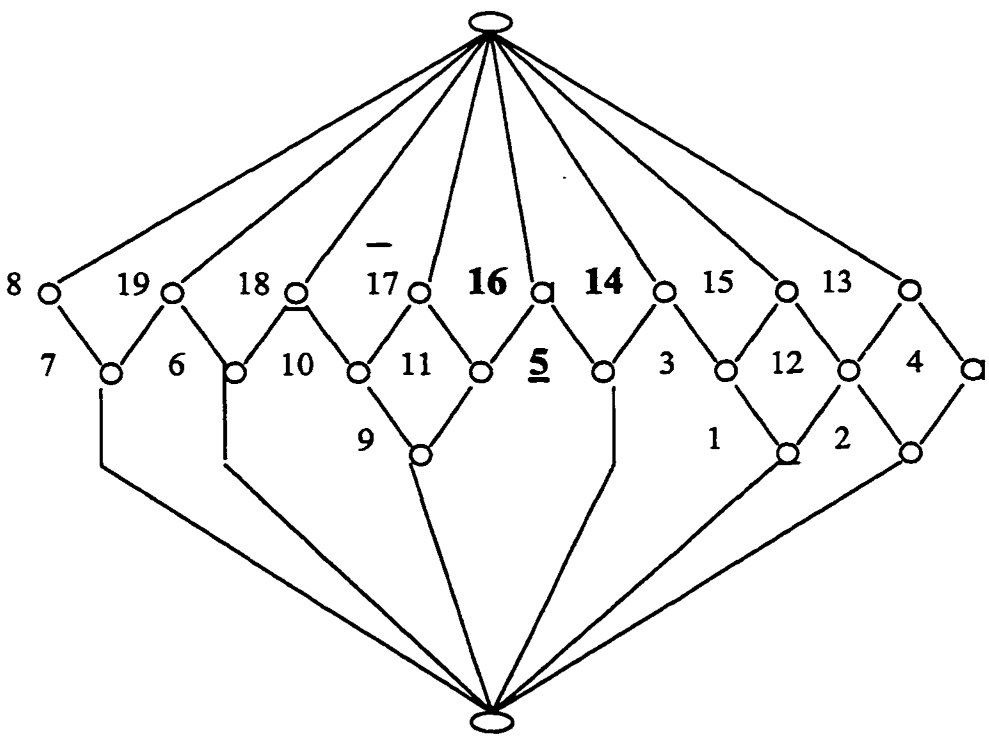

Figure 3.2.1.2 Super-sub relation neighborhood of concept 5.

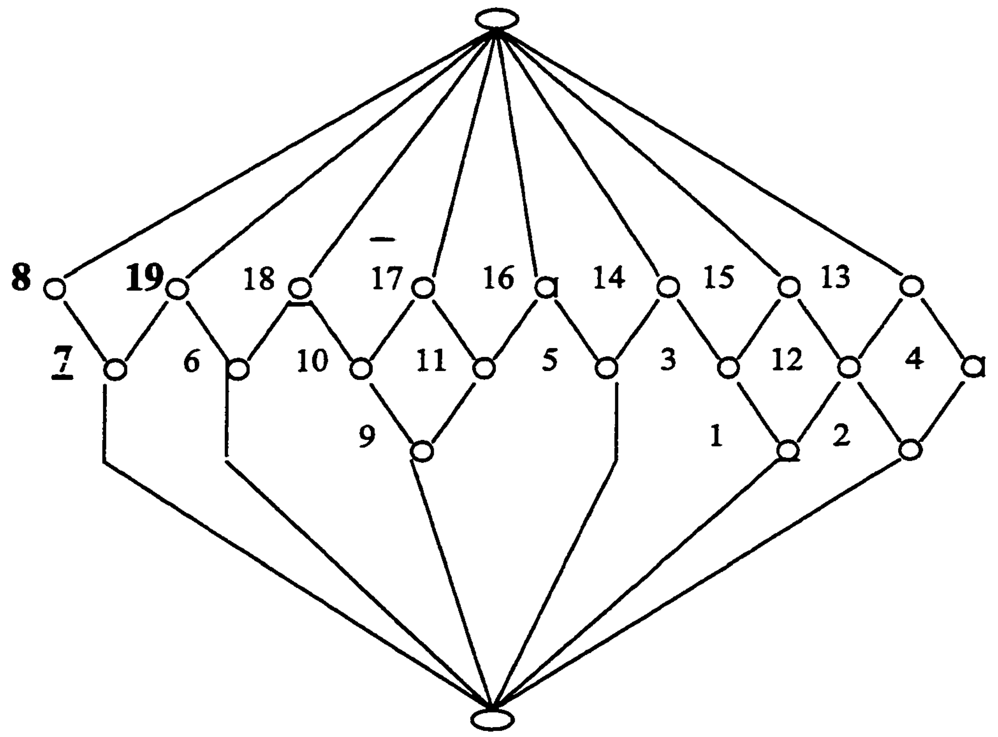

Figure 3.2.1.3 Super-sub relation neighborhood of concept 7.

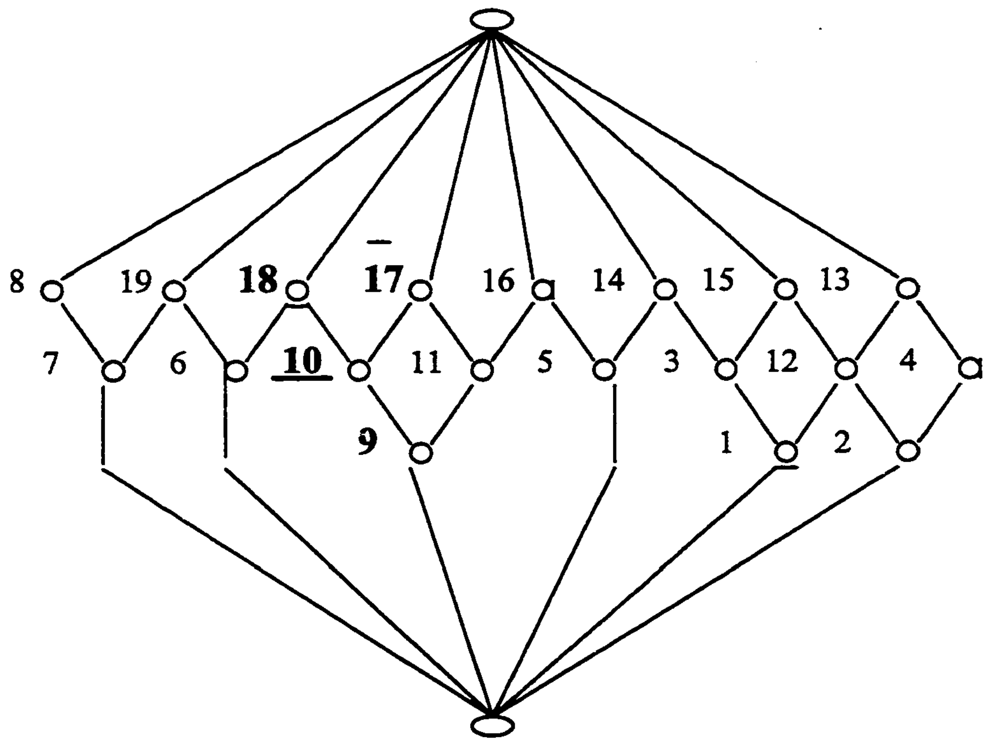

Figure 3.2.1.4 Super-sub relation neighborhood of concept 10.

From the definition, we can see that Na) and Nb) are true.

Proposition 3.2.1.3 Let B(G, M, I) be a concept lattice, and $(A_0, B_0) \in B(G, M, I)$. Then

(I) If $u \in NS_1(A_0, B_0)$, then $(A_0, B_0) \in u$,

(II) If $u, v \in NS_1(A_0, B_0)$, then $u \cup v, u \cap v \in NS_1(A_0, B_0)$.

Remark: Nc) and Nd) may fail.

For example, in example 3.2.1, $v = \{14, 15, 3\} \in NS_1(3)$ and $15 \in v$. But $v \notin NS_1(15)$ since $14 \in v$ but 14 is not in a neighborhood of 15, so Nc) fails. Also, $v = \{14, 15, 3, 16\} \supseteq v$. But $v \notin NS_1(3)$, so Nd) fails.

Although Nc) fails, we has the following:

Proposition 3.2.1.4 Let B(G, M, I) be a concept lattice and (A_1, B_1), (A_2, B_2) two concepts in B(G, M, I). If (A_1, B_1) is in a neighborhood of (A_2, B_2) then (A_2, B_2) is also in a neighborhood of (A_1, B_1).

Proof: If (A_1, B_1) is in a neighborhood of (A_2, B_2), then $(A_1, B_1) \leq (A_2, B_2)$ or $(A_1, B_1) \geq (A_2, B_2)$. So (A_2, B_2) is also in a neighborhood of (A_1, B_1) by definition.

Definition 3.2.1.5 Let (A_1, B_1) and (A_2, B_2) be two concepts in a concept lattice. If (A_1, B_1) is in a neighborhood of (A_2, B_2) or (A_2, B_2) is in a neighborhood of (A_1, B_1), we say that (A_1, B_1) and (A_2, B_2) are in a same neighborhood.

Remark: If two concept (A_1, B_1) and (A_2, B_2) are in a same neighborhood, then (A_1, B_1) and (A_2, B_2) are understood as "close to" or "adjacent to".

Example 3.2.1.6 Let G be the set of all people in a city, and let M be all cities in Canada. For $a \in G$ & $b \in M$, aIb means person a has visited city b in 1999. A concept (A_1, B_1)

$\in$ B(G, M, I) satisfies $A_0 = B_0'$ and $B_0 = A_0'$, i.e. A_0 is all people in the city who has visited all cities in B_0 and B_0 is the set of all cities visited by all people in A_0. (A, B) $\in$ B(G, M, I) is in a neighborhood of (A_0, B_0) means either $A \subseteq A_0$ or $A_0 \subseteq A$, i.e., A is either larger than A_0 or smaller than A_0, equivalently, the set of cities B is either smaller or larger than B_0.

3.2.2 Close-relation neighborhood system NS_2

The super-sub-relation neighborhood system defined in section 3.2.1 is an approach to analyze data mathematically. It is a set-theoretical model for describing the concepts in a concept lattice that are "close" or "near" each other. To be more applicable to the real world, we define a new neighborhood system in a concept lattice. It is not only suitable for the application, but also provide a mathematical model for formal concept analysis. Two concepts are in a same neighborhood if all the members in the two extents share same attributes.

Definition 3.2.2.1 Let B(G, M, I) be a concept lattice and (A_0, B_0) $\in$ B(G, M, I). A concept (A, B) in B(G, M, I) is said to be in a close-relation neighborhood of (A_0, B_0), if there exists an attribute $m_0 \in M$ such that aIm_0 & $a_0I\, m_0$ for every $a \in A$ and every $a_0 \in A_0$. Any subset of concepts in B(G, M, I) containing (A_0, B_0) is called a close-relation neighborhood of (A_0, B_0). The collection of all close-relation neighborhoods of (A_0, B_0) is denoted by $NS_2(A_0, B_0)$. The close-relation neighborhood system is denoted by NS_2.

Proposition 3.2.2.2 Let (A_0, B_0) be a concept in a concept lattice B(G, M, I), Then

(I) For every $v \in NS_2(A_0, B_0)$, $(A_0, B_0) \in v$.

(II) If $u, v \in NS_2(A_0, B_0)$, then $u \cup v$ and $u \cap v$ are in $NS_2(A_0, B_0)$.

Proof: (I) and (II) follow from the definition immediately.

Theorem 3.2.2.3 Let (A_0,B_0) and (A_1,B_1) be two concepts in a concept lattice B(G, M, I). Then the following statements are equivalent:

(I) (A_0, B_0) is in a close-relation neighborhood of (A_1, B_1).

(II) (A_1, B_1) is in a close-relation neighborhood of (A_0, B_0).

(III) $B_0 \cap B_1 \neq \varnothing$.

Proof: (I) ⇒ (II) Let $m_0 \in M$ such that aIm_0 & $a_0I\,m_0$ for all $a\in A_1$ and $a_0\in A_0$. Since (A_0, B_0) is in a close-relation neighborhood of (A_1, B_1), by definition, (A_1, B_1) is also in a close-relation neighborhood of (A_0, B_0).

(II) ⇒ (III) Since (A_1, B_1) is in a close-relation neighborhood of (A_0, B_0), there exists $m_0 \in M$ such that aIm_0 & a_0Im_0 for all $a\in A_1$ and $a_0\in A_0$, so $m_0\in A_0'$ and $m_0\in A_1'$. Since $A_0' = B_0$ and $A_1' = B_1$, $m_0\in B_0 \cap B_1$. Therefore, $B_0 \cap B_1 \neq \varnothing$.

(III) ⇒ (I) Let $m_0\in B_0 \cap B_1$, since $B_0 \cap B_1 \neq \varnothing$, hence aIm_0 for every $a_0\in A_0$, because $A_0 = B_0'$. Similarly, aIm_0 for every $a\in A_1$. Therefore (A_0, B_0) is in a close-relation of (A_1, B_1).

We say that two concepts (A_0, B_0) and (A_1, B_1) are in the same close-relation neighborhood if any one of (I), (II) and (III) is satisfied.

Example 3.2.2.4 Let G be the set of all students at Lakehead University and let M be all the hobbies of students. Two concepts (A_0, B_0) and (A_1, B_1) are in a same close-relation neighborhood if and only if $B_0 \cap B_1 \neq \varnothing$, if and only if the two groups of students A_0 and A_1 share some same hobby.

Remark: As in proposition 3.2.2.2, Na) and Nb) are satisfied, but Nc) and Nd) fail.

Example 3.2.2.5 In example 2.4.2,

$NS_2(3)$ = the family of all subsets of {1, 2, 3, 5, 12, 14, 15} containing {3}.

Interpretation: since the concept 3:= ({a, c}, $\{C_2, C_3, C_5\}$) as in example 2.4.2, a concept is in the neighborhood of ({a, c}, $\{C_2, C_3, C_5\}$) if and only if all people in the concept and a, c has been in the same city.

$NS_2(5)$ = the family of all subsets of {1, 3, 11, 14, 16} containing {5}.

$NS_2(7)$ = the family of all subsets of {6, 8, 19} containing {7}.

$NS_2(10)$ = the family of all subsets of {6, 9, 11, 17, 18} containing {10}.

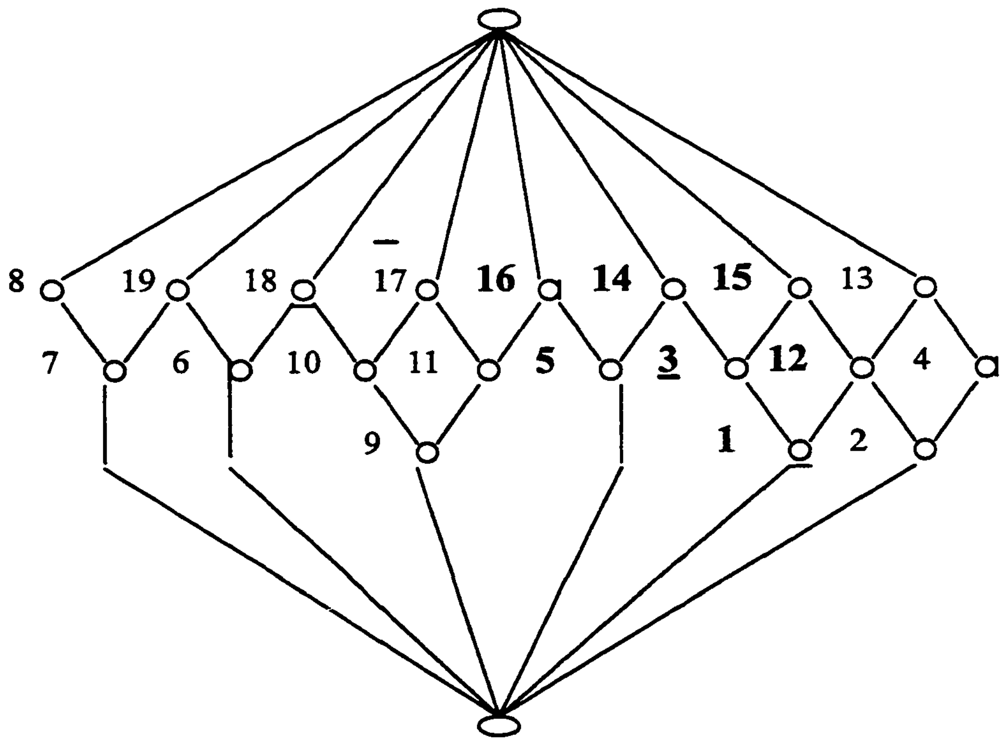

Figure 3.2.2.1 Close-relation neighborhood of concept 3.

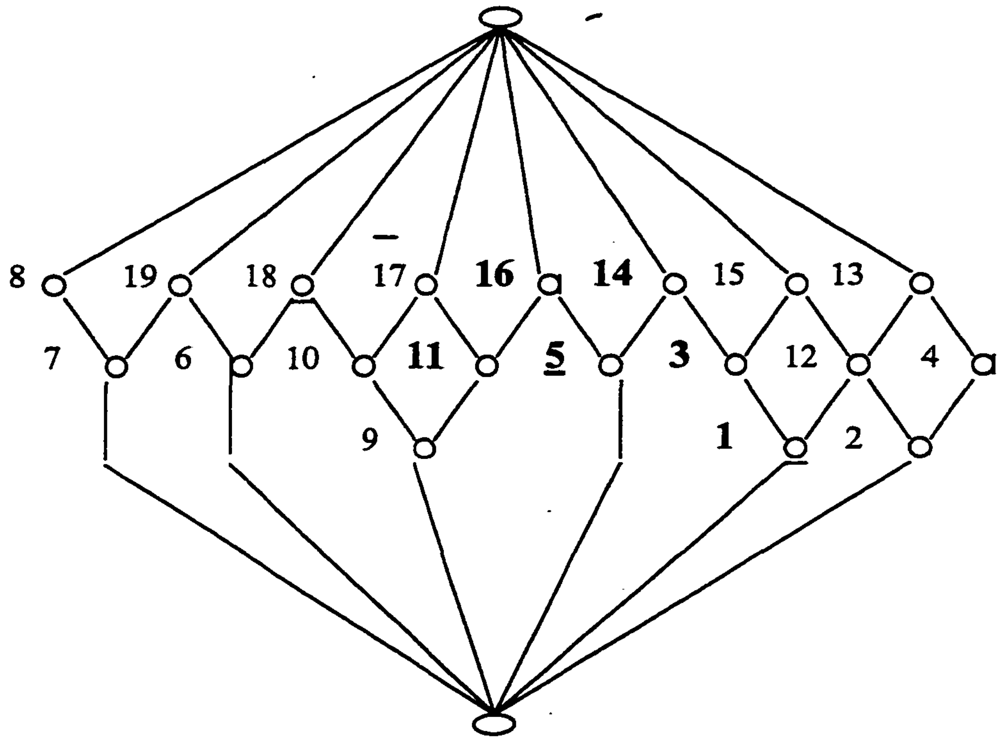

Figure 3.2.2.2 Close-relation neighborhood of concept 5.

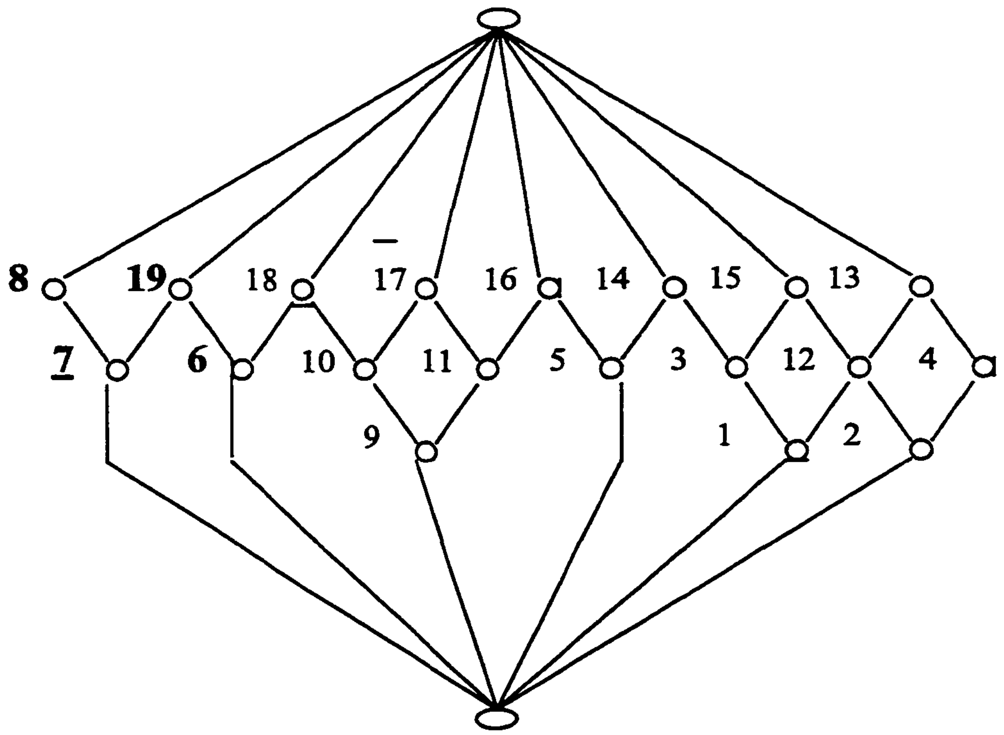

Figure 3.2.2.3 Close-relation neighborhood of concept 7.

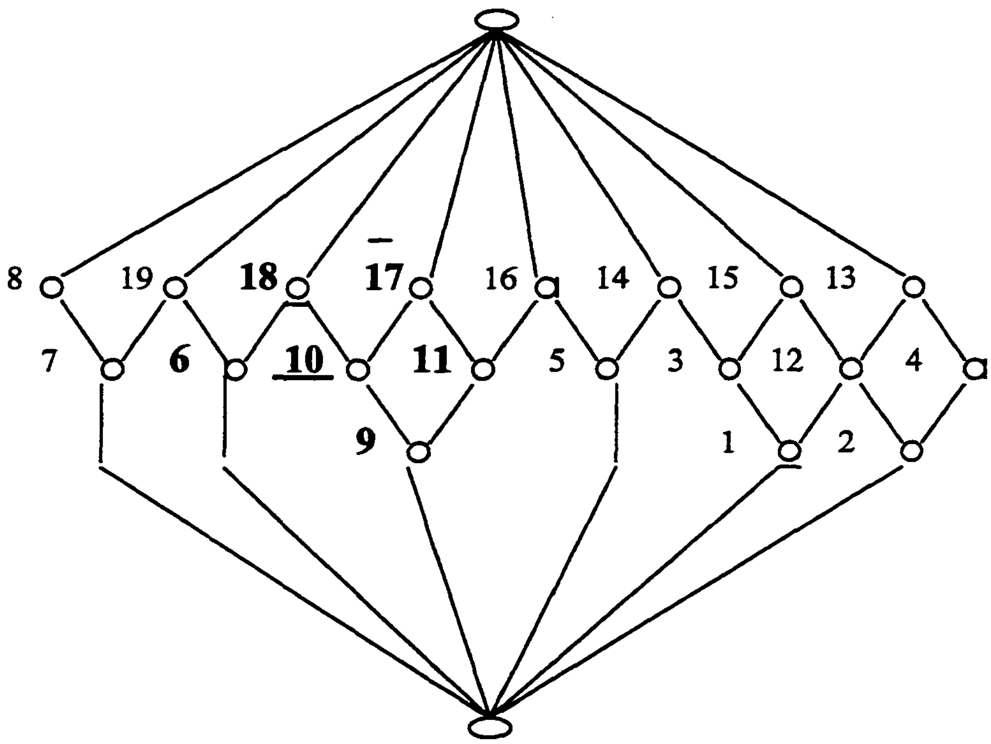

Figure 3.2.2.4 Close-relation neighborhood of concept 10.

3.2.3 Far-relation neighborhood system NS_3

We have introduced two types of neighborhood systems. We define another neighborhood system to simulate the relations among the concepts in a concept lattice. Roughly speaking, a concept (A, B) is in a neighborhood of a concept (A_0, B_0) in this system if every object in A shares some common attribute with some object in A_0. We give a formal definition as follows.

Definition 3.2.3.1 Let B(G, M, I) be a concept lattice and (A_0, B_0) $\in$ B(G, M, I). A concept (A, B) in B(G, M, I) is said to be in a far-relation neighborhood of (A_0, B_0), if for every $a \in A$, there exists $a_0 \in A_0$ and $m_0 \in M$ such that aIm_0 & a_0Im_0. Any subset of concepts in a far-relation neighborhood of (A_0, B_0) containing (A_0, B_0) is called a far-relation neighborhood of (A_0, B_0). The family of all far-relation neighborhoods is denoted by $NS_3(A_0, B_0)$. The far-relation neighborhood system is denoted by NS_3.

Example 3.2.3.2 Let G be the set of all students at Lakehead University and let M be the set of all hobbies of students. We define a relation I on G×M by aIm if the student a has the hobby m, where a ∈ G and m ∈ M. Let $(A_0, B_0) \in B(G, M, I)$. A concept (A, B) in B(G, M, I) is in a far-relation neighborhood of (A_0, B_0) if every student in the group A shares at least one common hobby with some student in group A_0.

Remark: From example 3.2.3.2, we see that if (A, B) is in a far-relation neighborhood of (A_0, B_0), then (A_0, B_0) may not be in a far-relation neighborhood of (A, B).

The following proposition follows from the definition.

Proposition 3.2.3.3 Let B(G, M, I) be a concept lattice and $(A_0, B_0) \in B(G, M, I)$, then

(I) For every $u \in NS_3(A_0, B_0)$, $(A_0, B_0) \in u$.

(II) If $u, v \in NS_3(A_0, B_0)$, then $u \cup v$ and $u \cap v$ are also in $NS_3(A_0, B_0)$.

Remark: Again, Nc) and Nd) fail for the far-relation neighborhood system.

Example 3.2.3.4 As in Example 2.4.2, we have the following neighborhood system:

$NS_3(3)$ = the collection of all subsets of W containing 3, where

W = {1, 2, 3, 4, 5, 12, 13, 14, 15},

is the set of concepts in far-relation neighborhood of concepts.

$NS_3(5)$ = the collection of all subsets of W containing 5, where

W = {1, 3, 5, 9, 11, 14, 16},

$NS_3(7)$ = the collection of all subsets of W containing 7, where

W = {6, 7, 8, 19},

$NS_3(10)$ = the collection of all subsets of W containing 10, where

W = {5, 6, 9, 10, 11, 16, 17, 18, 19},

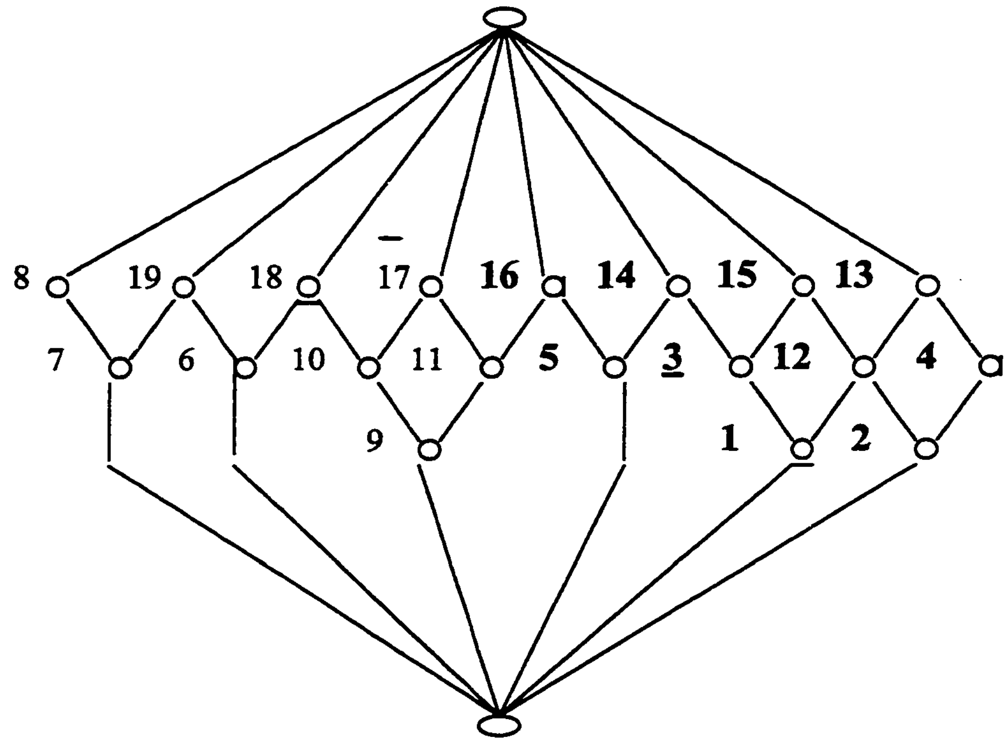

Figure 3.2.3.1 Far-relation neighborhood of concept 3.

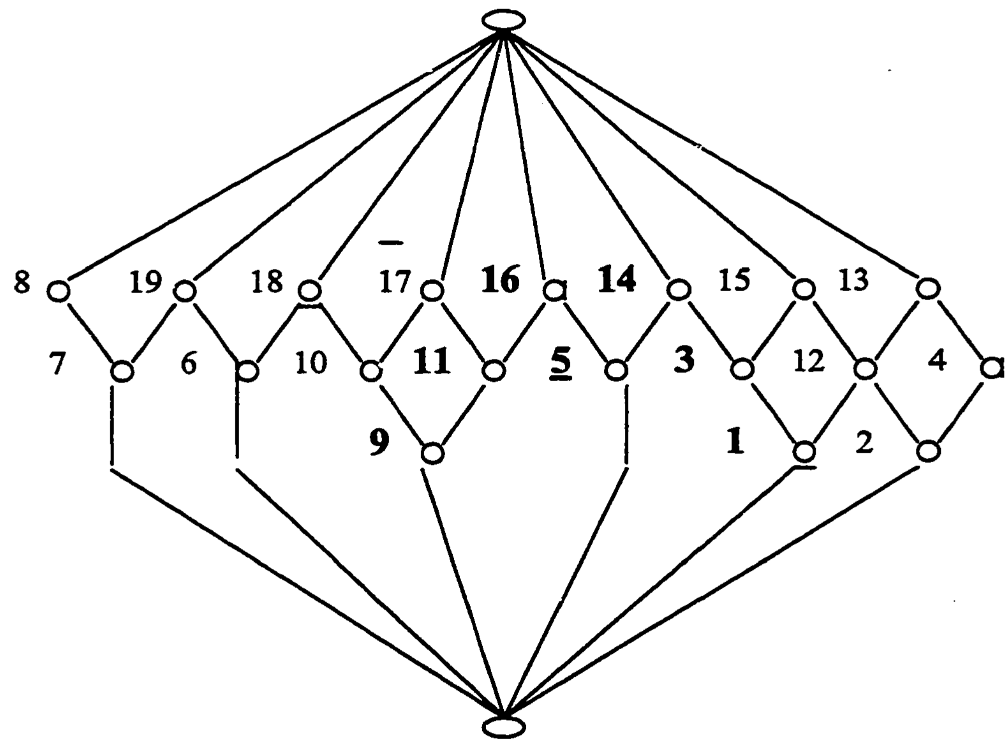

Figure 3.2.3.2 Far-relation neighborhood of concept 5.

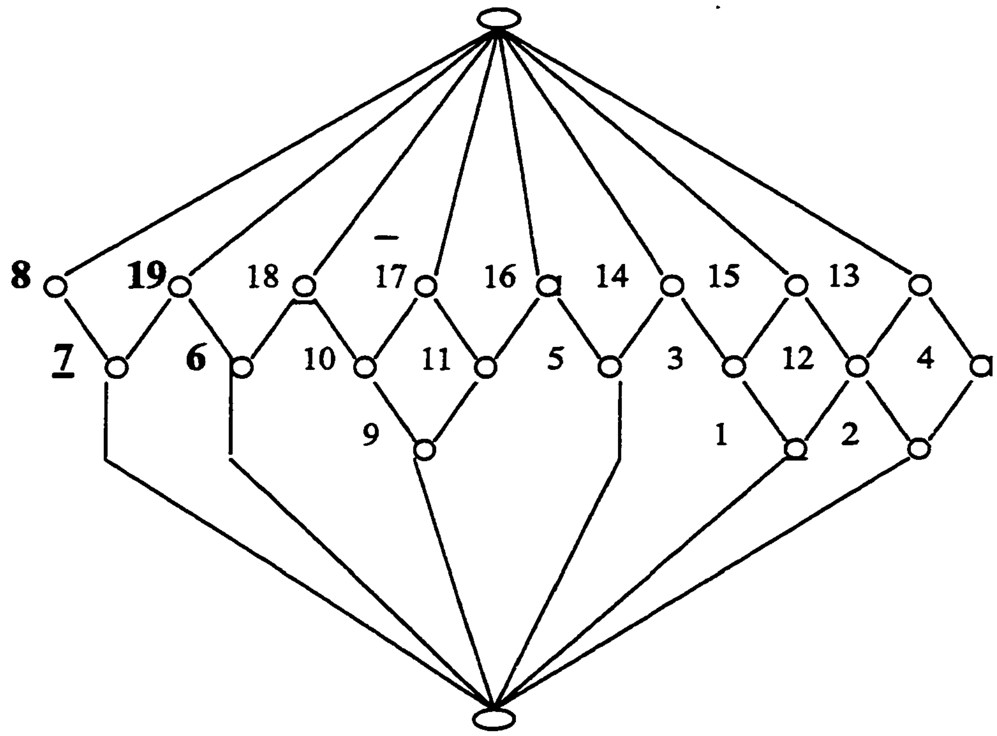

Figure 3.2.3.3 Far-relation neighborhood of concept 7.

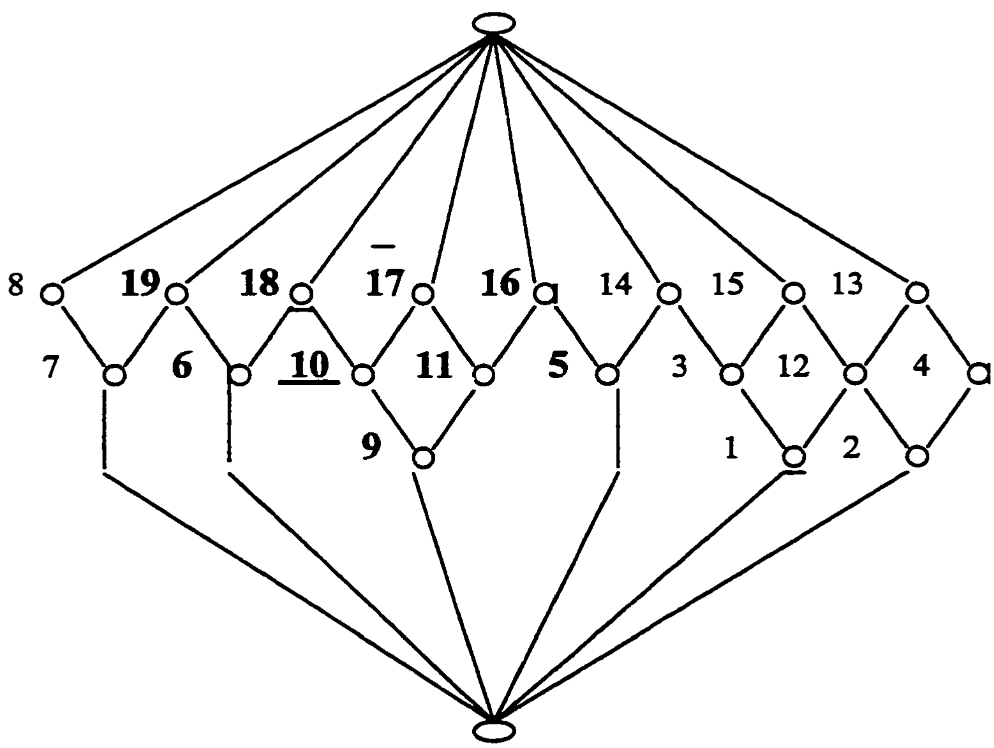

Figure 3.2.3.4 Far-relation neighborhood of concept 10.

3.2.4 Comparison of NS_1, NS_2 & NS_3

In this section, we compare the three different neighborhood systems defined above. Our conclusion is the following.

Theorem 3.2.4.1 Let B(G, M, I) be a concept lattice and $(A_0, B_0) \in B(G, M, I)$. We have $NS_1(A_0, B_0) \subseteq NS_2(A_0, B_0) \subseteq NS_3(A_0, B_0)$, where NS_1, NS_2, NS_3 are super-sub-relation, close-relation and far-relation neighborhood systems of (A_0, B_0), respectively.

Proof: Let $(A, B) \in NS_1(A_0, B_0)$, then $(A, B) \geq (A_0, B_0)$ or $(A, B) \leq (A_0, B_0)$. So $B \supseteq B_0$ or $B \subseteq B_0$. So $B \cap B_0 = B_0$ or B. Hence $B \cap B_0 \neq \varnothing$. By Theorem 3.2.2.3, $(A, B) \in NS_2(A_0, B_0)$. Hence $NS_1(A_0, B_0) \subseteq NS_2(A_0, B_0)$. Let $(A, B) \in NS_2(A_0, B_0)$. Then there exists $m_0 \in M$ such that aIm_0 & a_0Im_0 for all $a \in A$ and $a_0 \in A_0$. So for every $a \in A$, aIm_0 & a_0Im_0 for every $a_0 \in A_0$. By definition, $(A, B) \in NS_3(A_0, B_0)$, therefore $NS_2(A_0, B_0) \subseteq NS_3(A_0, B_0)$.

We use the following example to illustrate this theorem.

Example 3.2.4.2 Let B(G, M, I) be the concept lattice in example 2.4.2, then we have

$NS_1(3)$ = all subset of W containing 3, where W = {1, 3, 14, 15},
$NS_2(3)$ = all subset of W containing 3, where W = {1, 2, 3, 5, 12, 14, 15},
$NS_3(3)$ = all subset of W containing 3, where W = {1, 2, 3, 4, 5, 12, 13, 14, 15};
$NS_1(5)$ = all subset of W containing 5, where W = {5, 14, 16},
$NS_2(5)$ = all subset of W containing 5, where W = {1, 3, 5, 11, 14, 16},
$NS_3(5)$ = all subset of W containing 5, where W = {1, 3, 5, 9, 11, 14, 16};
$NS_1(7)$ = all subset of W containing 7, where W = {7, 8, 19},
$NS_2(7)$ = all subset of W containing 7, where W = {6, 7, 8, 19},
$NS_3(7)$ = all subset of W containing 7, where W = {6, 7, 8, 19};
$NS_1(10)$ = all subset of W containing 10, where W = {9, 10, 17, 18},

$NS_2(10)$ = all subset of W containing 10, where W = {6, 9, 10, 11, 17, 18},

$NS_3(10)$ = all subset of W containing 10, where W = {5, 6, 9, 10, 11, 16, 17, 18, 19};

We get the conclusion that $NS_1(a) \subset NS_2(a) \subset NS_3(a)$.

Chapter 4

NEIGHBORHOOD SYSTEMS IN CONCEPT LATTICES OF MULTI-VALUED CONTEXTS

4.1 Introduction

Often the notion of formal context as discussed in chapter 2 and 3 is not adequate for understanding and representing data since some data are not given by cross-tables. In formal concept analysis, a new approach has been introduced to represent the data (see [5, 7, 16]). This approach is based on the extension of the notion of the set-theoretic model of formal context to multi-valued context. This multi-valued context is used to formalize some data structures, which are represented in statistics by data matrices and in computer science by relational databases.

We point out here that a formal context may be understood as a special case of a multi-valued context. The approach to the multi-valued context is to derive a suitable formal context from a given multi-valued context. Such a derivation is always an action of interpretation. Although there is no general way for the derivation, in formal concept analysis, this is done by a method called conceptual scaling (see [7]). After a conceptual scaling, a new formal context is obtained: we can deal with the concept lattice of the new formal context as in Chapter 3. In this chapter, we introduce two neighborhood systems to deal with two conceptual scales, so called nominal scale and one-dimensional ordinal scale. We will give the definition of multi-value contexts first. It is a formal context together with the values for the attributes.

4.2 Multi-valued contexts

Definition 4.2.1 A multi-valued context is defined to be a quadruple (G, M, W, I), where G, M and W are sets and I is a ternary relation between G, M and W, i.e. I $\subseteq$ G×M×W such that (g, m, w_1) $\in$ I and (g, m, w_2) $\in$ I imply $w_1 = w_2$ for g $\in$ G, m $\in$ M and $w_1, w_2 \in$ W. The elements of G, M and W are called objects, (multi-valued) attributes and attribute values, respectively. (g, m, w) $\in$ I is read: the object g has the values w for the attribute m. The multi-valued context (G, M, W, I) is called an n-valued context if |W| = n. A formal context may be understood in this terminology as a special case: a 1-valued context.

Example 4.2.2 Let G be the set of all students at Lakehead University and let M be the set of courses offered at Lakehead University in 1999 ~ 2000 school year. W is the set of real numbers from 0 to 100. We define a ternary relation between G, M and W as follows, for any (g, m, w) $\in$ G×M×W, (g, m, w) $\in$ I if the student g has taken the course m in 1999 ~ 2000 school year and obtained w marks in the class.

This relation I satisfies the condition (g, m, w_1) $\in$ I and (g, m, w_2) $\in$ I imply $w_1 = w_2$ since every student obtains only one mark in the same class. Hence (G, M, W, I) is a multi-valued context.

Example 4.2.3 Table 4.2.1 is an example of a multi-valued context. Its objects are the eleven persons whose name are heading the rows and its attributes are the twelve cities which are represented by the columns; the value indicates that an object has the attribute of the value, i.e., which person has been in that city how many times.

C1	C3	C5	C7	C9	C11
20	19	8			
17	18	11			
19	20	8			
17	24	13			
15	25				
				30	4
				21	10
				17	10
			25		
			31		
			25		

Table 4.2.1. Multi-valued context

We need to derive a suitable formal context from a multi-valued context. Such a derivation is always an action of interpretation. There are certainly a lot of different ways to interpret. In the following, we are going to introduce a method called conceptual scaling.

Conceptual scaling (see [7]): derive from a multi-valued context to a suitable formal context. Let K:= (G, M, W, I) denote a multi-valued context.

(I) The first step of conceptual scaling is to interpret for each attribute m its values as objects of some separate formal context $\$_m := (G_m, M_m, I_m)$, i.e., the attribute m is understand as a partial map from G into G_m. The context $\$_m$ and their concept lattices should have a clear structure and should reflect some meaning of the data for interpretation.

(II) The second step, the scales $\$_m$ (m ∈ M) are combined to a common scale

$$\$:= (\times_{m \in M} G_m, \bigcup_{m \in M} (M_m \times \{m\}), \nabla)$$

Where ∇ is the relation with

$$(g_m)_{m\in M}\nabla(n,p)\Leftrightarrow g_pI_pn$$

(III) The third step, we obtain the formal context (G, $\bigcup_{m\in M}(M_m\times\{m\})$, J) with

$$gJ(n,p)\Leftrightarrow(m(g))_{m\in M}\nabla(n,p)$$

Or equivalently

$$P(g)I_pn$$

Definition 4.2.4 (G, $\bigcup_{m\in M}(M_m\times\{m\}$, J) is called the derived context of the scaled context (K, \$) and the concept lattice of the derived context is also called the concept lattice of the scaled context, i.e. B(G, $\bigcup_{m\in M}(M_m\times\{m\}$, J).

We use the following example to illustrate the conceptual scaling method.

Example 4.2.5 Let (G, M, W, I) be the multi-valued context in Example 4.2.3. For each m $\in$ M, let G_m be the set of all students taking the course m in 1999 ~ 2000 school year, M_m the set of real number between 0 and 100. We define I_m by, for g$\in G_m$ and n $\in M_m$, gI_mn $\Leftrightarrow$ g(v) $\geq$ 60, i.e. the student g in class m has obtained v marks which is at least 60. So (G, $\bigcup_{m\in M}(M_m\times\{m\}$, J) is the derived context. For g$\in$G and $(n_{m_0},m_0)\in\bigcup_{m\in M}(M_m\times\{m\})$, g J$(n_{m_0},m_0)$ if g$(m_0)I_{m_0}\ n_{m_0}$ $\Leftrightarrow$ the student g has taken the course m_0 in 1999 ~ 2000 school year and got n_{m_0} marks, and furthermore, the mark is at least 60.

We are going to consider two natural conceptual scales, the nominal scale (W, W, =) and the one-dimensional ordinal scale (W, W, $\geq$). The nominal scale (W, W, =) is a simple scale. We illustrate this scale with the following example:

Example 4.2.6 Let (G, M, W, I) be the multi-valued context in Example 4.2.3, then

G = {a, b, c, d, e, ..., i, j, k} is the set of tourists.

M = {C1, C2, ... C12} is the set of cities visited.

W = {0, 1, 2, ... 100}

In this scale method all G_m = G, all M_m = W and I_m = "=". For x ∈ G and

$(n_{m_0}, m_0) \in \bigcup_{m \in M} (M_m \times \{m\})$,

$$x \, J \, (n_{m_0}, m_0) \text{ if and only if } m_0(x) \; I_m \; n_{m_0} \Leftrightarrow m_0(x) = n_{m_0}$$

i.e. the tourist x has visited the city $m_0 \in M$, $n_{m_0} \in W$ times. For example, a J (20, C1), since tourist a has visited the city C1 exactly 20 times by example 4.2.3.

	C_1	C_1	C_1	C_1	C_2	C_2	C_2	C_2	C_3	C_3	C_3	C_3	C_3	C_4	C_4	C_4	C_4	C_4	C_5	C_5	C_5
	15		19		16		20		18		20		25		17		19		8		13
															×				×		
									×								×				
			×				×				×								×		
					×																×
	×												×								

Table 4.2.2-1. cross-table of derived context --- part I

	C_6	C_6	C_7	C_7	C_8	C_8	C_9	C_9	C_9	C_{10}	C_{10}	C_{11}	C_{11}	C_{11}	C_{12}	C_{12}
	5		25		32		17		30		18		6		8	
									×							
							×				×					
	×		×		×											
					×											
	×		×													

Table 4.2.2-2. cross-table of derived context --- part II

There are 18 concepts of the derived context:

1. ({ a }, { (20, C_1), (21, C_2), (19, C_3), (17, C_4), (8, C_5) }),
2. ({ b }, { (17, C_1), (18, C_2), (18, C_3), (19, C_4), (11, C_5) }),
3. ({ c }, { (19, C_1), (20, C_2), (20, C_3), (18, C_4), (8, C_5) }),
4. ({ d }, { (17, C_1), (16, C_2), (24, C_3), (15, C_4), (13, C_5) }),
5. ({ e }, { (15, C_1), (18, C_2), (25, C_3), (23, C_4) }),
6. ({ b, d }, { (17, C_1) }),
7. ({ b, e }, { (18, C_2) }),
8. ({ f }, { (14, C_6), (33, C_8), (30, C_9), (4, C_{11}) }),
9. ({ g }, { (21, C_9), (11, C_{10}), (10, C_{11}), (14, C_{12}) }),
10. ({ h }, { (17, C_9), (18, C_{10}), (10, C_{11}), (14, C_{12}) }),
11. ({ i }, { (5, C_6), (25, C_7), (32, C_8) }),
12. ({ j }, { (14, C_6), (31, C_7), (32, C_8) }),
13. ({ k }, { (5, C_6), (25, C_7), (33, C_8) }),
14. ({ i, k }, { (5, C_6), (25, C_7) }),
15. ({ f, j }, { (14, C_6) }),
16. ({ i, j }, { (32, C_8) }),
17. ({ f, k }, { (33, C_8) }),
18. ({ g, h }, { (10, C_{11}), (14, C_{12}) }).

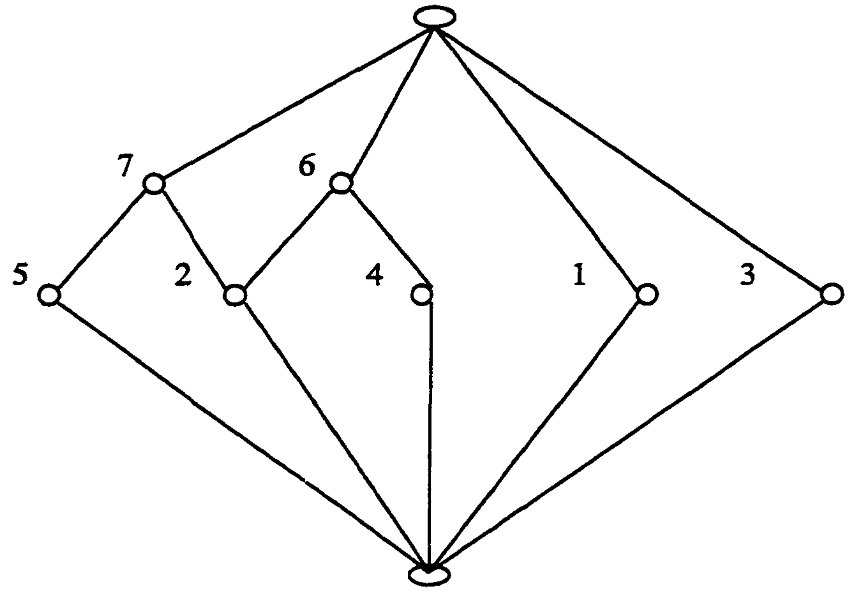

Figure 4.2.3-1. concept lattice of the derived context --- part I

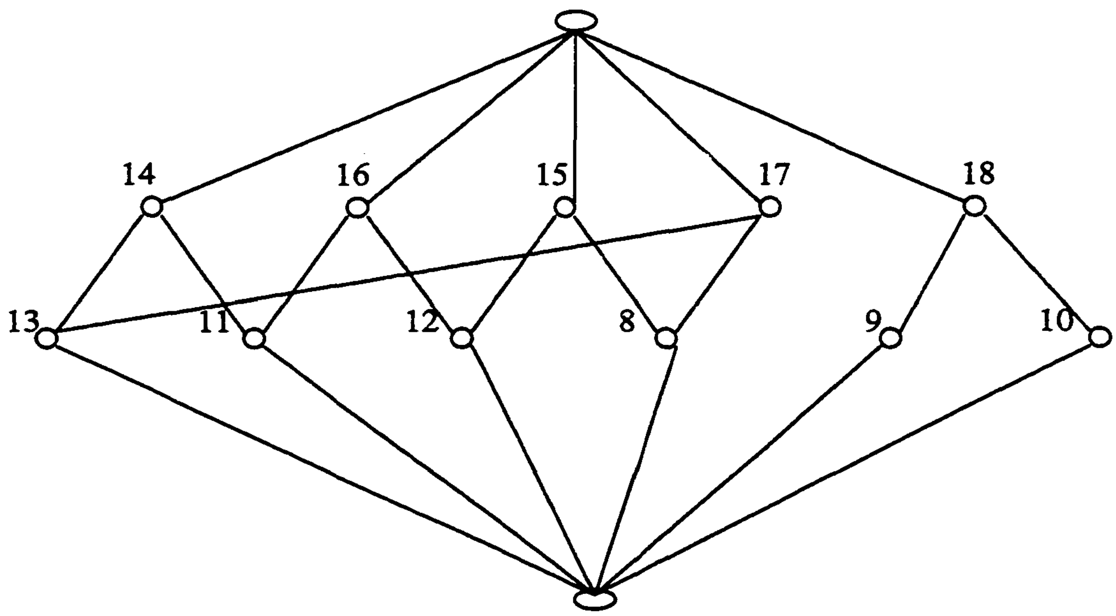

Figure 4.2.3-2. concept lattice of the derived context --- part II

The nominal scale (W, W, =) is used to conceptually separate different values. But it would not reflect the important order of the values. The one-dimensional ordinal scale (W, W, ≥) is more appropriate for the ordinal nature of the attribute values. Again, we use example 4.2.3 to illustrate this scale:

Example 4.2.7 Let (G, M, W, I) be the multi-valued context in example 4.2.3, then
G = {a, b, c, ... k} is the set of tourists.
M = {C1, C2, ... C12} is the set of cities visited.
W = {1, 2, 3, ..., 100} is the set of number of visiting times.

In this one-dimensional ordinal scale, all G_m = G, all M_m = W and I_m = "≥". For visitor x ∈ G and city m_0 ∈ M,

$$x\,J\,(n_{m_0}, m_0) \Leftrightarrow m_0(x)\,I_m\,\,m_0 \Leftrightarrow m_0(x) \geq n_{m_0}$$

⇔ visitor x has visited city m_0 at least n_{m_0} times. For example, a J (16, C1) since tourist a has visited city C1 exactly 20 times ≥ 16.

4.3 Neighborhood systems

Let (G, M, W, I) be a multi-valued context such that W is a subset of the set of real number R. In this section, we assume that we take the following conceptual scaling: (G, $\bigcup_{m\in M}(M_m \times \{m\}$, J) where M_m = W for all m ∈ M.

Definition 4.3.1 Let (A_0, B_0) be a concept in B(G, $\bigcup_{m\in M}(M_m \times \{m\})$, J). For a positive number δ > 0, we say that a concept (A,B) in B(G, $\bigcup_{m\in M}(M_m \times \{m\})$, J) is in the δ-neighborhood of (A_0, B_0) if for each $(v^0, m^0) \in B_0$, there exists (v, m) ∈ B such that m = m^0 and $|v - v^0| < \delta$ (note that B_0, B ⊆ $\bigcup_{m\in M}(M_m \times \{m\})$, where M_n = W is a subset of R).

Any subset of concepts in a δ-neighborhood of (A_0, B_0) and containing (A_0, B_0) is called a δ-neighborhood of (A_0, B_0). $NS_\delta(A_0, B_0)$ denotes the collection of all δ-neighborhood of (A_0, B_0), and is called the δ-neighborhood system of (A_0, B_0).

Remark: We define the δ-neighborhood system by the attribute values only. Roughly speaking, (A, B) is in a δ-neighborhood of a concept (A_0, B_0) if their attribute values are close by δ.

By the definition of concept (A,B) that A = B' and B = A', it follows that

Proposition 4.3.2 Let (A_0, B_0) be in the concept lattice B (G, $\bigcup_{m\in M}(M_m \times \{m\})$, J), then a concept (A,B) is in a δ-neighborhood of (A_0, B_0) ⇔ whenever g_0 J (r_0, m_0) for all $g_0 \in A_0$ and some $(r_0, m_0) \in B_0$ then there is $r \in M_{m_0}$ such that g J (r, m_0) for all $g \in A$ and $|r - r_0| < \delta$.

Proposition 4.3.3 Let (A_0, B_0) be a concept in B (G, $\bigcup_{m\in M}(M_m \times \{m\})$, J) and δ > 0, then the following conditions hold:

(I) For every $U \in NS_\delta(A_0, B_0)$, we have $(A_0, B_0) \in U$;

(II) If δ_1 and δ_2 are positive numbers such that $\delta_1 < \delta_2$, then

$$NS_{\delta_1}(A_0, B_0) \subseteq NS_{\delta_2}(A_0, B_0)$$

(III) If $U, V \in NS_\delta(A_0, B_0)$, then $U \cap V \in NS_\delta(A_0, B_0)$

Proof: (I) Follows from the definition.

(II) Let $U \in NS_\delta(A_0, B_0)$. For every $(A,B) \in U$, (A,B) is in a δ_1-neighborhood of (A_0, B_0). So for every (r_0, m_0) in B_0, there exists an $(r, m) \in B$ such that $m = m_0$ and

$|r - r_0| < \delta_1$. Since $\delta_1 < \delta_2$, $|r - r_0| < \delta_2$. By definition, (A,B) is in the δ_1-neighborhood of (A_0, B_0). Thus, U $\in NS_{\delta_2}(A_0, B_0)$.

Hence $NS_{\delta_1}(A_0, B_0) \subseteq NS_{\delta_2}(A_0, B_0)$.

(III) If U, V $\in NS_\delta(A_0, B_0)$, then every (A,B) $\in U \cap V$ is in a δ-neighborhood of the concept (A_0, B_0). Also, $(A_0, B_0) \in U$ and $(A_0, B_0) \in V$ imply $(A_0, B_0) \in U \cap V$. So $U \cap V \in NS_\delta(A_0, B_0)$ by definition.

Remark: Axioms Nc) and Nd) for neighborhood system in topology as in (3.1) may fail.

Proposition 4.3.4 Let (A,B) and (A_0, B_0) be concepts in B (G, $\bigcup_{m\in M}(M_m \times \{m\})$, J) and $\delta > 0$. If $(A,B) \leq (A_0, B_0)$, then (A,B) is in a δ-neighborhood.

Proof: Since $(A,B) \leq (A_0, B_0)$, we have $A \subseteq A_0$ or $B \supseteq B_0$. So for every $(r_0, m_0) \in B_0$, $(r_0, m_0) \in B$. If we take (r, m) as (r_0, m_0) in B, then $m = m_0$ and $|r - r_0| = 0 < \delta$. Thus, (A,B) is in a δ-neighborhood by definition.

We illustrate this definition by the following examples.

Example 4.3.5 Let G be the set of all students at Lakehead University and let M be all courses offered at the university in 1999 ~ 2000 school year. We choose the conceptual scaling (G, $\bigcup_{m\in M}(M_m \times \{m\})$, J), such that M_m is the set of nonnegative real numbers and for g $\in$ G and (r, m) $\in \bigcup_{m\in M} M_m \times \{m\}$, we define g J (r, m) $\Leftrightarrow$ the student g takes the course m and spends totally r minutes in the course m for the whole school year. Let (A_0, B_0) be a concept in B (G, $\bigcup_{m\in M}(M_m \times \{m\})$, J), a concept (A, B) is in a δ-neighborhood of $(A_0, B_0) \Leftrightarrow$ for every (r_0, m_0) in B_0, there exists (r, m) $\in$ B such that $m = m_0$ and $|r - r_0| < \delta \Leftrightarrow$ for every course m_0 taken by all students in A_0 and spent the same length of time r_0 by all the student in A_0, all the students in A take the same course m_0 as well and they spend r minutes totally in the course with $| r- r_0 |< \delta$. So (A, B) is "near" to (A_0, B_0) in the following sense: each course taken by all members in A_0 is also taken by all members in A and the time they spend is closed by δ minutes.

Example 4.3.6: In example 2.4.2, we take the nominal scaling and $\delta = 2$. Concept 3 is in 2-neighborhood of concept 1 by the definition, i.e. tourist c in concept 3 is "close to" tourist a in concept 1 in the following sense: for any city visited by tourist a, it is also visited by tourist c in concept 3. Moreover, the numbers of visiting times of tourist a and tourist c are closed by δ.

(I) δ-neighborhoods when $\delta = 2$

For a = 1: 3 is in the δ-neighborhood of 1; for a = 3: 1 is in the δ-neighborhood of 3; for a = 11: 13 is in the δ-neighborhood of 11; for a = 16: 17 is in the δ-neighborhood of 16; for a = 17: 16 is in the δ-neighborhood of 17.

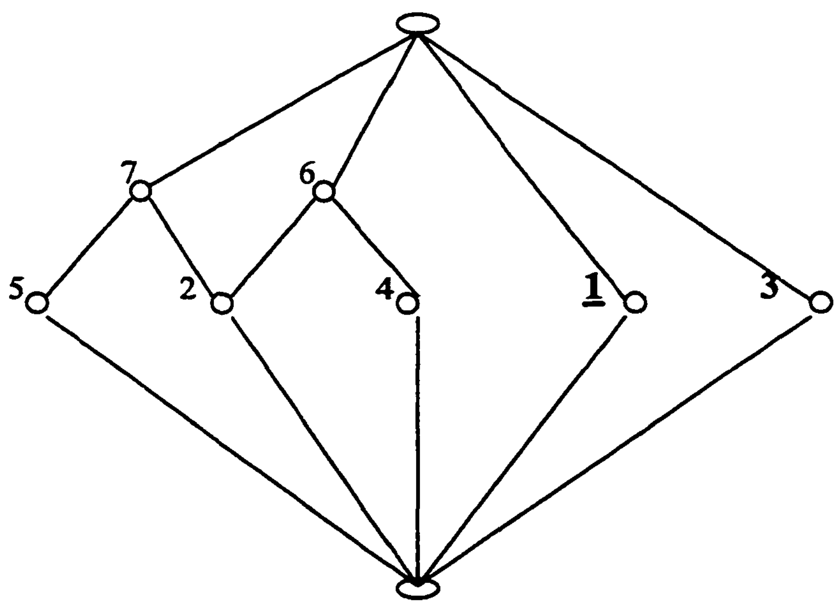

Figure 4.3.1. the $\delta = 2$ neighborhood of concept 1.

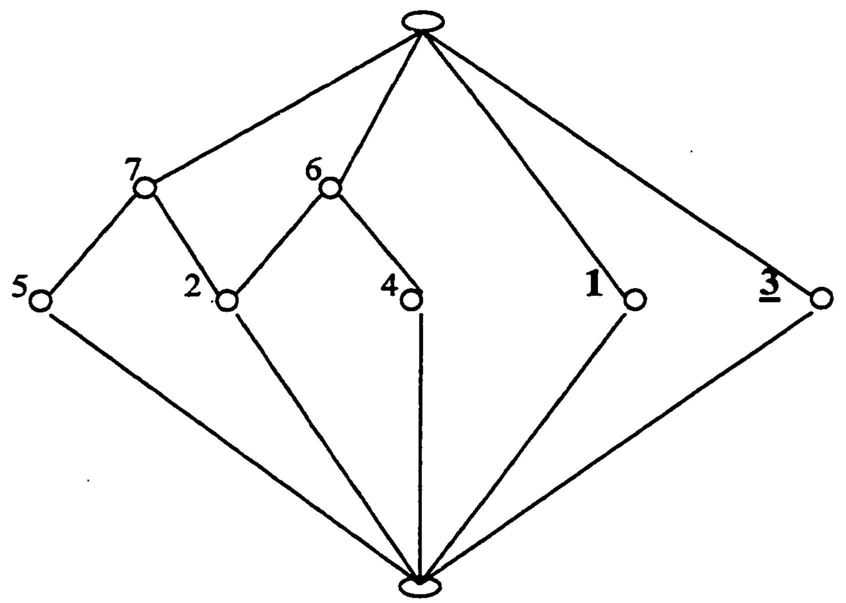

Figure 4.3.2. the $\delta = 2$ neighborhood of concept 3.

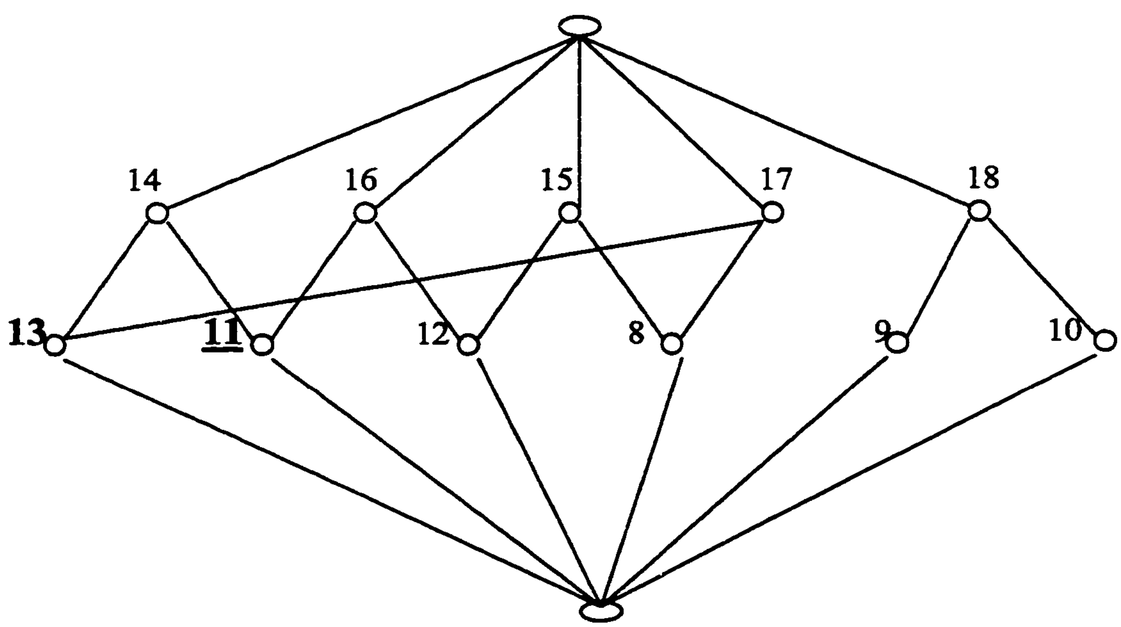

Figure 4.3.3. the $\delta = 2$ neighborhood of concept 11.

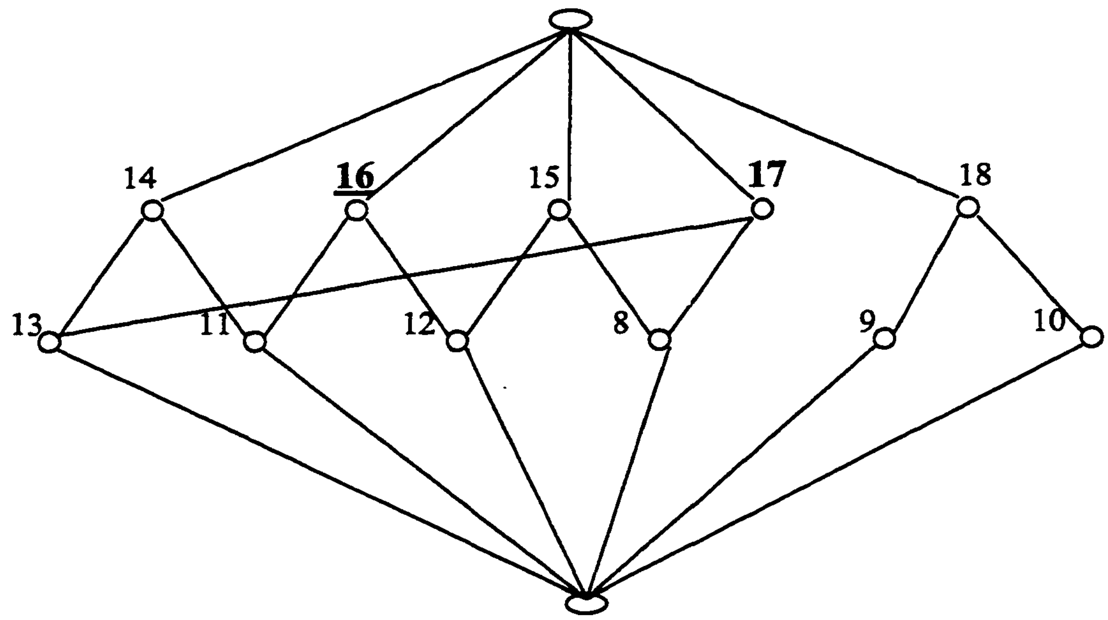

Figure 4.3.4. the $\delta = 2$ neighborhood of concept 16.

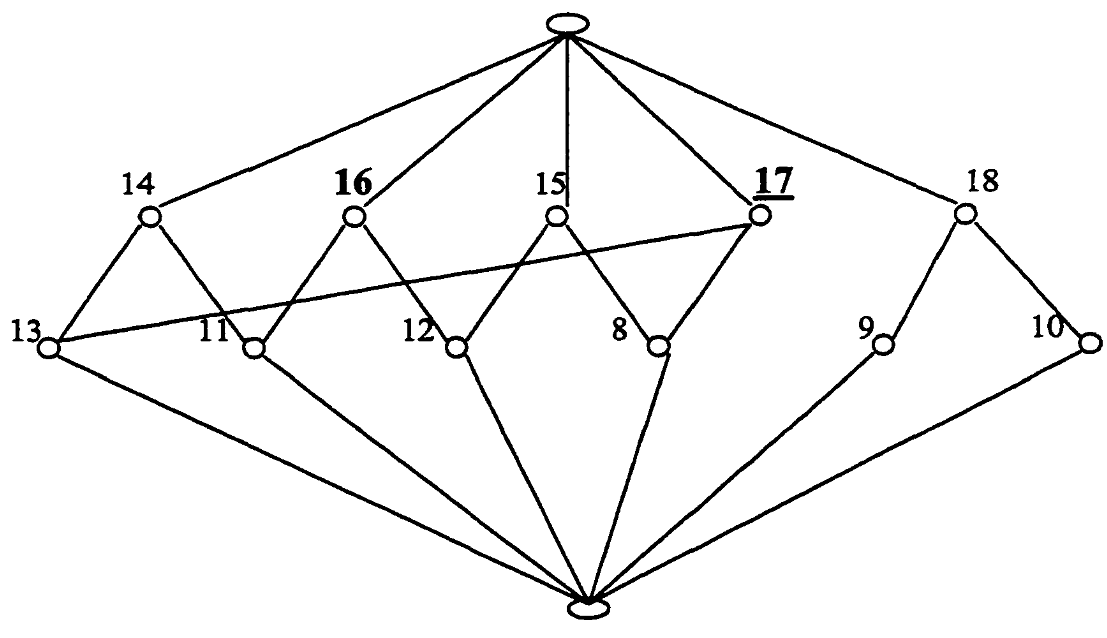

Figure 4.3.5. the $\delta = 2$ neighborhood of concept 17.

(II) δ-neighborhoods when $\delta = 4$

For a = 1, concepts 2 and 3 are in the δ-neighborhood of concept 1; for a = 3, concepts 1 and 2 are in the δ-neighborhood of concept 3.

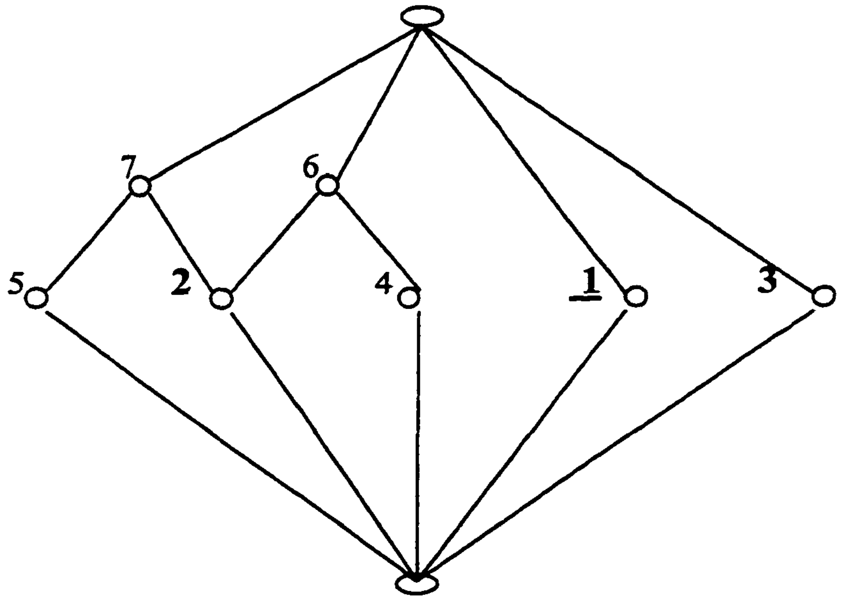

Figure 4.3.6. the δ = 4 neighborhood of concept 1.

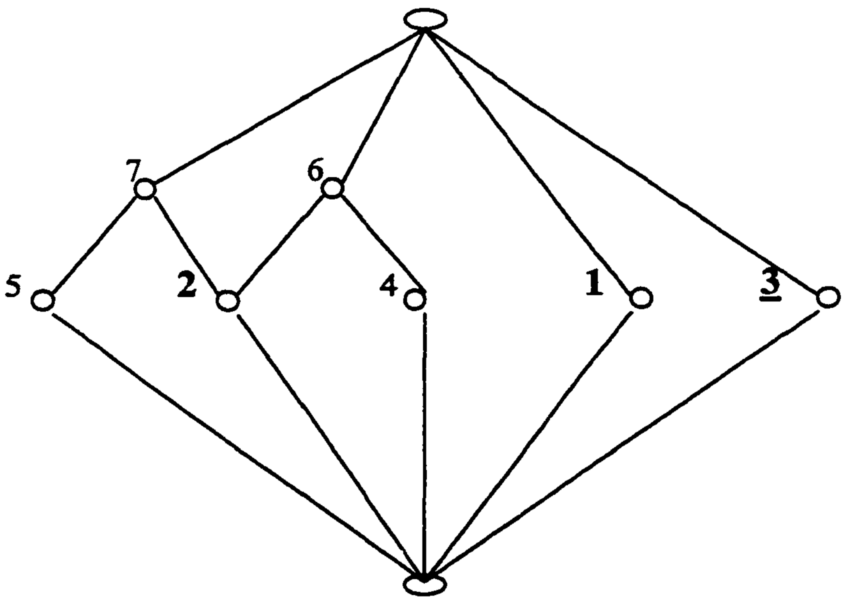

Figure 4.3.7. the δ = 4 neighborhood of concept 3.

(III) δ-neighborhoods when $\delta = 6$

For a = 1, concepts 2, 3 and 4 are in the δ-neighborhood of concept 1; for a = 3, concepts 1, 2 and 4 are in the δ-neighborhood of concept 3.

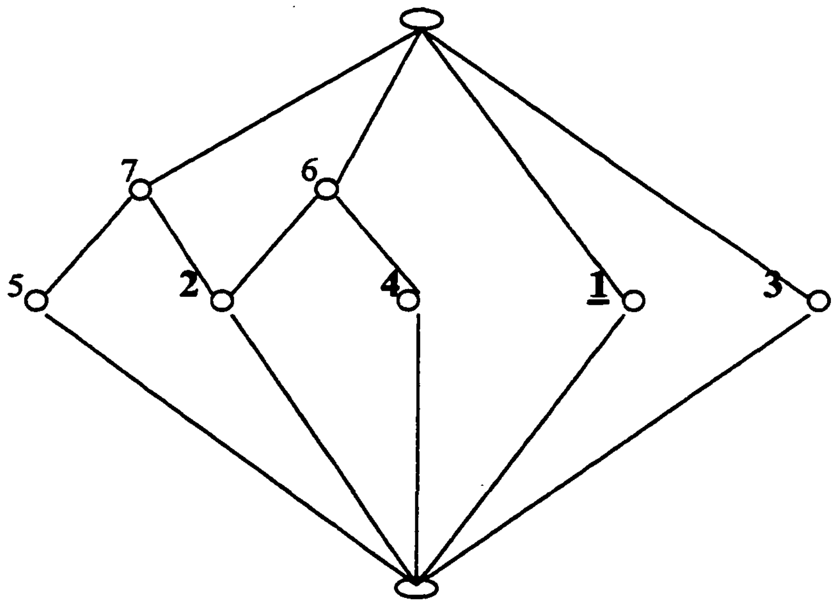

Figure 4.3.8. the $\delta = 6$ neighborhood of concept 1.

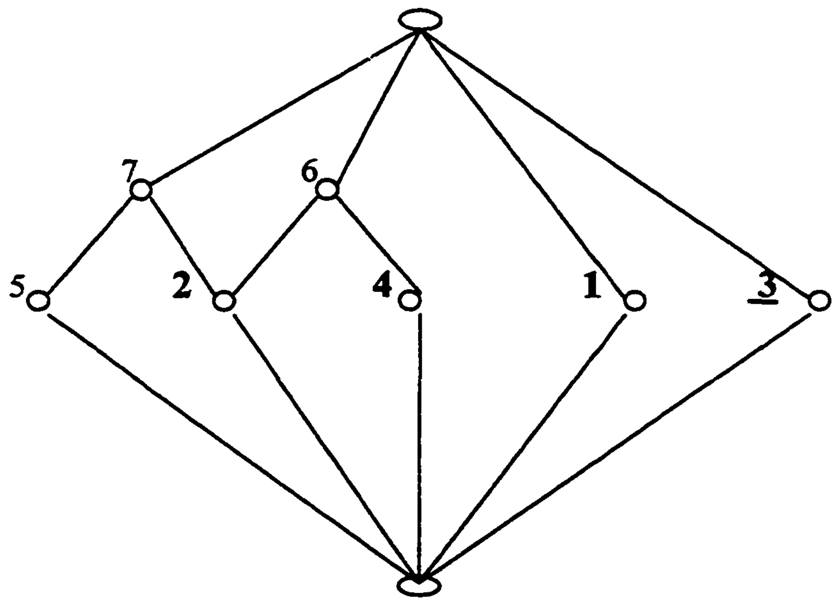

Figure 4.3.9. the $\delta = 6$ neighborhood of concept 3.

(IV) δ-neighborhoods when δ = 10

For a = 11, concepts 12 and 13 are in the δ-neighborhood of concept 11.

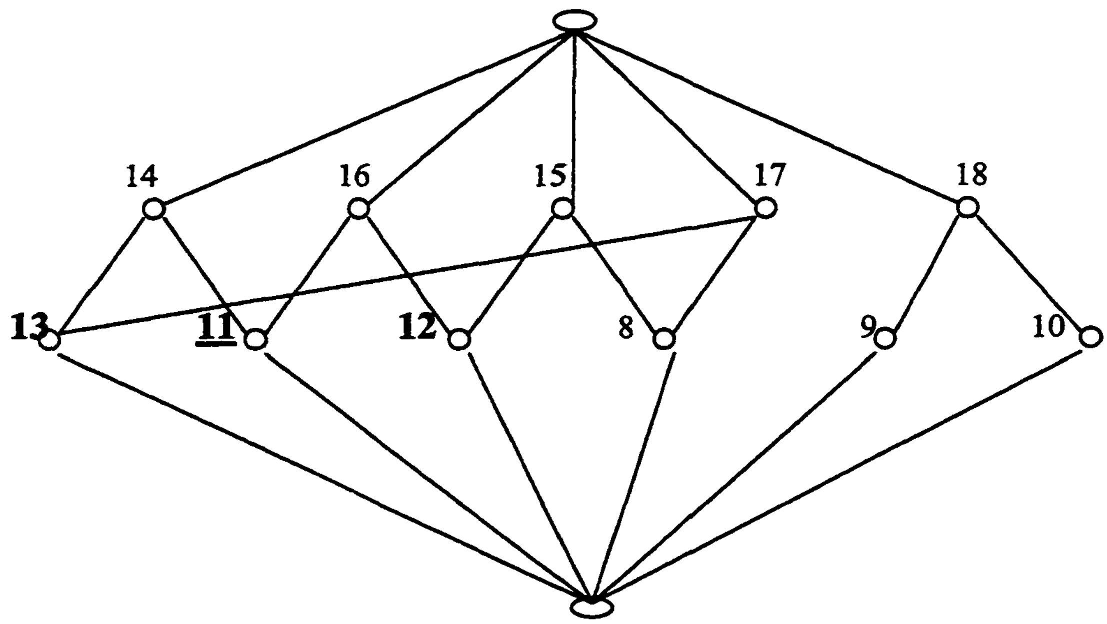

Figure 4.3.10. the δ = 10 neighborhood of concept 11.

Example 4.3.7

[redacted]	P_1	[redacted]	P_3	[redacted]	...	[redacted]	...	[redacted]	...	[redacted]	P_9	[redacted]
[redacted]	T1,1	[redacted]	T1,3	[redacted]		[redacted]		[redacted]		[redacted]	T1,9	[redacted]
[redacted]	T2,1	[redacted]	T2,3	[redacted]		[redacted]		[redacted]		[redacted]	T2,9	[redacted]
[redacted]	T3,1	[redacted]	T3,3	[redacted]		[redacted]		[redacted]		[redacted]	T3,9	[redacted]
[redacted]		[redacted]		[redacted]	...	[redacted]	...	[redacted]		[redacted]		[redacted]
[redacted]		[redacted]		[redacted]		[redacted]		[redacted]		[redacted]		[redacted]
[redacted]		[redacted]		[redacted]		[redacted]		[redacted]		[redacted]		[redacted]
[redacted]		[redacted]		[redacted]		[redacted]		[redacted]		[redacted]		[redacted]
[redacted]		[redacted]		[redacted]		[redacted]		[redacted]		[redacted]		[redacted]
[redacted]		[redacted]	...	[redacted]		[redacted]		[redacted]		[redacted]		[redacted]
[redacted]	T9,1	[redacted]	T9,3	[redacted]	...	[redacted]	...	[redacted]		[redacted]	T9,9	[redacted]
[redacted]	T10,1	[redacted]	T10,3	[redacted]		[redacted]		[redacted]		[redacted]	T10,9	[redacted]

Table 4.3.11. Multi-valued context

Table 4.3.11 is an example of a multi-valued context: its objects are the one hundred persons who dial the phone; its attributes are the other one hundred persons who receive the phone; its attribute values are the time of the phone call, furthermore, m(g) = w means that the person g called the person m for w minutes. If there is valuc 0 means the two persons do not communicate with each other.

Because they are all positive real numbers that may not be integers, so we can not use the table to describe the procedure of scaling.

Suppose G = {a, b, c, ..., i, j} is the set of 10 callers and M = {P_1, P_2, ..., P_{10}} is the set of 10 people receiving the call. Let W= R^+ be the set of nonnegative real numbers and let I be described as above. We use the following conceptual scale for (G, M, W, I): for each m $\in$ M, M_m = W and I_m is the usual order relation of real numbers. Let (A_0, B_0) be a concept in B(G, $\bigcup_{m\in M}(M_m \times \{m\})$, J) and let $\delta > 0$. Then a concept (A, B) is in the δ-neighborhood of (A_0, B_0) if for any (r^0, m^0)$\in B_0$, there is (r, m) $\in$B such that $m^0 = m$ and $| r-r^0 | < \delta$, i.e. for any phone call to person m^0 made by some one in A_0, then there is person in A who called the same person m^0. Moreover, the lengths of the two calls are close by δ units. So the two concepts, or two groups of callers are "close to" or "near to" each other in this sense.

Chapter 5
CONCLUSION

In this thesis, we proposed various neighborhood systems in a concept lattice of a context and of a multi-valued context as well. The technique is based on the notion of neighborhoods in topology, a well-established subject in pure mathematics. These proposals not only have a solid theoretical basis, but also offer practical retrieval when conducting retrieval process in databases that contain information measuring the relation among the original objects.

The concept of neighborhood systems is a useful and effective tool for representing and analyzing semantics information, just as used in topology. In this thesis, we started from a formal context and measured the distance (e.g. far or near) between the concepts in the formal context based on the attributes of the concepts, i.e., the characteristics or properties of the concept, in various ways. Therefore, these models have many applications in the areas related to information science, such as inference, databases, information approximation, and data mining and data analysis. This thesis is only an initial step of application of the neighborhood concept in this direction. The future work should be on the aspect of further investigation on each of the proposed neighborhood systems. It is also interesting to develop more special types of neighborhood systems with many-valued context for special applications in information sciences.

The way of providing a meaningful notion of neighborhood systems depends heavily on the attributes of the concepts in the concept lattice. So it could vary from one application to the other. But the technique is the same, as used in topology, representing the " far to" or "near to" relationship between objects or concepts by neighborhoods. It turns out that the notion of neighborhood systems is very effective in our case.

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