On the Spectrum $\hat{G}$ of a Locally Compact Group G

by

John Brian Whitfield

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Abstract

Separation properties of the Fell topology, on the spectrum $\hat{G}$ of a locally compact group G, characterize important properties of G. We will develop three equivalent ways to describe the Fell topology on the spectrum $\hat{A}$ of any C*-algebra A. Specifically, we show that both the relative weak*-topology on $P(A)$, the set of pure states of A, and the Jacobson topology on $\text{Prim}(A)$, the set of all primative ideals on A, can be mapped onto $\hat{A}$ so that both topologies agree with the Fell topology. We will also study the correspondences, both between the set of strongly continuous unitary representations of G and the irreducible representations of the group C*-algebra $C^*(G)$, and between the continuous functions of positive type on G and the set of pure states of $C^*(G)$. As well, we give a survey of results outlining the characterization of G by simple separation properties of the Fell topology on $\hat{G}$.

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Dedication

To my family.

Contents

Chapter 1

Introduction

1.1 Organization

This thesis has three main objectives. We first present a skeletal introduction to the Fell topology on the spectrum [or the dual space] of a general C*-algebra A. Next, we strive to extend this topology to the spectrum $\hat{G}$ of a locally compact group G. Then, finally, we give a survey of results displaying the relationship between topological properties of G and $\hat{G}$. It has been an age old technique of mathematicians, that when studying an object having little structure [such as G] and therefore affording few tools to unlock the mysteries within, to instead study a "related" object [in our case $\hat{G}$] which has more structure and as such is easier to work with. Indeed, this thesis outlines a living example of this tried and true technique. The following summarizes chapter 2 through to the appendix.

Chapter 2 is for the most part a technical chapter. It begins with a brief introduction to C*-algebras together with some examples and selective attributes of such algebras. Our main objective in this chapter is to introduce one of the most important concepts in harmonic analysis, namely representations. Following the lead of Dixmier [5] we first present the notion of positive linear forms and show how such forms are automatically continuous on involutive Banach algebras. This then leads to the Gelfand-Naimark-Segal construction of representations from positve forms. Our definition of representations is then formally given. An equivalence relation on representations is presented along with the definitions of cyclic vectors and nondegenerate representations. The strong inter-relationship between positive forms and representations is then developed. More importantly, in subsection 2.3, we go on to show the inter-relationship between pure positive forms, or equivalently the extreme points of the closed unit ball of positive forms, and topologically irreducible representations. In fact, it is here that we develop a well defined surjective map, from the set of pure forms onto the set of equivalence classes of topologically irreducible representations, which we make use of in chapter 3. Finally, we end chapter 2 with the so called equivalence theorem as found in Fell's paper *The Dual Spaces of C*-algebras* [9]. This is a theorem about C*-algebras. It states that, if the kernel of a representation π contains the intersection of the kernels of all the elements of a set of representaions S, then any positive form associated with π is weak*-approximated by sums of positive forms associated with elements of S.

We begin chapter 3 by defining $\hat{A}$ to be the set of equivalence classes of topologically irreducible representations of a C*-algebra A. We then show three different ways to endow $\hat{A}$ with a topology: *i*) the Fell topology obtained by defining a closure operator with the notion of weak equivalence, *ii*) by taking the quotient topology on $\hat{A}$ of the relative weak*-topology on the set of pure positive forms and the surjective map described in chapter 2, and *iii*) by transferring the Jacobson or hull-kernel topology from the set of primitive two-sided ideals of A. Due to the ground work and in particular the equivalence theorem in chapter 2 we proceed to show that the topologies *i*) and *ii*) coincide on $\hat{A}$. Further, by making use of preliminary work in chapter 2 we show that the set of primitive two-sided ideals of a C*-algebra A is exactly the set of kernels of the elements of $\hat{A}$. From this we go on to show that the Jacobson topology transferred to $\hat{A}$ via this relationship also coincides with the Fell topology.

In chapter 4 we transport the Fell topology introduced in chapter 3 onto the spectrum $\hat{G}$ of a locally compact group G. We begin the chapter with the definition of unitary repesentaions of a locally compact group G. We proceed to show that these unitary repesentaions strongly correspond with the non-degenerate representaions of $L^1(G)$. In fact, this correspondence preserves irreducibility. Next we introduce functions of positive type and show that when endowed with the topology of compact convergence these functions correspond to the positive forms on $L^1(G)$ endowed with the weak*-topology. Our next aim is to develop the group C*-algebra of G, denoted by $C^*(G)$, which is the completion of $L^1(G)$ with regard to a special norm. We then show that this completion to $C^*(G)$ extends the corresponce between the unitary representations of G and the non-degenerated representations of $L^1(G)$ upto the representations of $C^*(G)$. As well, we can extend the preceeding analogy from functions of positive type endowed with the topology of compact convergence to positive forms on $C^*(G)$ endowed with the weak*-topology. Finally, we can now endow $\hat{G}$, the set of equivalence classes of irreducible unitary representations of G, with the so called Fell topology by simpily importing it from $C^*(G)^\wedge$ via the bijective correspondence between $\hat{G}$ and $C^*(G)^\wedge$. The chapter ends with a brief introduction to the reduced dual of G. This proves to be a usefull space when we study the topological properties of $\hat{G}$ in chapter 5.

Chapter 5 is presented differently than the other chapters. Whereas the bulk of the theory is self contained in chapters 2 through 4, in this chapter we merely give a survey of results, sketch proofs, and frequently cite other sources. The main aim of this chapter is show certain characterizations, of properties on a locally compact group G, by simple separtion properties on the spectrum $\hat{G}$. We begin the chapter with a section showing that any spectrum of a C*-algebra paired with the Fell topology satisfies certain inherent topological properties, namely every such spectrum is a quasi-compact Baire space. The following section breaks into two subsections. We begin by studying locally compact Abelian groups and then move to the general [non-Abelian] case. The aim of §5.2 is to show that if a locally compact group G is compact then $\hat{G}$ is discrete. It is a corollary of the Pontrjagin duality theorem, that for Abelian groups $\hat{G}$ discrete implies G is compact. Bagget [1, theorem 3.4] improves this result to show that if G is separable [and non-abelian] and $\hat{G}$ is discrete then G is compact. Due to P.S. Wang [19, theorem 7.7] this result also holds for σ-compact groups. We end this chapter with a short section of comments on the separation properties of $\hat{G}$.

The first section of the appendix is presented in order to justify our assumption that all locally compact groups are Hausdorff. Indeed, we show that it is essentially of no restriction to make this assumption. The second section introduces the involutive convolutive algebra $L^1(G)$. Our aim here is merely to set the notation surrounding $L^1(G)$ to support chapter 4. Section three presents a definition of an approximate identity and the often used result that every C*-algebra possess an approximate identity. In the forth section we present a technical result, the so called transitivity theorem, which is vital to the proof of the equivalence theorem in chapter 2. In the final section, a brief glimpse of the theory of von Neumann algebras and the commutant of a C*-algebra is given. Here we touch only on what we needed to prove the equivalence theorem in chapter 2.

Finally, note that each section is ended with a list of references. The author acquiesce that no original work is presented here, except perhaps in the method of presentation. These references are therefore communicated to serve two purposes: first, to acknowledge the source of the ideas for parts and/or all of the proofs in the given section, and second, simply as references for further study.

1.2 Preliminaries

1.2.1 Topological Groups

Definition 1.2.1 A group G equipped with a topology such that the group operations are continuous with respect to the given topology is called a **topological group**. That is, both the maps $(x, y) \to xy$, from $G \times G$ to G, and $x \to x^{-1}$, from G to G, are continuous.

Definition 1.2.2 A **locally compact group** is a topological group G such that the topolology is locally compact. That is, G is a topological group where each point has at least one compact neighborhood. We will always assume that a locally compact group G is Hausdorff. This assumption is in fact not much of a restriction [cf §A.1].

Definition 1.2.3 Let f be a function on a topological group G and $y \in G$. Then the **left translate** of f through y is defined by $L_y f(x) = f(y^{-1}x)$. Similarly the **right translate** of f through y is defined by $R_y f(x) = f(xy)$. Clearly we have $L_{xy} = L_x L_y$ and $R_{xy} = R_x R_y$.

References: [3], [6], [10].

1.2.2 Normed involutive algebras

Definition 1.2.4 A **normed algebra** is an associative algebra A equipped with a norm on the vector space structure of A which satisfies $||xy|| \leq ||x||||y||$ for all $x, y \in A$. Moreover, if A is complete with respect to this norm metric then we say A is a **Banach algebra**.

Definition 1.2.5 Let A be an algebra over the field **C** of complex numbers. A mapping $x \to x^*$ from A into A is called an **involution** if it satisfies the following properties:

i) $(x^*)^* = x$

ii) $(x+y)^* = x^* + y^*$

iii) $(\lambda x)^* = \bar{\lambda} x^*$

iv) $(xy)^* = y^* x^*$

for any $x, y \in A$, and $\lambda \in \mathbf{C}$. A is said to be an **involutive algebra** if it is endowed with an involution.

Definition 1.2.6 A **normed involutive algebra** is a normed algebra A endowed with an involution $x \to x^*$ such that $\|x^*\| = \|x\|$ for all $x \in A$. If A is also complete with respect to this norm metric then A is called an **involutive Banach algebra**.

Definition 1.2.7 Let A be an involutive algebra, $x \in A$ and $S \subset A$. x^* is called the **adjoint** of x and S is **self-adjoint** if S is closed under involution. As well, $x \in A$ is said to be **hermitian** if $x = x^*$ and **normal** if $xx^* = x^*x$. Denote the set of all hermitian elements of A by A_h. An idempotent hermitian element is called a **projection**, that is, $x = x^* = x^2$.

A fact that we will use often is that each $x \in A$ can be uniquely written in the form $x_1 + ix_2$ with x_1, x_2 hermitian. We simply take $x_1 = (x+x^*)/2$ and $x_2 = (x - x^*)/2i$.

Definition 1.2.8 A multiplicative linear map π from an algebra A into an algebra B is called a **morphism**. That is, $\pi : A \to B$ such that

$$\pi(x+y) = \pi(x) + \pi(y), \quad \pi(\lambda x) = \lambda \pi(x), \quad \pi(xy) = \pi(x)\pi(y),$$

$\forall\, x, y \in A$, $\lambda \in \mathbf{C}$. If, in addition, A and B are involutive algebras and π satisfies the additional property

$$\pi(x^*) = \pi(x)^*$$

for all $x \in A$, then π is said to be an **involutive morphism**.

References: [5], [13], [10], [6].

Chapter 2

C*-Algebras

2.1 C*-Algebras

Definition 2.1.1 A **C*-algebra** is an involutive Banach algebra A such that

$$||x||^2 = ||x^*x||, \forall x \in A.$$

Definition 2.1.2 If A is a C*-algebra and $x = y^*y$ for some $y \in A$ then we say x is a **positive element**. We will denote the set of all positive elements of A by A^+.

2.1.3 If A is a Banach algebra with an involution such that $||x||^2 \leq ||x^*x||$ for all x, then A is a C*-algebra. $||x||^2 \leq ||x^*x||$ for all x implies $||x||^2 \leq ||x^*||||x||$ so that $||x|| \leq ||x^*||$ and interchanging x and x^* yields $||x^*|| = ||x||$. Thus

$$||x||^2 \leq ||x^*x|| \leq ||x^*||||x|| = ||x||^2,$$

and indeed A is a C*-algebra.

Example 2.1.4

i) Let $A = \mathbf{C}$ and set the norm on A to be the modulus $|\cdot|$. Then the map $x \to \bar{x}$, where $\bar{x}$ is the complex conjugate, is an involution on $\mathbf{C}$. By definition $|x| = |\bar{x}|$, hence A is a normed involutive algebra. In fact A is a C*-algebra since $|x|^2 = a^2 + b^2 = \sqrt{(a^2+b^2)^2} = |\bar{x}x|$ where $x = a + ib$.

ii) Let X be a locally compact space. The space $C_0(X)$ of continuous complex-valued functions vanishing at infinity on X with the usual pointwise algebraic operations, the involution $f \to \bar{f}$ and the uniform norm $||f|| = \sup_{t\in X}|f(t)|$ is also a C*-algebra.

iii) Let $\mathcal{H}$ be a complex Hilbert space and $A = \mathcal{L}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$. Then A with the operator norm and involution defined as the usual adjoint operation is a C*-algebra. To see this, from comment 2.1.3 we need only show $||T^*T|| \geq ||T||^2$ since $||T^*T|| \leq ||T^*||||T|| = ||T||^2$ always holds. For any unit vector $u \in \mathcal{H}$, $||T^*T|| \geq \langle T^*Tu, u\rangle = \langle Tu, Tu\rangle = ||Tu||^2$. Thus taking supremum over all such u yields $||T^*T|| \geq ||T||^2$. Thus A is a C*-algebra. It follows that any self-adjoint norm closed subalgebra of A is also a C*-algebra.

iv) Let G be a locally compact group with left Haar measure λ. Let A be the convolution algebra $L^1(G)$ [cf A.1] with an involution given by $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}, \forall f \in L^1(G)$, where Δ is the modular function of G. A equipped with the usual norm $||f|| = \int_G |f|d\lambda$ is an involutive Banach algebra which is *not* a C*-algebra.

2.1.5 If we have a family $(A_n)_{n\in\Delta}$ of C*-algebras we can define the product C*-algebra of the A_n's as follows. Set $A = \{(x_n)_{n\in\Delta} : x_n \in A_n \text{ and } \sup_{n\in\Delta}||x_n|| < +\infty\}$. Then A with the operations

$$(x_n + y_n) = (x_n) + (y_n), \quad (\lambda x_n) = \lambda(x_n),$$
$$(x_n y_n) = (x_n)(y_n), \qquad (x_n^*) = (x_n)^*$$

and norm

$$\|(x_n)\| = \sup_n \|x_n\|$$

is a C*-algebra.

2.1.6 A is said to be **unital** if A possesses a unit element or a multiplicative identity. If A is a unital Banach algebra then

$$\|1\|^2 = \|1^*1\| = \|1\|, \text{ hence } \|1\| = 1 \text{ or } 0.$$

Thus for $A \neq 0$, $\|1\| = 1$.

2.1.7 If A is an involutive Banach algebra then let A_1 be the set of ordered pairs (x, λ), such that $x \in A$ and $\lambda \in \mathbf{C}$. Then A_1 is easily seen to be a vector space with linear operations defined componentwise. Also A_1 is an involutive algebra if multiplication is defined by

$$(x, \lambda)(y, \rho) = (xy + \lambda y + \rho x, \lambda\rho)$$

and involution is defined by

$$(x, \lambda)^* = (x^*, \overline{\lambda}).$$

A_1 is said to be the involutive algebra obtained from A by the adjunction of an identity $\mathbf{1} = (0, 1)$.

The following proposition allows us the freedom to almost always assume that a C*-algebra is unital.

Proposition 2.1.8 *Let A be a C*-algebra. Then the norm on A can be extended to A_1 in exactly one way that makes A_1 a C*-algebra.*

<u>*Proof:*</u> Suppose A has an identity denoted e. Then A and $\mathbf{C}(1-e)$ are complementary self-adjoint closed subspaces of A_1. So there exists an involutive algebraic isomorphism of A_1 onto $A \times \mathbf{C}$ which maps A onto $A \times \{0\}$. $A \times \mathbf{C}$ can then be given the product C*-algebra structure 2.1.5. Then A_1 is isomorphic to a self-adjoint norm closed subalgebra of $A \times \mathbf{C}$ and is hence a C*-algebra.

Now assume A does not have an identity. Consider the following semi-norm of A_1:

$$||(x,\lambda)|| = \sup\{||xy + \lambda y|| : \ y \in A, \ ||y|| \leq 1\}.$$

This is the operator norm on the left multiplication operator on A induced by (x, λ). It is easily seen that $||(x,0)|| = ||x||$, $\forall x \in A$. As well, $\forall\, x, y \in A$ and $\forall\, \lambda, \rho \in \mathbf{C}$ we have

$$\begin{aligned}
||(x,\lambda)(y,\rho)|| &= ||(xy + \lambda y + \rho x, \lambda\rho)|| \\
&= \sup_{z \in (A)_1} ||(xy + \lambda y + \rho x)z + \lambda\rho z|| \\
&= \sup_{z \in (A)_1} ||x(yz + \rho z) + \lambda(yz + \rho z)|| \\
&\leq \sup_{z \in (A)_1} \{||(x,\lambda)||\,||yz + \rho z||\} \\
&\leq \sup_{z \in (A)_1} \{||(x,\lambda)||\,||(y,\rho)||\,||z||\} \\
&= ||(x,\lambda)||\,||(y,\rho)||
\end{aligned}$$

and

$$||(x,\lambda)^*|| \leq ||x^*|| + |\overline{\lambda}| = ||x|| + |\lambda| = ||(x,\lambda)||.$$

Hence $||(x,\lambda)|| = ||(x,\lambda)^{**}|| \leq ||(x,\lambda)^*||$, so $||(x,\lambda)^*|| = ||(x,\lambda)||$.

To show that $||\ \ ||$ is actually a norm on A_1 suppose $||(x,\lambda)|| = 0$. If $\lambda = 0$ then $||x|| = ||(x,0)|| = 0$ which implies $x = 0$. If $\lambda \neq 0$ we can consider $e = -x/\lambda$ in A. Since $xy + \lambda y = 0\ \forall y \in A$ we have $ey = y\ \forall y \in A$, so that e is a left identity for A.

Taking adjoints we see e^* is a right identity for A. Hence $e = ee^* = e^*$ is an identity for A which contradicts our assumption that A does not have an identity. Therefore $||(x,\lambda)|| = 0$ iff $(x,\lambda) = (0,0)$, hence $||\ ||$ is a norm on A_1.

Finally we show that

$$||(x,\lambda)||^2 = ||(x,\lambda)^*(x,\lambda)||, \quad x \in A, \lambda \in \mathbf{C}.$$

By 2.1.3 we need only show that

$$||(x,\lambda)||^2 \leq ||(x,\lambda)^*(x,\lambda)||, \quad x \in A, \lambda \in \mathbf{C}.$$

In fact it suffices to show this for $||(x,\lambda)|| = 1$. For each $0 < r < 1$ we can find a $y \in A$ such that $||y|| \leq 1$ and $||(x,\lambda)(y,0)|| \geq r$. Since each $(x,\lambda)(y,0)$ belongs to $[A \times \{0\}]$, we have

$$\begin{aligned} ||(x,\lambda)^*(x,\lambda)|| &\geq ||(y,0)^*(x,\lambda)^*(x,\lambda)(y,0)\ | \\ &= ||[(x,\lambda)(y,0)]^*[(x,\lambda)(y,0)]|| \\ &= |||(x,\lambda)(y,0)||^2 \geq r^2, \end{aligned}$$

and therefore

$$||(x,\lambda)^*(x,\lambda)|| \geq 1 = ||(x,\lambda)||^2.$$

□

Theorem 2.1.9 *If I is a closed two-sided ideal of a C*-algebra A, then I is a C*-subalgebra of A and the quotient A/I is a C*-algebra under its usual operations and the quotient norm.*

Proof: Let $\{e_i\}$ be an approximate identity for I as found in section A.3. Then for all

$a \in I$ we have

$$\|a^*e_i - a^*\| = \|e_ia - a\| \to 0,$$

and $a^*e_i \in I$, so $a^* \in \overline{I} = I$ and hence I is self-adjoint closed subalgebra of A. Thus I is a C*-subalgebra of A.

By 2.1.8, we assume A is unital. It is easily verified that the quotient is an involutive Banach algebra. Hence, we need only show that the quotient norm is a C*-norm. Let $a \in A$ and $\epsilon > 0$ then we can find a $b \in I$ such that $\|a+b\| < \|a+I\| + \epsilon/2$. Since $\|b - e_ib\| \to 0$ there exists i_0 such that $\|b - e_ib\| < \epsilon/2$ for all $i \geq i_0$, and therefore

$$\begin{aligned} \|a - e_ia\| &\leq \|(1-e_i)(a+b)\| + \|b - e_ib\| \\ &\leq \|a+b\| + \|b - e_ib\| \\ &< \|a+I\| + \epsilon/2 + \epsilon/2. \end{aligned}$$

Hence $\|a+I\| = \lim_i \|a - e_ia\|$. So now if $a \in A$ and $b \in I$, then

$$\begin{aligned} \|a+I\|^2 &= \lim_i \|a - e_ia\|^2 \\ &= \lim_i \|(1-e_i)a^*a(1-e_i)\| \\ &\leq \sup_i \|(1-e_i)(a^*a+b)(1-e_i)\| + \lim_i \|(1-e_i)b(1-e_i)\| \\ &\leq \|a^*a+b\| + \lim_i \|b - e_ib\| \\ &= \|a^*a+b\|. \end{aligned}$$

Thus, $\|a+I\|^2 \leq \|a^*a+I\|$ so by comment 2.1.3 A/I is a C*-algebra. □

References: [5], [13], [10], [6], [16], [20].

2.2 Positive Forms and Representations

The aim of this section is to study the relationship between positive forms and representations. As the reader shall see, representations give rise to positive forms in straight forward way [2.2.16]. However, the construction of a representation from a positive form is not as straight forward. The basic construction described here [2.2.5] is known as the Gelfand-Naimark-Segal [or GNS] construction.

Definition 2.2.1 Let A be a normed involutive algebra, let A' denote the algebraic dual of A, that is, all linear forms on A and let $A^* \subset A'$ denote the continuous dual of A, that is all continuous linear forms on A . $f \in A'$ is said to be **positive** if $f(x^*x) \geq 0$ for all $x \in A$, or, equivalently, $f(x) \geq 0$ for all $x \in A^+$. Set $A'^+ = \{f \in A' : f \text{ is positive}\}$ and $A^{*+} = \{f \in A^* : f \text{ is positive}\}$. A *continuous* linear form $f \in A^{*+}$ is called a **state** of A if $||f|| = 1$. We will denote the set of all states of A by $S(A)$. Further, we say $f \in A'^+$ is **extendable** if A is unital. If f can be extended to a $g \in {A_1}'^+$, where A_1 is the involutive algebra obtained from A by the adjunction of an identity.

2.2.2 Let A be an involutive Banach algebra with approximate identity and let $f \in A'^+$. We can define a sesquilinear form $(x, y) \rightarrow (x|y)$ on $A \times A$ such that $(x|y) = f(y^*x)$, $\forall\ x, y \in A$. By applying Schwarz's inequality and the polarization identity we see that this form satisfies for $x, y \in A$,

$$(x|y) = f(y^*x) = \overline{f(x^*y)} = \overline{(y|x)}, \tag{2.1}$$

$$|(x|y)|^2 = |f(y^*x)|^2 \leq f(x^*x)f(y^*y) = (x|x)(y|y). \tag{2.2}$$

Proposition 2.2.3 *Let A an involutive Banach algebra with an approximate identity. Then $A'^+ = A^{*+}$.*

Proof: **Claim:** *Let $f \in A'^{+}$ and let $a \in A$ be fixed. Then the linear form $x \to f(a^*xa)$ is an extendable element of A'^{+}.*

We can assume A is non-unital. Clearly for any $x \in A$ we have $f(a^*x^*xa) = f((xa)^*xa) \geq 0$, so the map $x \to f(a^*xa)$ is an element of A'^{+}. As well, since $(A, 0)$ is a two sided ideal of A_1 the equation $f_1((x, \lambda)) = f(a^*xa)$ defines a linear form $f_1 \in A_1'$ which extends f and $f_1((x, \lambda)^*(x, \lambda)) = f(a^*x^*xa) = f((xa)^*xa) \geq 0$. So $f_1 \in A_1'^{+}$.

Claim: *An extendable element of A'^{+} is continuous.*

Suppose A is unital. If $x \in A$ is hermitian and $||x|| \leq 1$ then $1-x$ is a positive element of A [5, Lemma 2.1.3]. Hence, for $f \in A'^{+}$ we have $f(1-x) \geq 0 \Rightarrow f(1) \geq f(x)$. Similarly $-f(x) = f(-x) \leq f(1)$. Thus $|f(x)| \leq f(1)$ for all hermitian $x \in A$ such that $||x|| \leq 1$. Now for arbitrary $y \in A$ with $||y|| \leq 1$ we have $||y^*y|| \leq 1$ [recall y^*y is hermitian for arbitrary y], so from 2.2.2 equation 2.2

$$|f(y)|^2 \leq f(1)f(y^*y) \leq f(1)^2$$

by what was proved above. Therefore $||f|| \leq f(1)$, so f is continuous.

Now if A is not unital then for any extendable $f \in A'^{+}$ we have a positive linear functional in the unital algebra A_1 which extends f, and from the above argument it is continuous.

Claim: *Let $f \in A'^{+}$. Then for any $a, b \in A$ the linear form $x \to f(axb)$ is continuous.*

¿From our first two claims and in light of the polarization identity,

$$4axb = \sum_{n=0}^{3} i^n(a + i^n b^*)x(a + i^n b^*)^*$$

it follows directly that $x \to f(axb)$ is continuous.

Claim: $A'^{+} = A^{*+}$.

Let $f \in A'^{+}$. Since f is linear it suffices to prove f is continuous at 0. Let $\{x_i\}$ be an arbitrary sequence in A such that $x_i \to 0$. The Cohen-Hewitt factorization theorem [13, V.9.2] basically states that if B is a Banach algebra with an approximate identity, then for $c \in B$ there are elements $a, b \in B$ such that $c = ab$, where b belongs to the closed left ideal generated by c, and b is arbitrarily close to c. A corollary of this factorization theorem [13, V.9.3] yields a factorization of null sequences of A. This factorization can be used to produce a sequence $\{y_i\}$ in A such that $y_i \to 0$ and for each i we have $x_i = ay_ib$ for fixed $a, b \in A$. Thus from the continuity of $x \to f(axb)$ we have $f(x_i) = f(ay_ib) \to 0$. Therefore f is continuous, that is, $f \in A^{*+}$. □

Corollary 2.2.4 *If A is a C*-algebra then $A'^{+} = A^{*+}$.*

<u>*Proof:*</u> By definition A is an involutive Banach algebra and Corollary A.2.2 guarantees the existence of an approximate identity. The result now follows directly from proposition 2.2.3. □

2.2.5 [The GNS construction] ¿From equation 2.2 it is easily seen that the subset $N_f := \{x \in A : f(x^*x) = 0\}$ of A is a left ideal of A. That is, $f(x^*x) = 0$ if and only if $f(yx) = 0$, $\forall y \in A$. Thus A/N_f is an inner product space constructed canonically from A. We will denote by $\mathcal{H}_f$ the Hilbert space which is the completion of A/N_f with this inner product norm. Now for $f \in A'^{+}$ and for $x \in A$ define an operator $\pi(x)$ from A/N_f into A/N_f such that

$$\pi(x)(y + N_f) = xy + N_f. \tag{2.3}$$

Claim: *If A is an involutive Banach algebra and π is the operator described by 2.3, then π can be uniquely extended to an involutive morphism π_f of A into $\mathcal{L}(\mathcal{H}_f)$.*

<u>*Proof:*</u> For $x, y \in A$ it is easily seen that $||\pi(x)(y + N_f)||^2 = f(y^*x^*xy)$. Now assuming that $f(y^*x^*xy) \neq 0$ then from our above observations we have $f(y^*y) > 0$. Define a

function $g : A \to \mathbf{C}$ such that $g(z) = f(y^*zy)/f(y^*y)$. Clearly g is positive and linear, so letting $\{e_i\}$ be the approximate identity for A we have

$$\|g\| = \lim_i g(e_i) = \lim_i f(y^*e_iy)/f(y^*y) = f(y^*y)/f(y^*y) = 1.$$

It follows that $g(x^*x) \leq \|x^*x\| \leq \|x\|^2$, so

$$f(y^*x^*xy) \leq \|x\|^2 f(y^*y) = \|x\|^2\|y + N_f\|^2.$$

Combining this inequality with our initial comment yields

$$\|\pi(x)(y + N_f)\|^2 \leq \|x\|^2\|y + N_f\|^2 \Rightarrow \|\pi(x)\| \leq \|x\|,$$

hence $\pi(x) \in \mathcal{L}(A/N_f)$, $\forall x \in A$. Thus $\pi(x)$ has a continuous extension to an operator $\pi_f(x) \in \mathcal{L}(\mathcal{H}_f)$. It is readily seen that π is an involutive morphism of the algebraic structure of A into $\mathcal{L}(A/N_f)$ and that the algebraic preserving properties of π extend to π_f. Finally, if $\xi \in \mathcal{H}_f$ then there exists a net $\{y_i + N_f\}$ in A/N_f such that $\xi = \lim_i(y_i + N_f)$. So

$$\begin{aligned}
\pi_f(x)^*(\xi) &= \lim_i \pi_f(x)^*(y_i + N_f) \\
&= \lim_i \pi(x)^*(y_i + N_f) \\
&= \lim_i \pi(x^*)(y_i + N_f) \\
&= \lim_i \pi_f(x^*)(y_i + N_f) \\
&= \pi_f(x^*)(\xi).
\end{aligned}$$

Hence π_f is an involutive morphism of A into $\mathcal{L}(\mathcal{H}_f)$. □

Definition 2.2.6 Let A be an involutive algebra and $\mathcal{H}$ a Hilbert space. A **representation** of A in $\mathcal{H}$ is an involutive morphism π of A into $\mathcal{L}(\mathcal{H})$. In other words $\pi : A \to \mathcal{L}(\mathcal{H})$ such that

$$\pi(x+y) = \pi(x) + \pi(y), \quad \pi(\lambda x) = \lambda\pi(x),$$
$$\pi(xy) = \pi(x)\pi(y), \qquad \pi(x^*) = \pi(x)^*$$

$\forall\, x, y \in A$, $\lambda \in \mathbf{C}$. We will denote the set of all representaions of A by $R(A)$. We will also often use $\mathcal{H}_\pi$ to denote the Hilbert space associated with π. **Note:** Many authors define a representation more generally on an algebra and exclude the condition $\pi(x^*) = \pi(x)^*$. Our definition therefore coincides with *-representations or involutive representations found elsewhere in the literature.

Proposition 2.2.7 *Let A be an involutive Banach algebra, B a C*-algebra and π a morphism of A into B. Then $||\pi(a)|| \le ||a||$ for all $a \in A$; so π is continuous.*

Proof: For any $b \in B_h$ we have $||b^2|| = ||b^*b|| = ||b||^2$ and hence by induction $||b^{2^n}||^{2^{-n}} = ||b||$. Then $\lim_{n\to\infty} ||b^{2^n}||^{2^{-n}} = r(b)$ where $r(b)$ is the spectral radius of b [20, theorem 5.5]. For any $a \in A$ we have the spectrum of $\pi(a)$ as an element of B_1 is contained in the spectrum of a as an element of A_1 so that $r(\pi(a)) \le r(a) \le ||a||$ [20, theorem 3.3]. Therefore,

$$||\pi(a)||^2 = ||\pi(a^*a)|| = r(\pi(a^*a)) \le ||a^*a|| \le ||a^*||\,||a|| = ||a||^2.$$

□

Corollary 2.2.8 *Let A be a C*-algebra. Then each $\pi \in R(A)$ is continuous.*

Proof: Since $\mathcal{L}(\mathcal{H})$ is a C*-algebra [2.1.4 *iii*)] the result follows directly from proposition 2.2.7. □

2.2.9 ¿From 2.2.5 we can see that π_f is a representation associated with the Hilbert space $\mathcal{H}_f$. This representation is the so called *Gelfand-Naimark-Segal representation* or GNS representaion associated to f.

2.2.10 Let π, $\pi' \in R(A)$ and let $\mathcal{H}$ and $\mathcal{H}'$ be the corresponding Hilbert spaces. We say π is equivalent to π' if there exists an isomorphism U from $\mathcal{H}$ onto $\mathcal{H}'$ such that $U\pi(x) = \pi'(x)U$, $\forall\, x \in A$. It is easily seen that this is indeed an equivalence relation on $R(A)$. Therefore we can talk about a *class of representations*. We will denote the equivalence class of π by $[\pi]$.

As well, any continuous linear operator $T : \mathcal{H} \to \mathcal{H}'$ such that $T\pi(x) = \pi'(x)T$, for all $x \in A$, is called an *intertwining operator* for π and π'. The set of all intertwining operators for π and π' is a vector space whose dimension is called the *intertwining number* of π and π'.

Definition 2.2.11 Let $S \subset R(A)$. Then we can construct a continuous linear operator $\pi(x)$ in $\bigoplus_{\rho\in S}\mathcal{H}_\rho$ which induces $\rho(x)$ in each $\mathcal{H}_\rho$ since $\{||\rho(x)|| \,:\, \rho \in S\}$ is bounded for each $x \in A$. It follows that $x \to \pi(x)$ is a representation of A in $\bigoplus_{\rho\in S}\mathcal{H}_\rho$. We will denote this representation, called the **Hilbert sum** of $\rho \in S$, by $\bigoplus_S \rho$.

Definition 2.2.12 Let A be an involutive algebra and let $\pi \in R(A)$.

i) If $\xi \in \mathcal{H}_\pi$ then the closure of $\pi(A)\xi$ is a closed subpace of $\mathcal{H}$. If this closure $\overline{\pi(A)\xi} = \mathcal{H}_\pi$ then we say that ξ is a **cyclic vector** for π.

ii) We say that π is **non-degenerate** or that $\pi(A)$ acts non-degenerately on $\mathcal{H}_\pi$ if the closure of the linear span of the set $\{\pi(x)\xi \,:\, x \in A,\ \xi \in \mathcal{H}_\pi\}$ which we will denote as $[\pi(A)\mathcal{H}_\pi]$, is equal to $\mathcal{H}_\pi$. Equivalently, π is non-degenerate if for each non-zero $\xi \in \mathcal{H}_\pi$ there exists $x \in A$ such that $\pi(x)\xi \neq 0$. **Note:** by definition any representation that admits a cyclic vector is non-degenerate.

Proposition 2.2.13 *Let A be an involutive algebra. Let π and π' be representations of A in $\mathcal{H}$ and $\mathcal{H}'$ respectively, and let $\xi\{\xi'\}$ be a cyclic vector for $\pi\{\pi'\}$. If $(\pi(x)\xi|\xi) = (\pi'(x)\xi'|\xi')$ $\forall x \in A$ then π and π' are equivalent, that is, there is a unique isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$ mapping $\pi(x)$ to $\pi'(x)$ and ξ to ξ'.*

Proof: Suppose the conditions of the proposition are satisfied. Thus for any $x, y \in A$ we have

$$(\pi(x)\xi|\pi(y)\xi) = (\pi(y^*x)\xi|\xi) = (\pi'(y^*x)\xi|\xi) = (\pi'(x)\xi'|\pi'(y)\xi').$$

Since $\xi\{\xi'\}$ is a cyclic vector for $\pi\{\pi'\}$ we have $\pi(A)\xi$ $\{\pi'(A)\xi'\}$ is dense in $\mathcal{H}\{\mathcal{H}'\}$. Thus there is an isomorphism U of $\mathcal{H}$ onto $\mathcal{H}'$ such that $U(\pi(x)\xi) = \pi'(x)\xi'$ for all $x \in A$. So for any $y \in A$ we have

$$(U\pi(x))(\pi(y)\xi) = U\pi(xy)\xi = \pi'(xy)\xi' = \pi'(x)(\pi'(y)\xi') = (\pi'(x)U)(\pi(y)\xi).$$

Thus $U\pi(x) = \pi'(x)U$ for each $x \in A$ since $\pi(y)\xi$ are dense in $\mathcal{H}$. As well we have for all $x \in A$

$$(\pi'(x)\xi'|\xi') = (\pi(x)\xi|\xi) = (U\pi(x)\xi|U\xi) = (\pi'(x)\xi'|U\xi),$$

which implies $U\xi = \xi'$. Finally the uniquness of U is apparent since the values that U takes on $\pi(A)\xi$, which is dense in A, are predetermined. □

In the following, we continue to use the notation and the results from the GNS construction, 2.2.5.

Proposition 2.2.14 *Let A be an involutive Banach algebra with an approximate identity and $f \in S(A)$. Then there exists a unique vector $\xi_f \in \mathcal{H}_f$ such that for all $x \in A$*

i) $f(x) = (x + N_f|\xi_f)$

ii) $\pi_f(x)\xi_f = x + N_f$ *and* ξ_f *is a unit cyclic vector for* π_f.

Proof: The map $x + N_f \to f(x)$ is a well defined norm-decreasing linear form on A/N_f and as such can be extended to a norm-decreasing linear form on $\mathcal{H}_f$. Thus by the Riesz representation theorem there exists a unique vector ξ_f in $\mathcal{H}_f$ such that this linear form maps $\eta \to (\eta|\xi_f)\ \forall \eta \in \mathcal{H}_f$. Therefore $f(x) = (x + N_f|\xi_f)$ for all $x \in A$ and ξ_f is the desired vector for i).

Now let $x, y \in A$. Then

$$(y + N_f|\pi_f(x)\xi_f) = (x^*y + N_f|\xi_f) = f(x^*y) = (y + N_f|x + N_f).$$

Since y was arbitrary we have $\pi_f(x)\xi_f = x + N_f$. That $\pi_f(A)\xi_f = A/N_f$ is dense in $\mathcal{H}_f$ implies that ξ_f is cyclic for π_f. It follows that $[\pi_f(A)\xi_f] \subseteq [\pi_f(A)\mathcal{H}_f]$ is equal to $\mathcal{H}_f$. That is, π_f is non-degenerate. If $\{e_i\}$ is an approximate identity in A then $\{\pi_f(e_i)\}$ is an approximate identity for $\mathcal{L}(\mathcal{H}_f)$ and therefore converges to I, the identity in $\mathcal{L}(\mathcal{H}_f)$. Thus

$$\|\xi_f\|^2 = (\xi_f|\xi_f) = \lim_i(\pi_f(e_i)\xi_f|\xi_f) = \lim_i f(e_i) = \|f\| = 1$$

and therefore ξ_f is a unit vector. □

Proposition 2.2.15 *Let A be an involutive Banach algebra with an approximate identity. If* $\pi \in R(A)$ *and* ξ *is a unit cyclic vector for* π *then the map* $f : A \to \mathbf{C}$ *such that* $f(x) = (\pi(x)\xi|\xi)$ *is in* $S(A)$. *Moreover, we have* π *is equivalent to* π_f *(see 2.2.5).*

Proof: Consider

$$(\pi(x^*x)\xi|\xi) = (\pi(x)^*\pi(x)\xi|\xi) = \|\pi(x)\xi\|^2 \geq 0,$$

so $f \in A'^{+}$. Since π admits a cyclic vector, namely ξ, it is non-degenerate. Thus taking the approximate identity $\{e_i\}$ for A we have that the net $\{\pi(e_i)\}$ is strongly convergent to I, the identity operator in $\mathcal{H}_\pi$. So

$$||f|| = \lim_i f(e_i) = \lim_i (\pi(e_i)\xi|\xi) = (\xi|\xi) = 1,$$

showing that $f \in S(A)$. Finally since

$$(\pi(x)\xi|\xi) = f(x) = (\pi_f(x)\xi_f|\xi_f) \ \forall x \in A$$

proposition 2.2.13 states π is equivalent to π_f. □

2.2.16 Note that in the above proof we did not need ξ to be normalized or cyclic to show that the map $x \to (\pi(x)\xi|\xi)$ is in A^{*+}. Thus for any $\pi \in R(A)$ and $\xi \in \mathcal{H}_\pi$ we have $x \to (\pi(x)\xi|\xi)$ is in A^{*+}. We call this map the *form defined by* π and ξ. If π is fixed and we allow ξ to vary then we obtain the *forms associated with* π.

References: [5], [8], [13], [10], [6], [16], [20].

2.3 Pure States and Irreducible Representations

In §2.2 we have shown how to associate representions and positive forms. We now set out to see what type of positive forms relate specifically to irreducible representations. In fact, as we shall state in 3.2.1, the set of pure states is exactly the subset of the positive forms we desire.

Definition 2.3.1 Let A be a normed involutive algebra. $f \in A^{*+}$ is called **pure** if $f \neq 0$ and every $g \in A^{*+}$ dominated by f [f dominates g iff $f - g \in A^{*+}$] is of the form $g = \lambda f$ where $0 \leq \lambda \leq 1$. $P(A)$ will denote the set of pure states of A.

Lemma 2.3.2 *Let A be a C^*-algebra, $f \in P(A)$ and $g \in A^{*+}$ such that f dominates g. Then there exists a unique operator T in the commutant [cf. A.4.3] of $\pi_f(A)$ such that $0 \leq T \leq I$ and*

$$g(x) = (\pi_f(x)T\xi_f|\xi_f), \ \forall x \in A.$$

Proof: Let ξ_f be the cyclic unit vector associated with the representation π_f [2.2.14]. For $x, y \in A$

$$|g(y^*x)|^2 \leq g(x^*x)g(y^*y) \leq f(x^*x)f(y^*y) = \|\pi_f(x)\xi_f\|^2\|\pi_f(y)\xi_f\|^2.$$

So $(x + N_f, y + N_f)_g = g(y^*x)$ defines a unique positive continuous sesquilinear form on A/N_f. We can therefore extend $(\ , \ )_g$ to a bounded sesquilinear form $(\ | \ )_g$ on $\mathcal{H}_f$ with norm ≤ 1. Hence by [16, theorem 2.3.6] there is an operator T on $\mathcal{H}_f$ such that for all $\xi, \eta \in \mathcal{H}_f$, $(T\xi|\eta) = (\xi|\eta)_g$ and $\|T\| \leq 1$. So

$$g(y^*x) = (x + N_f|y + N_f)_g = (T(x + N_f)|y + N_f) = (T\pi_f(x)\xi_f|\pi_f(y)\xi_f).$$

Hence $(T(x + N_f)|x + N_f) \geq 0$, $\forall x \in A$, so T is positive. For $x, y, z \in A$, we have

$$\begin{aligned}(\pi_f(x)T(y + N_f)|z + N_f) &= (T(y + N_f)|x^*z + N_f) = g(z^*xy) \\ &= (T(xy + N_f)|z + N_f) = (T\pi_f(x)(y + N_f)|z + N_f)\end{aligned}$$

Hence, $\pi_f(x)T = T\pi_f(x)$ for all $x \in A$. Hence T is in the commutant of $\pi_f(A)$. As well,

$$g(y^*x) = (T(x + N_f)|y + N_f) = (T\pi_f(x)\xi_f|\pi_f(y)\xi_f) = (T\pi_f(y^*x)\xi_f|\xi_f);$$

so if $\{e_i\}$ is an approximate identity for A then $g(e_ix) = (T\pi_f(e_ix)\xi_f|\xi_f)$ and taking limits yields $g(x) = (T\pi_f(x)\xi_f|\xi_f)$. Finally uniquness follows because if T_0 also satisfies the above conditions of T then

$$\begin{aligned}(T_0(x+N_f)|y+N_f) &= (T_0\pi_f(y^*x)\xi_f|\xi_f)\\ &= g(y^*x)\\ &= (T\pi_f(y^*x)\xi_f|\xi_f)\\ &= (T(x+N_f)|y+N_f).\end{aligned}$$

for all $x, y \in A$. So $T_0 = T$. □

Proposition 2.3.3 *Let A be an involutive Banach algebra with an approximate identity.*

i) The norm closed unit ball $(A'^+)_1$ of A'^+ is convex and compact in the weak-topology of A^*.*

ii) $S(A)$ is convex and compact in the weak-topology of A^*.*

iii) The set of extreme points of $(A'^+)_1$ is equal to $P(A)\bigcup\{0\}$.

Proof: Suppose f is in the weak*-closure of $(A'^+)_1$. Then there is a net $\{f_i\}$ in $(A'^+)_1$ such that $f_i(x) \to f(x)$, $\forall x \in A$. Thus for all $x \in A$ we have $f(x^*x) = \lim_i f_i(x^*x) \geq 0$, which shows $f \geq 0$. For all $x \in (A)_1$ we see $|f(x)| = \lim_i |f_i(x)| \leq 1$. Thus f is in $(A'^+)_1$, so $(A'^+)_1$ is weak*-closed. Now Let $g(x) = \lambda f_1(x) + (1-\lambda)f_2(x)$ for all $x \in A$, where $f_1, f_2 \in (A'^+)_1$ and $0 \leq \lambda \leq 1$. Then $g(x^*x) = \lambda f_1(x^*x) + (1-\lambda)f_2(x^*x) \geq 0$ so g is positive. As well, $|g(x)| = |\lambda f_1(x) + (1-\lambda)f_2(x)| \leq \lambda|f_1(x)| + (1-\lambda)|f_2(x)| \leq 1$ for all $x \in A$, so $g \in (A'^+)_1$. Thus $(A'^+)_1$ is convex. Since $(A'^+)_1$ is a weak*-closed subset of $(A')_1$, by Alaoglu's theorem $(A')_1$ is weak*-compact. This proves i).

With a similar argument as above and with the addition of

$$\begin{aligned} \lim_i |g(e_i)| &= \lim_i |\lambda f_1(e_i) + (1-\lambda) f_2(e_i)| \\ &\leq \lambda \lim_i |f_1(e_i)| + (1-\lambda) \lim_i |f_2(e_i)| = 1 \end{aligned}$$

for $f_1, f_2 \in S(A)$, and $\{e_i\}$ an approximate identity of A, we have $g \in S(A)$, which proves $ii)$.

To show $iii)$ we first show that 0 is an extreme point of $(A'^+)_1$. Suppose $0 = \lambda f_1 + (1-\lambda) f_2$, where $\lambda \in (0,1)$ and $f_1, f_2 \in (A'^+)_1$. Then $0 \geq -\lambda f_1(x^*x) = (1-\lambda) f_2(x^*x) \geq 0$ for all $x \in A$. Hence $f_1 = f_2 = 0$ on the positive elements of A and therefore on A. So 0 is an extreme point of $(A'^+)_1$.

To see that $P(A)$ is also contained in the set of extreme points of $(A'^+)_1$ let $f \in P(A)$ and suppose $f = \lambda f_1 + (1-\lambda) f_2$, where $0 < \lambda < 1$ and $f_1, f_2 \in (A'^+)_1$. Then λf_1 is dominated by f, so $\lambda f_1 = \mu f$ for $0 \leq \mu \leq 1$. Since

$$1 = ||f|| = \lambda ||f_1|| + (1-\lambda) ||f_2||$$

we see that $||f_1|| = ||f_2|| = 1$. So $\lambda = \mu$ and $f_1 = f = f_2$. Thus f is an extreme point.

Finally, to see the reverse inclusion, let g be a non-zero extreme point of $(A'^+)_1$. Since $g = ||g||(g/||g||) + (1 - ||g||)0$ and $0, g/||g|| \in (A'^+)_1$ we have $||g|| = 1$. Let $h \in A'^+$ be non-zero and such that g strictly dominates h. Then $||h|| \in (0,1)$. Since both g and $g-h$ are positive, $1 - ||h|| = ||g-h||$. It follows from the facts that $g = ||h||[h/||h||] + (1-||h||)[(g-h)/||g-h||]$ and g is an extreme point that $g = [h/||h||]$. So $h = ||h||g$. Thus $g \in P(A)$. □

Definition 2.3.4 Let A be an involutive algebra. If $\pi \in R(A)$, $\mathcal{H}_\pi \neq 0$, and the only closed subspaces of $\mathcal{H}_\pi$ invariant under $\pi(A)$ are 0 and $\mathcal{H}_\pi$, then π is said to be

topologically irreducible and we say that $\pi(A)$ acts irreducibly on $\mathcal{H}_\pi$. Let $r(A)$ denote the set of non-trivial topologically irreducible representaions of A.

Theorem 2.3.5 *Let A be an involutive Banach algebra with an approximate identity. Then $f \in A'^{+}$ is pure iff $\pi_f \in r(A)$.*

Proof: First, suppose $f \in P(A)$. Then $f(x) = (\pi_f(x)\xi_f|\xi_f)$. Since $f(x) \neq 0$ for some $x \in A$ it follows that $\pi_f(x) \neq 0$; so π_f is non-zero. Now let P be a projection in $\mathcal{H}_f$ which commutes with $\pi_f(A)$. Then the form g on A such that $g(x) = (\pi_f(x)P\xi_f|\xi_f) = (\pi_f(x)P\xi_f|P\xi_f)$ is in A'^{+}. As well, for $x \in A$

$$\begin{aligned} g(x^*x) &= (\pi_f(x^*x)P\xi_f|P\xi_f) = \|\pi_f(x)P\xi_f\|^2 \\ &= \|P\pi_f(x)\xi_f\|^2 \leq \|\pi_f(x)\xi_f\|^2 = f(x^*x), \end{aligned}$$

so that g is dominated by f. Hence there exists $\lambda \in [0,1]$ such that $g = \lambda f$ and therefore $(\pi_f(x)P\xi_f|\xi_f) = (\lambda\pi_f(x)\xi_f|\xi_f)$ for all $x \in A$. Thus

$$\begin{aligned} (P(x+N_f)|y+N_f) &= (P\pi_f(x)\xi_f|\pi_f(y)\xi_f) \\ &= (P\pi_f(y^*x)\xi_f|\xi_f) \\ &= (\lambda\pi_f(y^*x)\xi_f|\xi_f) \\ &= (\lambda(x+N_f)|y+N_f) \end{aligned}$$

for all $x, y \in A$. Since A/N_f is dense in $\mathcal{H}_f$ we have $\lambda I = P = P^2$, where I is the identity for $\mathcal{H}_f$. Hence $\lambda = 0$ or 1; so $P = 0$ or I. Thus $\pi_f \in r(A)$

Now assume $\pi_f \in r(A)$. Then there exists an $x \in A$ such that $(\pi_f(x)\xi_f|\xi_f) \neq 0$; so $f \neq 0$. Let $g \in A'^{+}$ be dominated by f. Then by lemma 2.3.2 there exists a unique operator T in the commutant of $\pi_f(A)$ such that $0 \leq T \leq I$ and $g(x) = (\pi_f(x)T\xi_f|\xi_f)$

for all $x \in A$. However by A.5.5 the commutant of $\pi_f(A)$ is exactly $\mathbf{C}I$; so $T = \lambda I$ for $\lambda \in [0,1]$. It follows that $g = \lambda f$. Hence $f \in P(A)$. □

Definition 2.3.6 **Algebraic irreducibility** of $\pi \in R(A)$ means that the *only* [not necessarily closed] vector subspaces of $\mathcal{H}_\pi$ invariant under $\pi(A)$ are 0 and $\mathcal{H}_\pi$. Denote the set of all algebraically irreducible $\pi \in R(A)$ by $a(A)$. Clearly if $\dim\mathcal{H}_\pi < \infty$ then all subspaces of $\mathcal{H}_\pi$ are closed so that $r(A) = a(A)$. However if $\dim\mathcal{H}_\pi = \infty$ then algebraic irreducibility is far more restrictive then topological irreducibility. However we will show [cf proposition 2.3.8] that if A is a C*-algebra then algebraic and topological irreducibility coincide.

2.3.7 Note that if $\pi \in r(A)$ then π is non-degenerate. For if $\pi \in r(A)$, we have $\pi(A)[\pi(A)\mathcal{H}_\pi] \subseteq [\pi(A)\mathcal{H}_\pi]$, that is, $[\pi(A)\mathcal{H}_\pi]$ is invariant under $\pi(A)$. Since π is not trivial, $[\pi(A)\mathcal{H}_\pi] \neq 0$. Hence $[\pi(A)\mathcal{H}_\pi] = \mathcal{H}_\pi$ so that π is non-degenerate. In fact, for any non-zero vector $\xi \in \mathcal{H}_\pi$, the closed linear span of $\{\pi(x)\xi : x \in A\}$ is non-zero and invariant under π. Therefore, if $\pi \in r(A)$ then every non-zero $\xi \in \mathcal{H}_\pi$ is a cyclic vector for π.

Proposition 2.3.8 *Let A be a C*-algebra. Then* $r(A) = a(A)$.

<u>*Proof:*</u> By definition we have $r(A) \supset a(A)$. To see the reverse inclusion let $\pi \in r(A)$ and let S be a non-zero subspace of $\mathcal{H}_\pi$ invarient under $\pi(A)$. Let $\xi \in S$ be non-zero and let $\eta \in \mathcal{H}_\pi$ be arbitrary. Then by the transitivity theorem A.4.2 there exists $x \in A$ such that $\pi(x)\xi = \frac{\eta}{\|\eta\|}$, which implies $\eta \in S$. Therefore, $S = \mathcal{H}_\pi$ and since S was arbitrary we have $\pi \in a(A)$. □

Proposition 2.3.9 *Let A be a C*-algebra and let* $f \in P(A)$. *Then* $A/N_f = \mathcal{H}_f$.

Proof: We simply need to show that A/N_f is complete. By proposition 2.3.5 $\pi_f \in r(A)$. Also $\pi_f \in a(A)$ by 2.3.8. Hence the vector subspace A/N_f of $\mathcal{H}_f$, which by 2.3.7 is invariant for $\pi_f(A)$, must equal 0 or $\mathcal{H}_f$. Since $f \neq 0$ it follows that $N_f \neq A$. Therefore $A/N_f = \mathcal{H}_f$. □

References: [5], [8], [13], [6], [16], [20].

2.4 Equivalence Theorem

2.4.1 If B is an involutive subalgebra of $\mathcal{L}(\mathcal{H})$ and $\xi \in \mathcal{H}$ then the positive form $x \to (x\xi|\xi)$ for $x \in B$ is denoted by f_ξ. Further, if A is C*-algebra, $\pi \in R(A)$ is nondegenerate, $\xi \in \mathcal{H}_\pi$, and f the positive form defined by π and ξ then

$$||f|| = \lim_{i\to\infty} f(e_i) = \lim_{i\to\infty}(\pi(e_i)\xi|\xi) = (\xi|\xi)$$

where $\{e_i\}$ is an approximate identity [A.3.2] for A. So if f_1 and π_1 are the canonical extensions of f and π to A_1 then

$$f_1(x) = (\pi_1(x)\xi|\xi) \quad \forall x \in A_1.$$

In particular, if π is the non-degenerate identical representation of a non-unital C*-subalgebra A of $\mathcal{L}(\mathcal{H})$ then the canonical extension of $f_\xi|_A$ is $f_\xi|_{A_1}$.

2.4.2 We say that $Q \subset S(A)$ satisfies condition **COND Q** if Q satisfies the following: *if* $x \in A_h$ is such that $f(x) \geq 0$ for each $f \in Q$ *then* $x \in A^+$.

Lemma 2.4.3 *Let A be a unital C*algebra, and $Q \subset S(A)$ such that Q satisfies* **COND** *Q. Then the weak*-closed convex hull of Q, $\overline{co(Q)}^{w*}$, is $S(A)$.*

Proof: In view of proposition 2.3.3 $\overline{co(Q)}^{w*} \subset S(A)$ is obvious. Suppose $\overline{co(Q)}^{w*} \not\supset S(A)$, so there is a $g \in S(A)$ such that $g \notin \overline{co(Q)}^{w*}$. Then by the Hahn-Banach theorem there exists a weak*-continuous linear form φ on $S(A)$, and a real number α such that $Re\,\varphi(g) < \alpha$ and $\min_{f\in Q} Re\,\varphi(f) \geq \alpha$. Hence, there exists an $x \in A$, in fact, an $x \in A_h$ because we only look at the real part of φ, such that $g(x) < \alpha$ and $\min_{f\in Q} f(x) \geq \alpha$. However, since Q satisfies **CONDQ** and $f(x) \geq \alpha$ for all $f \in Q$, $f(x - \alpha 1) \geq 0$ for all $f \in Q$. Thus $(x - \alpha 1) \in A^+$, so $g(x) \geq \alpha$, which contradicts the strict inequality $g(x) < \alpha$. Therefore $\overline{co(Q)}^{w*} \supset S(A)$. □

Lemma 2.4.4 *Let A be a C*-algebra, and $S \subset R(A)$. Then each $f \in S(A)$ such that* $\ker(\pi_f) \supset \bigcap_{\rho\in S} \ker(\rho)$ *is a weak*-limit of a net $\{g_i\}$ in $S(A)$, where each g_i is of the form* $\sum_Q f_\xi \circ \rho$, *Q is a finite subset of S, $\rho \in Q$, and $\xi \in \mathcal{H}_\rho$.*

Proof: We can assume the representation $\bigoplus_S \rho$ of A in $\bigoplus_S \mathcal{H}_\rho$ is injective and therefore A can be identified with the C*-subalgebra $\rho(A)$ in $\mathcal{L}(\mathcal{H}_\rho)$. If $\bigoplus_S \rho$ is not injective, that is $\bigcap_S \ker(\rho) \neq 0$, then we can simply identify $\bigoplus_S \rho$ with the quotient representation of $A/\{\bigcap_S \ker(\rho)\}$ in $\bigoplus_S \mathcal{H}_\rho$ and therefore assume $\bigcap_S \ker(\rho) = 0$.

From our comment 2.4.1 we can replace A by A_1. Let $Q = \{g \in S(A) : g = f_\xi \circ \rho,\ \rho \in S,\ \xi \in \mathcal{H}_\rho,\ \|\xi\| = 1\}$. If $x \in A_h$ such that $g(x) \geq 0$ for every $g \in Q$, then $f_\xi(\rho(x)) \geq 0$ for every $\rho \in S$ and for each unit vector $\xi \in \mathcal{H}_\rho$. Thus we have $\rho(x) \geq 0$ for every $\rho \in S$ which in turn implies $x \in A^+$. Therefore Q satisfies **COND Q** [2.4.2]. So we now only need to apply lemma 2.4.3. □

Theorem 2.4.5 (Equivalence Theorem) *Let A be any C*-algebra, $\pi \in R(A)$, and $S \subset R(A)$. Then the following are equivalent:*

i) $\ker(\pi) \supset \bigcap_{\rho\in S} \ker(\rho)$ *[ie. π is weakly contained in S, cf 3.1.2];*

ii) Every $f \in A'^{+}$ associated with π is a weak-limit of linear combinations of positve forms associated with S;*

iii) Every $f \in S(A)$ associated with π is a weak-limit of finite sums of $h \in S(A)$ which are sums of positive forms on A associated with S.*

Proof: Obviously we have $iii) \Rightarrow ii)$. We will now show $ii) \Rightarrow i)$. To this end assume the conditions of $ii)$. If $\rho(x) = 0$ for each $\rho \in S$ then $g(x^*x) = 0$ for every $g \in A'^{+}$ associated with S. Thus for any f associated with π we have f is a weak*-limit of linear combinations of g which are associated with S; hence $f(x^*x) = 0$, which implies that $\pi(x) = 0 \Rightarrow \ker(\pi) \supset \bigcap_{\rho \in S} \ker(\rho)$.

We now show $i) \Rightarrow iii)$; so assume $i)$. Then for every $f \in S(A)$ associated with π we have $\ker(\pi_f) \supset \bigcap_{\rho\in S} \ker(\rho)$. Hence every $f \in S(A)$ associated with π satisfies the conditions of lemma 2.4.4, which in turn implies $iii)$.

Moreover when π has a cyclic vector ξ, each condition of the equivalence theorem is equivalent to:

ii') The form f such that $f(x) = (\pi(x)\xi|\xi)$ in A'^{+} is a weak-limit of linear combinations of $g \in A'^{+}$ associated with S.*

Clearly $ii) \Rightarrow ii')$. To see the converse assume $ii')$. Let $g \in A'^{+}$ be associated with π, that is, $g(x) = (\pi(x)\eta|\eta)$ for some $\eta \in \mathcal{H}_\pi$ and for all $x \in A$. For $\epsilon > 0$ we can find a $y \in A$ such that $\|\pi(y)\xi - \eta\| < \epsilon$. Now define $h \in A'^{+}$ such that $h(x) = (\pi(x)\pi(y)\xi|\pi(y)\xi) = f(y^*xy)$. Then for each $x \in A$

$$\begin{aligned} |g(x) - h(x)| &= |(\pi(x)\eta|\eta) - (\pi(x)\pi(y)\xi|\pi(y)\xi)| \\ &\leq \|\pi(x)\eta\|\|\eta - \pi(y)\xi\| \\ &\quad + \|\pi(x)\eta - \pi(x)\pi(y)\xi\|\|\pi(y)\xi\| \\ &< \|x\|\|\eta\|\epsilon + \|x\|\epsilon(\|\eta\| + \epsilon). \end{aligned}$$

Hence h approximates g in norm and therefore also weak*-approximates g. From ii') f is the weak*-limit of a net $\{f_i\}$ of linear combinations of positive functionals associated with S. Letting $h_i(x) = f_i(y^*xy)$ we again have a net $\{h_i\}$ of linear combinations of positive functionals associated with S and h is the weak*-limit of h_i. So $h_i \overset{W^*}{\to} h \overset{W^*}{\to} g$. □

References: [5], [9], [13], [7].

Chapter 3

The Spectrum of a C*-Algebra

3.1 The Fell Topology

3.1.1 If A is a C*-algebra we will denote by $\hat{A}$ the set of eqivalence classes of $r(A)$, that is, the set of equivalence classes of non-trivial topological irreducible representations of A.

Definition 3.1.2 Let A be any C*-algebra, $\pi \in R(A)$, and $S \subset R(A)$. If π and S satisfy

$$\ker(\pi) \supset \bigcap_{\rho \in S} \ker(\rho);$$

then we say that π is **weakly contained** in S. If S and T are subsets of $R(A)$ then T is said to be **weakly contained** in S if each $\pi \in T$ is weakly contained in S. If S is also weakly contained in T then S and T are said to be **weakly equivalent**.

Proposition 3.1.3 *The map $S \to \overline{S}$ from the power set of $\hat{A}$ into itself, where $\overline{S} = \{[\pi] \in \hat{A} : [\pi]$ is weakly contained in $S\}$ satisfies Kuratowski closure axioms and therefore defines a unique topology on $\hat{A}$.*

Proof: Clearly $\overline{\emptyset} = \emptyset$ so we need only show $S \subset \overline{S}$, $\overline{S} = \overline{\overline{S}}$, and $\overline{S_1 \bigcup S_2} = \overline{S_1} \bigcup \overline{S_2}$. For $[\pi] \in S$ we obviously have $\ker[\pi] \supset \bigcap_{[\rho]\in S} \ker[\rho]$; so $S \subset \overline{S}$. Thus $\overline{S} \subset \overline{\overline{S}}$. If $[\pi] \in \overline{\overline{S}}$ then $\ker[\pi] \supset \bigcap_{[\rho]\in\overline{S}} \ker[\rho]$ and for each $[\rho] \in \overline{S}$ we have $\ker[\rho] \supset \bigcap_{\zeta\in S} \ker(\zeta)$. Hence $\ker([\pi]) \supset \bigcap_{[\rho]\in S} \ker[\rho]$ so $[\pi] \in \overline{S} \Rightarrow \overline{\overline{S}} \subset \overline{S}$. Combining the containments we have $\overline{S} = \overline{\overline{S}}$. Since $S_i \subset S_1 \bigcup S_2$ for $i = 1, 2$ we have $\overline{S_i} \subset \overline{S_1 \bigcup S_2}$, hence $\overline{S_1 \bigcup S_2} \supset \overline{S_1} \bigcup \overline{S_2}$.

Finally to show the reverse inclusion we show the contrapositive. Suppose $[\pi] \notin \overline{S_1} \bigcup \overline{S_2}$, that is, $\ker[\pi] \not\supset \bigcap_{[\rho]\in S_i} \ker[\rho]$ for $i = 1, 2$. So there must be an $x_i \in A$ such that $\pi(x_i) \neq 0$ and $\rho(x_i) = 0$, $\forall[\rho] \in S_i$ for $i = 1, 2$. We can therefore find a vector $\xi \in \mathcal{H}$ such that $\pi(x_1)\xi \neq 0$. Since π is irreducible and $\pi(x_2) \neq 0$ we can find a $y \in A$ such that $\pi(x_2)\pi(y)\pi(x_1)\xi \neq 0 \Rightarrow \pi(x_2 y x_1)\xi \neq 0 \Rightarrow x_2 y x_1 \notin \ker[\pi]$. However for all $[\rho] \in S_1 \bigcup S_2$ we have $\rho(x_2 y x_1) = 0$; thus $x_2 y x_1 \in \bigcap_{[\rho]\in S_1 \bigcup S_2} \ker[\rho]$. Therefore $\ker[\pi] \not\supset \bigcap_{[\rho]\in S_1 \bigcup S_2} \ker[\rho]$. So $\overline{S_1 \bigcup S_2} \subset \overline{S_1} \bigcup \overline{S_2}$. □

Definition 3.1.4 The topology $\mathcal{F}$ associated with the above closure operator is the so called **Fell topology** [9]. $\hat{A}$ paired with the Fell topology $\mathcal{F}$ is called the **spectrum of** A. Unless stated otherwise, we will denote $(\hat{A}, \mathcal{F})$, the spectrum of A, simply by $\hat{A}$. The spectrum of A is also referred to in many other works as the dual or the structure space of A.

References: [5], [9], [10], [13].

3.2 The Weak* Topology on $P(A)$

3.2.1 Let A be a C*-algebra. We will let $\mathcal{W}^*$ denote the relative weak*-topology on $P(A)$. That is, a f is the $\mathcal{W}^*$-limit of $\{f_i\}$ if and only if $f_i(x) \to f(x)$ for all $x \in A$.

Consider the map $\Phi: P(A) \to \hat{A}$ such that

$$\Phi(f) = [\pi_f].$$

By 2.2.13 and 2.2.14 Φ is well defined and 2.3.5 shows Φ to be surjective.

So we can endow $\hat{A}$ with the strongest topology, denoted $\mathcal{F}^{\mathcal{W}^*}$, that makes Φ continuous. In other words $\mathcal{F}^{\mathcal{W}^*}$ is the *quotient topology* relative to Φ and $\mathcal{W}^*$.

The following is a corollary to the equivalence theorem presented in §2.4.

Corollary 3.2.2 *Let A be any C*-algebra, $[\pi] \in \hat{A}$, and $S \subset \hat{A}$. Then the following are equivalent:*

i) $[\pi] \in \overline{S}^{\mathcal{F}}$;

ii) Every $f \in S(A)$ associated with $[\pi]$ is a weak-limit of a net $\{h_i\}$ in $S(A)$ associated with S.*

Proof: First we will show $ii) \Rightarrow i)$. It is easily seen that $ii) \Rightarrow \{2.4.5\ ii')\}$ and by the equivalence theorem $\{2.4.5\ ii')\} \Rightarrow i)$.

Now to show $i) \Rightarrow ii)$, suppose $i)$. Then every $f \in S(A)$ associated with $[\pi]$ is such that $\ker(\pi_f) \supset \bigcap_{[\rho] \in S} \ker[\rho]$. Therefore $ii)$ follows from lemma 2.4.4. □

Proposition 3.2.3 $\mathcal{F}^{\mathcal{W}^*} = \mathcal{F}$

Proof: It will suffice to show that for any C* algebra A the map Φ from $(P(A), \mathcal{W}^*)$ onto $(\hat{A}, \mathcal{F})$ is continuous and open [14, theorem 3.8]. To this end let $S \subset \hat{A}$. Then from corollary 3.2.2, $[\pi_f] \in \overline{S}^{\mathcal{F}}$ if and only if f is a weak*-limit of of a net $\{h_i\}$ in $S(A)$ where the h_i are associated with S. Hence $S = \overline{S}^{\mathcal{F}}$ if and only if $\Phi(S) = \overline{\Phi(S)}^{\mathcal{W}^*}$. So Φ is continuous.

Finally, to see that Φ is open, let $U \in \mathcal{W}^*$ and we show $\Phi(U) \in \mathcal{F}$. If $[\pi_f] \in \Phi(U)$ then

$$f \notin P(A) \sim U = \overline{P(A) \sim U}^{\mathcal{W}^*} = \overline{\Phi^{-1}(\hat{A} \sim \Phi(U))}^{\mathcal{W}^*} \Rightarrow [\pi_f] \notin \overline{\hat{A} \sim \Phi(U))}^{\mathcal{F}}.$$

Hence $\Phi(U) \in \mathcal{F}$. Therefore Φ is an open map. □

We can now say that the Fell topology $\mathcal{F}$ is the strongest topology on $\hat{A}$ that makes the map $\Phi : (P(A), \mathcal{W}^*) \to (\hat{A}, \mathcal{F})$ continuous.

References: [3], [5], [8], [9], [10], [13], [14].

3.3 The Jacobson Topology

Definition 3.3.1 A two-sided ideal of a C*-algebra A is said to be **primative** if it is the kernel of an algebraically irreducible morphism of A into a non-zero vector space. We will denote the set of all primative two sided ideals of A by $\text{Prim}(A)$.

Lemma 3.3.2 *Let A be a C*algebra. Then Prim$(A) = \{\ker(\pi) : \pi \in r(A)\}$.*

Proof: Since $r(A) = a(A)$ from proposition 2.3.8 we need only show that any algebraically irreducible morphism ϖ of A in a complex vector space V is algebraically equivalent to some $\pi \in R(A) \Rightarrow \pi \in a(A) = r(A)$.

Let $\nu \in V$ be non-zero and set $L = \{x \in A : \varpi(x)\nu = 0\}$. Clearly L is a left ideal. Since ϖ is irreducible it follows that $\varpi(A)\nu = V$ and L is a proper left ideal. We can then see that the map $x + L \to \varpi(x)\nu$ from A/L to V is an isomorphism of vector spaces and it identifies ϖ with the representation π of A/L in itself by left multiplication. So ϖ is algebraically equivalent to π. Also since $\varpi(A)\nu = V$ there is at least one $x_0 \in A$ such that $\varpi(x_0)\nu = \nu$. Hence for all $x \in A$

$$\varpi(x - xx_0)\nu = \varpi(x)\nu - \varpi(x)\varpi(x_0)\nu = 0 \implies x - xx_0 \in L,$$

so L is modular. As well, suppose J is a left ideal of A such that L is strictly contained in J. Let $y \in J \sim L$. Then $\varpi(y)\nu \neq 0$, and since ϖ is irreducible we have $\varpi(A)\nu = V$. Hence for any $x \in A$ there exists a $z \in J$ such that $\varpi(z)\nu = \varpi(x)\nu \Rightarrow \varpi(z - x)\nu = 0 \Rightarrow z - x \in L \subset J \Rightarrow x = -(y - x) + y \in J$. We have shown that if $x \in A$ then $x \in J$, thus $J = A$, which shows L is maximal. Therefore L is a modular maximal proper left ideal of A.

Let $F = \{f \in (A'^{+})_1 : f(x) = 0, \ \forall x \in L\}$. If $f \in F$ then because $x^*x \in L$ for all $x \in L$ we have $f(x^*x) = 0 \Rightarrow L \subset N_f$, so $L \subset \bigcap_F N_f$. It is easily shown that F is convex and weak*-compact, and is therefore the weak*-closed convex hull of the set F_e of its extreme points. So for $x \in A$ such that for every $f \in F_e$, $f(x^*x) = 0$ then we must have $f(x^*x) = 0$ for all $f \in F$. Thus $\bigcap_{F_e} N_f = \bigcap_F N_f \supset L$. As well, note that if any $f \in F$ can be written in the form $\alpha f_1 + (1 - \alpha)f_2$ for $f_1, f_2 \in A'^{+}$ and $\alpha \in (0, 1)$ then we have for all $x \in L$

$$0 \leq \alpha f_1(x^*x) + (1 - \alpha)f_2(x^*x) = f(x^*x) = 0 \Rightarrow f_1(x^*x) = f_2(x^*x) = 0;$$

hence by the inequality 2.2, $f_1(x) = f_2(x) = 0$; so $f_1, f_2 \in F$. This shows F is a face of $(A'^{+})_1$, and therefore $F_e \subset P(A) \bigcup \{0\}$. If $F_e = \{0\}$ then $F = \{0\} \Rightarrow L = A$, which contradicts L is proper. Thus there must exist at least one $f_0 \in F_e \bigcap P(A)$ such that $L \subset N_{f_0}$.

Finally, since L is maximal $L = N_{f_0}$. Therefore by proposition 2.3.9 $A/L = A/N_{f_0}$ can be given a Hilbert space structure $\mathcal{H}_{f_0}$ such that π is a representation of the C*-algebra A in the Hilbert space $A/L = \mathcal{H}_{f_0}$. □

Lemma 3.3.3 *If* π, $\pi' \in R(A)$ *are equivalent then* $\ker(\pi) = \ker(\pi')$.

Proof: Recall if π, $\pi' \in R(A)$ are equivalent, that is, $\pi' \in [\pi]$, then there exists a U such that $U\pi(x) = \pi'(x)U$, $\forall\, x \in A$. Thus for all $x \in \ker(\pi)$ we have $\pi'(x) = 0$, so $\ker(\pi) \subset \ker(\pi')$. Exchanging π and π' yields the reverse inclusion hence $\ker(\pi) = \ker(\pi')$. □

Theorem 3.3.4 *Let A be a C*algebra. Then* $Prim(A) = \{\ker([\pi]) : [\pi] \in \hat{A}\}$.

Proof: This follows directly from lemmas 3.3.2 and 3.3.3. □

3.3.5 Thus theorem 3.3.4 allows us to define a surjective map Ψ from $\hat{A}$ onto $\text{Prim}(A)$ such that

$$\Psi([\pi]) = \ker([\pi]).$$

In general the converse of lemma 3.3.3 does not hold. Hence Ψ is not in general injective. An example of a C*-algebra that has two [in fact many] distinct [not equivalent] representations with the same kernal, is the *CAR (canonical anticommutation relations) algebra* [4, p. 87].

Proposition 3.3.6 *The map* $F \to \overline{F}$ *from the power set of* $Prim(A)$ *into itself, where* $\overline{F} = \{\mathcal{I} \in Prim(A) : \mathcal{I} \supset \bigcap_{\mathcal{K} \in F} \mathcal{K}\}$ *and* $\overline{\emptyset} = \emptyset$, *satisfies Kuratowski closure axioms and therefore defines a unique topology on* $Prim(A)$.

Proof: Again, as in proposition 3.1.3, we need only show that $F \subset \overline{F}$, $\overline{F} = \overline{\overline{F}}$, and $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$. Clearly $F \subset \overline{F}$, $\overline{F} = \overline{\overline{F}}$ and $\overline{F_1 \cup F_2} \supset \overline{F_1} \cup \overline{F_2}$. Finally to show the reverse inclusion suppose $\mathcal{I} \notin \overline{F_1} \cup \overline{F_2}$, that is, $\ker[\pi] \notin \overline{F_1} \cup \overline{F_2}$ for some [perhaps more than one] $[\pi] \in \hat{A}$. So similar to the proof of 3.1.3 there must be an $x_i \in \overline{F_i}$ such that $\pi(x_i) \neq 0$ for $i = 1, 2$. We can therefore find a vector $\xi \in \mathcal{H}$

such that $\pi(x_1)\xi \neq 0$. Since π is irreducible and $\pi(x_2) \neq 0$ we can find a $y \in A$ such that $\pi(x_2)\pi(y)\pi(x_1)\xi \neq 0 \Rightarrow \pi(x_2yx_1)\xi \neq 0 \Rightarrow x_2yx_1 \notin \ker[\pi]$. However $x_2yx_1 \in \bigcap_{\mathcal{K}\in F_1\bigcup F_2}\mathcal{K}$. So $\ker[\pi] \not\supset \bigcap_{\mathcal{K}\in F_1\bigcup F_2}\mathcal{K}$ and hence $\mathcal{I} \notin \overline{F_1 \bigcup F_2}$. □

Definition 3.3.7 The topology $\mathcal{J}$ associated with the above closure operator is called the **Jacobson topology** on Prim(A) [12]. Many authors also refer to this topology as the *hull-kernel topology.*

3.3.8 We can create a topology $\mathcal{F}^{\mathcal{J}}$ on $\hat{A}$ by transfering the topology $\mathcal{J}$ from Prim(A) via the surjective map Ψ [3.3.5] onto $\hat{A}$ as follows: $U \in \mathcal{F}^{\mathcal{J}}$ iff $\Psi(U) \in \mathcal{J}$.

Claim: $\mathcal{F}^{\mathcal{J}} = \mathcal{F}$

__Proof:__ Consider $F \subset \hat{A}$. Then

$$\begin{aligned}
\overline{F}^{\mathcal{F}} &= \{[\pi] \in \hat{A} : \ \ker[\pi] \supset \bigcap_{[\rho]\in F} \ker[\rho]\} \\
&= \{[\pi] \in \hat{A} : \ \Psi[\pi] \supset \bigcap_{\ker[\rho]\in\Psi(F)} \ker[\rho]\} \\
&= \{[\pi] \in \hat{A} : \ [\pi] \in \overline{F}^{\mathcal{F}^{\mathcal{J}}}\} \\
&= \overline{F}^{\mathcal{F}\mathcal{J}}
\end{aligned}$$

□

So we can say that the Fell topology $\mathcal{F}$ is the weakest topology on $\hat{A}$ that makes the map $\Psi : (\hat{A}, \mathcal{F}) \to (\text{Prim}(A), \mathcal{J})$ continuous.

Proposition 3.3.9 *(Prim$(A), \mathcal{J}$) is a T_0-space.*

__Proof:__ Let $\mathcal{I}_1, \mathcal{I}_2 \in$ Prim(A) be distinct so that, say, $\mathcal{I}_1 \not\subseteq \mathcal{I}_2$. Then let $F = \{\mathcal{I} \in \text{Prim}(A) : \ \mathcal{I} \supseteq \mathcal{I}_1\}$. So we have $\mathcal{I}_1 \subseteq \bigcap_{\mathcal{I}\in F}\mathcal{I}$, which implies $\overline{F} \subseteq F$. So $\overline{F} = F$. Hence $\mathcal{I}_1$ is contained in a closed subset $\overline{F}$ of Prim(A) and $\mathcal{I}_2 \notin \overline{F}$. □

Proposition 3.3.10 *The following are equivalent:*

i) $(\hat{A}, \mathcal{F})$ *is a* T_0*-space*

ii) For any $[\pi]$, $[\pi'] \in \hat{A}$, *if* $\ker[\pi] = \ker[\pi']$ *then* π *is equivalent to* π'.

iii) The map $\Psi : (\hat{A}, \mathcal{F}) \to (Prim(A), \mathcal{J})$ *is a homeomorphism.*

Proof: We'll show $ii) \Rightarrow iii) \Rightarrow i) \Rightarrow ii)$.

$ii) \Rightarrow iii)$ If $ii)$ holds then clearly Ψ is injective and thus a continous bijection.

$iii) \Rightarrow i)$ This follows directly from Proposition 3.3.9.

$i) \Rightarrow ii)$ Suppose $[\pi]$, $[\pi'] \in \hat{A}$ and $\ker[\pi] = \ker[\pi']$. Then for any $U \in \mathcal{F}$ containing $[\pi]$ we have $\ker[\pi] \in \Psi(U) \Rightarrow \ker[\pi'] \in \Psi(U) \Rightarrow [\pi'] \in U$. Hence π is equivalent to π'. □

References: [5], [9], [10], [12], [14], [13], [16].

Chapter 4

The Spectrum $\hat{G}$ of a Locally Compact Group G

4.1 Representations of G and $L^1(G)$

Definition 4.1.1 Let G be a locally compact group. A **unitary representation** π of G is a morphism of G into the group $\mathcal{U}(\mathcal{H}_\pi) \subset \mathcal{L}(\mathcal{H}_\pi)$ of unitary operators on some Hilbert space $\mathcal{H}_\pi$ such that π is continous in the strong operator topology.

In other words, a map $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ such that for all $x, y \in G$

$$\pi(xy) = \pi(x)\pi(y) \quad \text{and} \quad \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$$

and $x \to \pi(x)\xi$ from G into $\mathcal{H}_\pi$ is continuous for any $\xi \in \mathcal{H}_\pi$. Since the strong and weak operator topologies coincide on $\mathcal{U}(\mathcal{H}_\pi)$ [A.5.2] the strong operator continuity condition can be replaced by weak operator continuity, that is, $x \to (\pi(x)\xi|\eta)$ is continuous from G to $\mathbf{C}$ for each $\xi, \eta \in \mathcal{H}_\pi$. $R(G)$ will denote the set of unitary representations of G.

4.1.2 Most of the notions regarding representations as presented in chapter 2 can be carried over to $R(G)$. Specifically we need speak of equivalent representations, class of representations, direct sum of representations, cyclic vector for a representation. As well, as in section 2.3 we say $\pi \in R(G)$ is **topologically irreducible** if any of the following equivalent conditions hold:

i) the only closed subspaces invariant under $\pi(G)$ are the trivial ones,

ii) the commutant of $\pi(G)$ in $\mathcal{L}(\mathcal{H}_\pi)$ is $\mathbf{C}I$, where I is the identity operator in $\mathcal{L}(\mathcal{H}_\pi)$,

iii) every non-zero $\xi \in \mathcal{H}_\pi$ is a cyclic vector for π.

In keeping with our earlier notation we will denote by $r(G)$ the subset of all topologically irreducible unitary representations of $R(G)$.

4.1.3 Let G be a locally compact group and fix now and forever the left Haar measure dx on G [cf A.2.1]. For $x \in G$ consider the operator $\pi_l(x) \in \mathcal{L}(L^2(G))$ defined by

$$(\pi_l(x)f)(y) = L_x f(y) = f(x^{-1}y) \qquad f \in L^2(G), y \in G$$

where L_x is the left translate of f [1.2.2]. π_l is easily shown to be in $R(G)$. We will call π_l the ***left regular representation*** of G in $L^2(G)$. As well, for $x \in G$ consider the operator $\pi_r(x) \in \mathcal{L}(L^2(G))$ defined by

$$(\pi_r(x)f)(y) = R_x f(y) = \Delta(x)^{-1/2} f(yx) \qquad f \in L^2(G), y \in G,$$

where Δ is the modular function of G [cf A.2.3] and R_x is the right translate [1.2.2]. Again it is readily seen that $\pi_r \in R(G)$. π_r is called the ***right regular representation.***

Proposition 4.1.4 *Let $\pi \in R(G)$. For $f \in L^1(G)$, define the operator $\pi'(f) \in \mathcal{L}(\mathcal{H}_\pi)$*

by

$$\pi'(f) = \int f(x)\pi(x)dx,$$

where this operator valued integral is interpreted as follows: For each $\xi \in \mathcal{H}_\pi$, the element $\pi'(f)\xi$ in $\mathcal{H}_\pi$ has an inner product with $\eta \in \mathcal{H}_\pi$ given by

$$(\pi'(f)\xi|\eta) = \int f(x)(\pi(x)\xi|\eta)dx. \tag{4.1}$$

Then π' is a non-degenerate representation of $L^1(G)$ in $\mathcal{H}_\pi$.

Proof: Since $(\pi(x)\xi|\eta)$ is a bounded continuous function of $x \in G$, $\int f(x)(\pi(x)\xi|\eta)dx$ is the ordinary integral of a function in $L^1(G)$. So it is easily seen that π' is linear and $|(\pi'(f)\xi|\eta)| \leq \|f\|_1\|\xi\|\|\eta\|$ so π' is bounded. As well,

$$\begin{aligned}
\pi'(f * h) &= \int\int f(y)h(y^{-1}x)\pi(x)dy\,dx = \int\int f(y)h(x)\pi(yx)dx\,dy \\
&= \int\int f(y)h(x)\pi(y)\pi(x)dx\,dy = \pi'(f)\pi'(g), \\
\pi'(f^*) &= \int \Delta(x^{-1})\overline{(f(x^{-1})}\pi(x)dx \\
&= \int [f(x)\pi(x)]^*\,dx = \pi'(f)^*
\end{aligned}$$

Hence $\pi' \in R(L^1(G))$.

Finally, to show that π' is non-degenerate consider a non-zero $\xi \in \mathcal{H}_\pi$. Then by the continuity of π we can choose a compact neighborhood V of e in G such that $\|\pi(x)\xi - \xi\| < \|\xi\|$ for all $x \in V$. Then set $f = |V|^{-1}\chi_V$ where $|V| = \int_V dx$ and χ_V is the characteristic function of V. Clearly $f \in L^1(G)$ and

$$\|\pi'(f)\xi - \xi\| = \frac{1}{|V|}\|\int_V [\pi(x)\xi - \xi]\,dx\| < \|\xi\|,$$

so we have shown $\pi'(f)\xi \neq 0$. □

4.1.5 $\pi' \in R(L^1(G))$ in the above theorem, is referred to as the representation associated with $\pi \in R(G)$.

Proposition 4.1.6 *If $\pi' \in R(L^1(G))$ is non-degenerate then π' is associated with exactly one $\pi \in R(G)$.*

Proof: Uniqueness follows easily from equality 4.1 in proposition 4.1.4. If π' is associated with both π and π_2 then for all $x \in G$ and all $\xi, \eta \in \mathcal{H}_{\pi'}$ we have $(\pi(x)\xi|\eta) = (\pi_2(x)\xi|\eta) \Rightarrow \pi(x) = \pi_2(x)$.

To show existence let B be the subspace of $\mathcal{H}_{\pi'}$ generated by $\pi'(L^1(G))\mathcal{H}_{\pi'}$. Since π' is non-degenerate B is dense in $\mathcal{H}_{\pi'}$. If $f \in L^1(G)$ and $\{e_U\}$ an approximate identity in $L^1(G)$ then

$$\begin{aligned} e_U * f \to f \in L^1(G) \quad &\Rightarrow \quad (L_x e_U) * f = L_x(e_U * f) \to L_x f \in L^1(G) \\ &\Rightarrow \quad \pi'(L_x e_U)\pi'(f)\xi \to \pi'(L_x f)\xi \end{aligned}$$

for all $\xi \in \mathcal{H}_{\pi'}$. Thus $\pi'(L_x e_U)$ converges strongly on B to a well defined operator $\dot{\pi}(x) : B \to B$ such that $\dot{\pi}(x)\pi'(f)\xi = \pi'(L_x f)\xi$. Since $\|L_x e_U\| \leq \|L_x e_U\|_1 = 1$ we can extend $\dot{\pi}(x)$ to an operator π on $\mathcal{H}_{\pi'}$ such that $\|\pi(x)\| \leq 1$ and $\pi(x)\pi'(f) = \pi(L_x f)$. As well,

$$\dot{\pi}(xy)\pi'(f) = \pi'(L_{xy}f) = \pi'(L_x L_y f) = \dot{\pi}(x)\pi'(L_y f) = \dot{\pi}(x)\dot{\pi}(y)\pi'(f),$$

$$\dot{\pi}(1) = I \Rightarrow \|\xi\| = \|\dot{\pi}(x^{-1})\dot{\pi}(x)\xi\| \leq \|\dot{\pi}(x)\xi\| \leq \|\xi\|$$

it then follows that $\pi(xy) = \pi(x)\pi(y)$ and π is a unitary representation of G in $\mathcal{H}_{\pi'}$. Finally, if $x_i \to x$ in G then $L_{x_i} f \to L_x f$ in $L^1(G)$, so $\dot{\pi}(x_i)\pi'(f) = \pi'(L_{x_i} f) \to$

$\pi'(L_x f) = \dot{\pi}(x)\pi'(f)$ strongly. Thus $\dot{\pi}(x_i)\xi \to \dot{\pi}(x)\xi$ for all $\xi \in B$ and since $||\dot{\pi}(x_i)|| = 1$ for all i it follows that $\pi(x_i)\xi \to \pi(x)\xi$ for all $\xi \in \mathcal{H}_{\pi'}$. Therefore π is continuous and hence $\pi \in R(G)$.

All that remains is to show that π' is precisely the representation associated with π. Let $f, g \in L^1(G)$ then

$$\begin{aligned}\pi'(f)\pi'(g) &= \pi'(f * g) = \int f(y)\pi'(L_y g)dy = \int f(y)\dot{\pi}(y)\pi'(g)dy \\ &= \left[\int f(y)\dot{\pi}(y)dy\right]\pi'(g) = \dot{\pi}(f)\pi(g).\end{aligned}$$

Again by the density of B in $\mathcal{H}_{\pi'}$ it follows that π' is the representation associated with π. □

4.1.7 ¿From propsitions 4.1.4 and 4.1.6 we have shown $\pi \to \pi'$ is a bijective correspondence between $R(G)$ and the non-degenerate representations of $L^1(G)$.

Proposition 4.1.8 *Let $\pi \in R(G)$.*

i) $\pi(G)$ and $\pi'(L^1(G))$ generate the same von Neumann algebra [cf A.5.1] in $\mathcal{L}(\mathcal{H}_\pi)$.

ii) $\pi(G)$ and $\pi'(L^1(G))$ have the same commutant [cf A.5.3] in $\mathcal{L}(\mathcal{H}_\pi)$.

iii) A closed subspace F in $\mathcal{H}_\pi$ is invariant for $\pi(G)$ if and only if F is invariant for $\pi'(L^1(G))$.

Proof: *i)* In the proof of proposition 4.1.6 we see that each $\pi(x)$ is the strong limit of $\pi'(L_x e_U)$ as $U \to \{1\}$. It follows that the strong closure of the algebra generated by $\pi(G)$ in $\mathcal{L}(\mathcal{H}_\pi)$ is contained in the strong closure of the algebra generated by $\pi'(L^1(G))$.

Now, if $f \in L^1(G)$ then f is the L^1-limit of functions $g \in C_C(G)$, that is g is a continuous function of compact support. Thus $\pi'(f)$ is the norm limit, hence strong limit, of $\pi'(g)$ with $g \in C_C(G)$. Finally, if $\epsilon > 0$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}_\pi$ then by the uniform continuity of the maps $x \to g(x)\pi(x)\xi_m$ we can find a finite partition $E = \{E_j\}$ of the supp g such that $\|g(x)\pi(x)\xi_m - g(y)\pi(y)\xi_m\| < \epsilon$ for $m = 1, \ldots, n$ when x and y are in the same E_j. Therefore for $m = 1, \ldots, n$ we have $\| \left(\sum_j g(x_j)\pi(x_j)|E_j|\right)\xi_m - \pi'(g)\xi_m\| < \epsilon|\text{supp } g|$ where $x_j \in E_j$. In other words, every strong neighborhood of $\pi'(g)$ contains sums of the form $\sum_j g(x_j)\pi(x_j)|E_j|$. So it follows that the strong closure of the algebra generated by $\pi'(L^1(G))$ in $\mathcal{L}(\mathcal{H}_\pi)$ is contained in the strong closure of the algebra generated by $\pi(G)$.

ii) T is in the commutant of $\pi(G)$ $\Leftrightarrow$ T commutes with every element in the von Neumann algebra generated by $\pi(G)$ $\Leftrightarrow$ [from *i)*] T commutes with every element in the von Neumann algebra generated by $\pi(L^1(G))$ $\Leftrightarrow$ T is in the commutant of $\pi(L^1(G))$.

iii) Suppose the closed subspace F in $\mathcal{H}_\pi$ is invariant for $\pi(G)$. If P_F is the orthogonal projection onto F then for any $x \in G$, $\pi(x)P_F\xi = \pi(x)\xi = P_F\pi(x)\xi$ for $\xi \in F$ and since $F^\perp$ is also invariant for $\pi(G)$ we have $\pi(x)P_F\xi = 0 = P_F\pi(x)\xi$ for $\xi \in F^\perp$. It follows that P_F is in the commutant of $\pi(G)$ and is therefore in the commutant of $\pi'(L^1(G))$ by *ii)*. Thus, if $\xi \in F$ and $f \in L^1(G)$ then $\pi'(f)\xi = \pi'(f)P_F\xi = P_F\pi'(f)\xi \in F$ so F is invariant for $\pi'(L^1(G))$. Now simply exchanging $\pi(G)$ and $\pi'(L^1(G))$ in the above argument completes the proof. □

4.1.9 It now follows from proposition 4.1.8 that the correspondence $\pi \to \pi'$ is a bijection from $r(G)$ onto $r(L^1(G))$. Henceforth we will denote both $\pi \in R(G)$ and non-degenerate $\pi' \in R(L^1(G))$ simpily by π.

References: [5], [6], [8], [10], [13].

4.2 Functions of Positive Type

Definition 4.2.1 Let G be a locally compact group. A **function of positive type** is a function $\phi \in L^\infty(G)$ that defines a positive linear form on $L^1(G)$, that is

$$\int (f^* * f)(x)\phi(x)dx \geq 0, \quad \forall f \in L^1(G).$$

$\mathcal{P}_0(G)$ will denote the set of all functions of positve type on G. Functions of positive type are not in general continuous, however we shall see in proposition 4.2.6 that they are locally almost everywhere equal to a continous function. We will denote the set of all *continuous* functions of positive type on G by $\mathcal{P}(G)$.

Note that

$$\begin{aligned} \int (f^* * f)(x)\phi(x)dx &= \int\int \Delta(y^{-1})\overline{f(y^{-1})}f(y^{-1}x)\phi(x)dy\,dx \\ &= \int\int \overline{f(y)}f(yx)\phi(x)dy\,dx \end{aligned}$$

so reversing integration and substituting $y^{-1}x$ for x yields ϕ is of positive type if and only if

$$\int\int f(x)\overline{f(y)}\phi(y^{-1}x)dy\,dx \geq 0, \quad \forall f \in L^1(G).$$

Proposition 4.2.2 *Let $\phi \in L^\infty(G)$, and $\omega \in L^1(G)^*$ be defined by ϕ [cf A.2.6]. Then $\phi \in \mathcal{P}(G)$ if and only if there exists $\pi \in R(G)$, $\xi \in \mathcal{H}_\pi$ such that $\phi(x) = (\pi(x)\xi|\xi)$ for all $x \in G$.*

Proof: First suppose $\phi \in \mathcal{P}(G)$, that is $\int (f^* * f)\phi \geq 0$ for all $f \in L^1(G)$ or equivalently $\omega \in L^1(G)^{*+}$. So we can form the representation $\pi_\omega \in R(L^1(G))$ and the vector ξ_ω [2.2.5]. From our comment 4.1.9 we can also consider $\pi_\omega \in R(G)$. So for any $f \in L^1(G)$ we have

$$\int \phi(x)f(x)dx = \omega(f) = (\pi_\omega(f)\xi_\omega|\xi_\omega) = \int(\pi_\omega(x)\xi_\omega|\xi_\omega)f(x)dx,$$

which implies

$$\phi(x) = (\pi_\omega(x)\xi_\omega|\xi_\omega) \quad \text{locally almost everywhere.}$$

In fact, this equality holds for all $x \in G$ since both ϕ and $(\pi_\omega(x)\xi_\omega|\xi_\omega)$ are continuous.

Conversly, suppose there exists $\pi \in R(G)$, $\xi \in \mathcal{H}_\pi$ such that $\phi(x) = (\pi(x)\xi|\xi)$ for all $x \in G$. It follows immediately from the continuity of π that ϕ is continuous. So for $f \in L^1(G)$ we have,

$$\begin{aligned}
\int\int f(x)\overline{f(y)}\phi(y^{-1}x)dy\,dx &= \int\int f(x)\overline{f(y)}(\pi(y^{-1}x)\xi|\xi)dy\,dx \\
&= \int\int f(x)\overline{f(y)}(\pi(x)\xi|\pi(y)\xi)dy\,dx \\
&= \int\int(f(x)\pi(x)\xi|f(y)\pi(y)\xi)dy\,dx \\
&= (\pi(f)\xi|\pi(f)\xi) = \|\pi(f)\xi\|^2 \geq 0
\end{aligned}$$

Hence $\phi \in \mathcal{P}(G)$. □

4.2.3 For $\pi \in R(G)$ and $\xi \in \mathcal{H}_\pi$ we say $x \to (\pi(x)\xi|\xi)$ is the function of positive type *defined by* π and ξ. If we fix π and allow ξ to vary we get the functions of positive type *associated with* π.

4.2.4 If $\phi \in \mathcal{P}_0(G)$ then ϕ defines a $\omega \in L^1(G)^{*+}$ [cf A.2.6] and therefore a pair (π_ω, ξ_ω), as seen in the proof of proposition 4.2.2, where $\pi_\omega \in R(G)$ and ξ_ω is a cyclic vector for π_ω. We will also denote this pair by (π_ϕ, ξ_ϕ). Conversely if

$\omega \in L^1(G)'^+$ $(= L^1(G)^{*+})$ then ω defines a $\phi \in \mathcal{P}(G)$, namely $\phi(x) = (\pi_\omega(x)\xi_\omega|\xi_\omega)$. As we shall see in the following proposition, analogous to proposition 2.2.13, the associated representations of ϕ are characterised up to equivalence.

Proposition 4.2.5 *If $\pi, \pi' \in R(G)$ with cyclic vectors ξ and ξ', respectively, and $(\pi(x)\xi|\xi) = (\pi'(x)\xi'|\xi')$ for all $x \in G$ then π is equivalent to π'.*

Proof: Assuming the hypothesis. By 4.1.9 we have

$$(\pi(f)\xi|\xi) = \int (\pi(x)\xi|\xi)f(x)dx = \int (\pi'(x)\xi'|\xi')f(x)dx = (\pi'(f)\xi'|\xi') \quad \forall f \in L^1(G).$$

Thus from proposition 2.2.13 π and π' are equivalent as representations in $R(L^1(G))$ and now due to 4.1.8 we can pass back to equivalence in $R(G)$. □

Proposition 4.2.6 *Let $\phi \in L^\infty(G)$, and $\omega \in L^1(G)^*$ be defined by ϕ. Then $\phi \in \mathcal{P}_0(G)$ if and only if ϕ is equal locally almost everywhere to a $\psi \in \mathcal{P}(G)$.*

Proof: If $\phi \in \mathcal{P}_0(G)$ then from comment 4.2.4 ϕ defines a pair (π_ϕ, ξ_ϕ). Then as seen in the proof of 4.2.2 $\phi(x) = (\pi_\phi(x)\xi_\phi|\xi_\phi)$ locally almost everywhere. Letting $\psi(x) = (\pi_\phi(x)\xi_\phi|\xi_\phi)$ then from proposition 4.2.2 $\psi \in \mathcal{P}(G)$.

Now assume ϕ is equal locally almost everywhere to some $\psi \in \mathcal{P}(G)$. Then for every $f \in L^1(G)$ we have

$$\int (f^* * f)\phi = \int (f^* * f)\psi \geq 0;$$

hence $\phi \in \mathcal{P}_0(G)$. □

4.2.7 Thus $\mathcal{P}_0(G)$ modulo equality locally almost everywhere is in a bijective correspondence with $R(G)$ [or equivalently the set of nondegenerate representaions in $R(L^1(G))$]

modulo equivalence. Further, we have $\mathcal{P}_0(G)$ modulo equality locally almost everywhere is in a bijective correspondence with $L^1(G)^{*+}$.

Now consider the set $\mathcal{P}_1(G) = \{\phi \in \mathcal{P}(G) : \phi(1) = 1$ [or equivalently $\|\phi\|_\infty = 1$]$\}$.

Claim: *The preceding bijective correspondence restricted to $\mathcal{P}_1(G)$ has a range equal to the state space $S(L^1(G))$.*

Proof: First let $\phi \in \mathcal{P}_1(G)$ and let $\omega \in L^1(G)^*$ be defined by ϕ. Then

$$\|\omega\| = \sup_{\|f\|_1 \le 1} |\omega(f)| | = \sup_{\|f\|_1 \le 1} \left| \int \phi f dx \right| \le \|\phi\|_\infty = 1.$$

Now for any $\epsilon > 0$ by continuity of ϕ there exists a neighborhood U of 1 in G such that $\phi(x) > 1 - \epsilon$ for all $x \in U$. Hence

$$\|\omega\| \ge \left| \int \phi(\frac{1}{|U|}\chi_U)dx \right| = \frac{1}{|U|} \left| \int_U \phi dx \right| \ge 1 - \epsilon;$$

so $\|\omega\| = 1 \Rightarrow \omega \in S(L^1(G))$.

Conversly, if $\omega \in S(L^1(G))$ and $\phi(x) = (\pi_\omega(x)\xi_\omega|\xi_\omega)$ then $\phi(1) = (\xi_\omega|\xi_\omega) = \|\xi_\omega\|^2 = 1$ [2.2.14]. Hence $\phi \in \mathcal{P}_1(G)$. □

Lemma 4.2.8 *Let $f \in L^1(G)$ and $\{\phi_i\}$ be a net in $\mathcal{P}_1(G)$. If ϕ_i weak*-converges to $\phi_0 \in \mathcal{P}_1(G)$ then $f * \phi_i$ converges to $f * \phi_0$ for the topology of compact convergence.*

Proof: We have

$$\begin{aligned}(f * \phi_i)(x) &= \int f(y)\phi_i(y^{-1}x)dy = \int f(xy)\phi_i(y^{-1})dy \\ &= \int f(xy)(\pi_{\phi_i}(y^{-1})\xi_{\phi_i}|\xi_{\phi_i})dy = \int f(xy)(\xi_{\phi_i}|\pi_{\phi_i}(y)\xi_{\phi_i})dy \\ &= \int f(xy)\overline{\phi_i(y)}dy = \int L_{x^{-1}}f(y)\overline{\phi_i(y)}dy.\end{aligned}$$

Now for any compact $K \subset G$ the set $K_f = \{L_{x^{-1}}f : x \in K\}$ is compact since $x \to L_{x^{-1}}f$ is continuous from G to $L^1(G)$. Let $\epsilon > 0$. By compactness there exist $x_1, \ldots, x_n \in K$ such that the balls $B(L_{x_j^{-1}}f, \frac{\epsilon}{3})$ cover K_f. Thus for $x \in K$ there is some j such that $\|L_{x^{-1}}f - L_{x_j^{-1}}f\| \leq \frac{\epsilon}{3}$. Hence, for any $x \in K$ we have

$$\begin{aligned} \|L_{x^{-1}}f\overline{\phi_i} - L_{x^{-1}}f\overline{\phi_0}\| &< \|L_{x^{-1}}f\overline{\phi_i} - L_{x_j^{-1}}f\overline{\phi_i}\| + \|L_{x_j^{-1}}f\overline{\phi_i} - L_{x_j^{-1}}f\overline{\phi_0}\| + \\ &\quad \|L_{x_j^{-1}}f\overline{\phi_0} - L_{x^{-1}}f\overline{\phi_0}\| \\ &= \|L_{x^{-1}}f - L_{x_j^{-1}}f\|\,\|\overline{\phi_i}\| + \|L_{x_j^{-1}}f\|\,\|\overline{\phi_i} - \overline{\phi_0}\| + \\ &\quad \|L_{x^{-1}}f - L_{x_j^{-1}}f\|\,\|\overline{\phi_0}\| \\ &< \epsilon/3 + \|L_{x_j^{-1}}f\|\,\|\overline{\phi_i} - \overline{\phi_0}\| + \epsilon/3. \end{aligned}$$

Since by hypothesis ϕ_i weak*-converges to ϕ_0 we can find a weak*-neighborhood $\mathcal{V}_j$ of ϕ_0 in $\mathcal{P}_1(G)$ such that $\|L_{x_j^{-1}}f\|\,\|\overline{\phi_i} - \overline{\phi_0}\| \leq \epsilon/3$. Set $\mathcal{V} = \bigcap_1^n \mathcal{V}_j$. Then $\mathcal{V}$ is again a weak*-neighborhood of ϕ_0 in $\mathcal{P}_1(G)$ and for $\phi_i \in \mathcal{V}$ and $x \in K$ we have $\|L_{x^{-1}}f\overline{\phi_i} - L_{x^{-1}}f\overline{\phi_0}\| < \epsilon$ which implies, due to our first equality, that $\|f * \phi_i - f * \phi_0\| < \epsilon$. Thus since K was arbitrary, we have $f * \phi_i$ converges to $f * \phi_0$ for the topology of compact convergence. □

Theorem 4.2.9 *On $\mathcal{P}_1(G)$ the relative weak*-topology $\sigma(L^\infty(G), L^1(G))$ coincides with the topology of compact convergence.*

Proof: Let $\epsilon > 0$. If $f \in L^1(G)$ then we can find a compact $K \subset G$ such that $\int_{G\sim K}|f| < \epsilon/4$. Now suppose the net $\{\phi_i\}$ in $\mathcal{P}_1(G)$ converges to $\phi_0 \in \mathcal{P}_1(G)$ for the topology of compact convergence. Then we can find an i_0 such that $|\phi_i(x) - \phi_0(x)| < \epsilon/(2\|f\|_1)$ on K for all $i \geq i_0$; so

$$\left|\int f\phi_i - \int f\phi_0\right| = \left|\int f(\phi_i - \phi_0)\right|$$

$$\begin{aligned} &\leq \int_K |f||\phi_i - \phi_0| + \int_{G\sim K} |f||\phi_i - \phi_0| \\ &< \|f\|_1 \frac{\epsilon}{2\|f\|_1} + \frac{\epsilon}{4}(\|\phi_i\|_\infty + \|\phi_0\|_\infty) \\ &= \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Thus we see that compact convergence on G implies weak*-convergence.

Conversely, let $\phi_0 \in \mathcal{P}_1(G)$ and $K \subset G$ be compact. The idea now is to find a weak*-neighborhood $\mathcal{V}$ of ϕ_0 in $\mathcal{P}_1(G)$ such that for $\phi \in \mathcal{V}$ we have $|\phi(x) - \phi_0(x)| < \epsilon + 4\sqrt{\epsilon}$ on K.

First, there exists a compact neighborhood V of e in G such that $|\phi_0(x) - 1| < \epsilon$ on V. Let $\mathcal{V}'$ be the weak*-neighborhood of ϕ_0 in $\mathcal{P}_1(G)$ such that

$$\mathcal{V}' = \left\{\phi \in \mathcal{P}_1(G) : \left|\int_V (\phi - \phi_0)\right| < \epsilon|V|\right\}.$$

Indeed $\mathcal{V}'$ is a weak*-neighborhood since $\chi_V \in L^1(G)$. If $\phi \in \mathcal{V}'$ then

$$\dagger \quad \left|\int_V (\phi(x) - 1)dx\right| \leq \left|\int_V (\phi_0(x) - 1)dx\right| + \left|\int_V (\phi(x) - \phi_0(x))dx\right| \leq 2\epsilon|V|.$$

Moreover, for $\phi \in \mathcal{V}'$ and $x \in G$ we have

$$\begin{aligned} |\chi_V * \phi(x) - |V|\phi(x)| &= \left|\int_V \left[\phi(y^{-1}x) - \phi(x)\right] dy\right| \leq \int_V \left|\phi(y^{-1}x) - \phi(x)\right| dy \\ &= \int_V \left|([\pi_\phi(y^{-1}x) - \pi_\phi(x)]\xi_\phi|\xi_\phi)\right| dy \\ &= \int_V \left|(\xi_\phi|[\pi_\phi(x^{-1}y) - \pi_\phi(x^{-1})]\xi_\phi)\right| dy \\ &\leq \int_V \|\pi_\phi(x^{-1}y)\xi_\phi - \pi_\phi(x^{-1})\xi_\phi\| dy \\ &= \int_V [2 - 2\mathrm{Re}(\pi_\phi(x^{-1}y)\xi_\phi|\pi_\phi(x^{-1})\xi_\phi)]^{1/2} dy \\ &= \int_V [2 - 2\mathrm{Re}(\pi_\phi(y)\xi_\phi|\xi_\phi)]^{1/2} dy = \int_V [2 - 2\mathrm{Re}\phi(y)]^{1/2} dy \end{aligned}$$

$$\leq \left(\int_V [2 - 2\mathrm{Re}\phi(y)]dy\right)^{1/2} |V|^{1/2} < 2|V|\sqrt{\epsilon},$$

where the last inequality follows from †. By lemma 4.2.8 we can find a weak*-neighborhood $\mathcal{V}''$ of ϕ_0 in $\mathcal{P}_1(G)$ such that for all $\phi \in \mathcal{V}''$ we have $|\chi_V * \phi - \chi_V * \phi_0| < \epsilon|V|$ on K.

We now claim that $\mathcal{V} = \mathcal{V}' \bigcap \mathcal{V}''$ is the desired weak*-neighborhood. For any $\phi \in \mathcal{V}$ and any $x \in K$ we have

$$\begin{aligned} |\phi(x) - \phi_0(x)|\,|V| &\leq \big||V|\phi(x) - \chi_V * \phi(x)\big| + |\chi_V * \phi - \chi_V * \phi_0| + \\ &\quad |\chi_V * \phi_0(x) - |V|\phi_0(x)| \\ &\leq 2|V|\sqrt{\epsilon} + |V|\epsilon + 2|V|\sqrt{\epsilon} = (\epsilon + 4\sqrt{\epsilon})|V|. \end{aligned}$$

□

Definition 4.2.10 Let $\phi \in \mathcal{P}(G)$ then ϕ is said to be **pure** if $\pi_\phi \in r(G)$. The set of all pure functions in $\mathcal{P}_1(G)$ will be denoted by $\mathcal{P}_P(G)$.

4.2.11 Let $\phi \in \mathcal{P}(G)$ and $\omega \in L^1(G)^*$ be defined by ϕ. Note that $\phi \in \mathcal{P}(G)$ is pure if and only if $\pi_\phi \in r(G)$ if and only if [4.1.9] $\pi_\omega \in r(L^1(G))$ if and only if [proposition 2.3.5] ω is pure. Combining the fact that $\phi \in \mathcal{P}(G)$ is pure if and only if ω is pure and our claim in 4.2.7 yields a bijective correspondence between $\mathcal{P}_P(G)$ and $P(L^1(G))$, the set of pure states of $L^1(G)$. Thus analogous to proposition 2.3.3 we have $\mathcal{P}_P(G) \bigcup \{0\}$ is the set of extreme points of $\mathcal{P}_1(G)$.

References: [5], [8], [10], [13], [14].

4.3 The Enveloping C*-algebra

Proposition 4.3.1 *Let A be an involutive Banach algebra with an approximate identity. Then we have*

$$\sup_{\pi\in R(A)} \|\pi(x)\| = \sup_{\pi\in r(A)} \|\pi(x)\| = \sup_{f\in(A^{*+})_1} f(x^*x)^{1/2} = \sup_{f\in P(A)} f(x^*x)^{1/2},$$

and denoting this common value by $\|x\|'$, we have $\|x\|' \leq \|x\|$. Moreover, the map $x \to \|x\|'$ is a seminorm on A such that

$$\|xy\|' \leq \|x\|'\|y\|', \quad \|x^*\|' = \|x\|', \quad \|x^*x\| = \|x\|'^2$$

for any $x, y \in A$.

Proof: Claim $\sup_{\pi\in R(A)} \|\pi(x)\| \leq \sup_{f\in(A^{*+})_1} f(x^*x)^{1/2}$. If $\pi \in R(A)$ then from our remark 2.2.16 all $f \in (A^{*+})_1$ associated with π are of the form $f(x^*x) = (\pi(x)\xi|\pi(x)\xi)$, where $\xi \in \mathcal{H}_\pi$ and $\|\xi\| \leq 1$. Hence $\|\pi(x)\|^2 = \sup_{\|\xi\|\leq 1}(\pi(x)\xi|\pi(x)\xi) = \sup_{\|\xi\|\leq 1} f(x^*x)$, where $f \in (A^{*+})_1$ is associated with π. Since the set of all $f \in (A^{*+})_1$ associated with π is a subset of $(A^{*+})_1$ the inequality follows.

Claim $\sup_{f\in(A^{*+})_1} f(x^*x)^{1/2} \leq \sup_{f\in P(A)} f(x^*x)^{1/2}$. For $f \in (A^{*+})_1$ and any $x \in A$ by definition $f(x^*x) \geq 0$. As well from 2.3.3 we have $f(x^*x) \leq \sup_{g\in P(A)} g(x^*x)$.

Claim $\sup_{f\in P(A)} f(x^*x)^{1/2} \leq \sup_{\pi\in r(A)} \|\pi(x)\|$. If $f \in P(A)$ then $\pi_f \in r(A)$ by proposition 2.3.5 and hence the proof of the first claim yeilds $f(x^*x) \leq \|\pi_f(x)\|^2$.

Claim $\sup_{\pi\in r(A)} \|\pi(x)\| \leq \sup_{\pi\in R(A)} \|\pi(x)\|$. This is obvious since $r(A) \subset R(A)$.

The four claims above show that indeed $\|x\|'$ has a common value and since $\|\pi(x)\| \leq \|x\|$ for all $\pi \in R(A)$ we have $\|x\|' \leq \|x\|$. Moreover, $x \to \|\pi(x)\|$ is a seminorm on A, so $x \to \|x\|'$ is also a seminorm on A. As well, for each $\pi \in R(A)$ and $x, y \in A$ we have

$$\|\pi(x^*)\| = \|\pi(x)\| \;\Rightarrow\; \|x^*\|' = \|x\|',$$
$$\|\pi(x^*x)\| = \|\pi(x)\|^2 \;\Rightarrow\; \|x^*x\|' = \|x\|'^2,$$
$$\|\pi(xy)\| = \|\pi(x)\|\|\pi(y)\| \;\Rightarrow\; \|xy\|' = \|x\|'\|y\|'.$$

□

Definition 4.3.2 Let $N = \{x \in A : \|x\|' = 0\}$ then clearly N is a closed self-adjoint two-sided ideal of A. $x + N \to \|x + N\|'$ is a norm on the involutive algebra A/N such that $\|(x + N)^*(x + N)\| = \|(x + N)\|^2$. Thus the completion B of A/N with regard to this norm is a C*-algebra. B is called the **enveloping C*-algebra** of A.

Definition 4.3.3 Let G be a locally compact group. $L^1(G)$ is an involutive Banach algebra with an approximate identity [A.3.1] and as such we can form its enveloping C*-algebra. We call this C*-algebra the **group C*-algebra of** G and denote it by $C^*(G)$.

4.3.4 For $f \in L^1(G)$ set $\|f\|' = \sup \|\pi(f)\| \leq \|f\|_1$, where we take the supremum as π runs over the non-degenerate representations of $L^1(G)$ or equivalently $R(G)$. Then $f \to \|f\|'$ is a seminorm on $L^1(G)$ [4.3.1]. In fact, we claim $f \to \|f\|'$ is a norm on $L^1(G)$. To see this, suppose $\|f\|' = 0$ then $\pi(f) = 0$ for every $\pi \in R(G)$. However taking the left regular representation $\pi_l \in R(G)$ as found in 4.1.3 then $\pi_l(f)$ is the operator $g \to f * g$ in $L^2(G)$ and we have for all $g \in L^1(G)$, $0 = \pi_l(f)g = f * g$. Thus when g is an approximate identity in $L^1(G)$ we have $f = 0$. In other words $\pi_l \in R(L^1(G))$ is an injective representation. Hence $C^*(G)$ is just the completion of $L^1(G)$ for this norm.

Proposition 4.3.5 *Let A be an involutive Banach algebra with an approximate identity and let κ be the canonical map of A into B, the enveloping C*-algebra of A. If $\pi \in R(A)$*

then there is exactly one $\rho \in R(B)$ such that $\pi = \rho \circ \kappa$ and $\rho(B)$ is the C*-algebra generated by $\pi(A)$.

Proof: Let $\pi \in R(A)$ then, in the notation of 4.3.2, $x \in N$ if and only if $\pi(x) = 0$ and π defines a representation $\pi' \in R(A/N)$ such that $\|\pi'(y+N)\|_B \leq \|y+N\|_B$ for $y \in A$. Hence π' extends to a $\rho \in R(B)$ such that $\pi = \rho \circ \kappa$. Since $\kappa(A)$ is dense in B it follows that ρ is unique and that $\pi(A)$ is operator-norm dense in $\rho(B)$. Finally, since B is a C*-algebra $\rho(B)$ is also a C*-algebra, hence the C*-algebra $\rho(B)$ must be the C*-algebra generated by $\pi(A)$. □

4.3.6 Using the notation of proposition 4.3.5 it is clear that the map $\pi \to \rho$ is a bijection from $R(A)$ onto $R(B)$ that preserves non-degeneracy and topological irreducibility.

4.3.7 It now follows from proposition 4.3.5 that the bijective correspondence $\pi \to \pi'$ from $R(G)$ onto the non-degenerate representations in $R(L^1(G))$ can be extended to a bijective correspondence from $R(G)$ onto $R(C^*(G))$. Moreover, the bijective correspondence $\pi \to \pi'$ from $r(G)$ onto $r(L^1(G))$ [4.1.9] can be extended to a bijective correspondence from $r(G)$ onto $r(C^*(G))$. In fact, we can now replace $L^1(G)$ by $C^*(G)$ in section 4.1.

Proposition 4.3.8 *Let A be an involutive Banach algebra with an approximate identity and let κ be the canonical map of A into B, the enveloping C*-algebra of A. If $f \in A^{*+}$ then there exists a unique $g \in B^{*+}$ such that $f = g \circ \kappa$ and $\|g\| = \|f\|$.*

Proof: Clearly if such a $g \in B^{*+}$ exists it must be unique since $\kappa(A)$ is dense in B. So we need only show existence. Let $f \in A^{*+}$ and $\{e_i\}$ be an approximate identity for A then for each $x \in A$,

$$|f(x)|^2 = \lim_i |f(xe_i)|^2 \leq f(x^*x)\overline{\lim_i} f(e_i^*e_i) \leq \|f\| f(x^*x) \leq \|f\|^2 \|x\|'^2,$$

so there is a $g \in B^*$ such that $f = g \circ \kappa$ and $||g|| \leq ||f||$. Since for any $y \in B$ there exists a net $\{x_i\}$ in A such that $\kappa(x_i) \to y$ we have

$$g(y^*y) = \lim_i f(x_i^*x_i) \geq 0,$$

so $g \in B^{*+}$. As well, if $x \in A$ such that $||x|| \leq 1$ then

$$|f(x)| = |g(\kappa(x))| \leq ||g||||\kappa(x)|| \leq ||g||||x|| \leq ||g||,$$

therefore $||f|| \leq ||g||$. □

4.3.9 ¿From proposition 4.3.8 it follows that the map $f \to g$ of A^{*+} onto B^{*+} is a bijection. Moreover, if $M \subset A^{*+}$ and N is its image under the map $f \to g$ then clearly the restriction of $f \to g$ from M to N is bicontinuous for the relative weak*-topologies $\sigma(A^*, A)$ on A^{*+}, and $\sigma(B^*, \kappa(A))$ on B^{*+}. If further $M \subset A^{*+}$ is bounded then N is bounded, since the map $f \to g$ is norm preserving. Therefore since $\kappa(A)$ is dense in B then the relative weak*-topologies $\sigma(B^*, \kappa(A))$ and $\sigma(B^*, B)$ coincide on N.

4.3.10 Specifically we now have a bijective correspondence between $L^1(G)^{*+}$ and $C^*(G)^{*+}$. We are now able replace $L^1(G)^{*+}$ by $C^*(G)^{*+}$ in section 4.2.

As well, since $\mathcal{P}_1(G)$ cooresponds to the bounded subset of $S(L^1(G))$ [4.2.7] and hence $S(C^*(G))$, this bijective correspndence restricted to $\mathcal{P}_1(G)$ is bicontinuous for the relative weak*-topology $\sigma(C^*(G)^*, C^*(G))$ on $S(C^*(G))$. ¿From 4.2.9 it follows that the topology of compact convergence on $\mathcal{P}_1(G)$ agrees with the relative weak*-topology $\sigma(C^*(G)^*, C^*(G))$ on $S(C^*(G))$.

References: [3], [5], [8], [10], [13], [16].

4.4 The Fell Topology on $\hat{G}$

Definition 4.4.1 Let $\hat{G}$ denote the set of equivalance classes of $r(G)$. So $[\pi] \in \hat{G}$ if π is topologically irreducible. ¿From comment 4.1.9 and proposition 4.2.5 it follows that there is a canonical bijection Υ of $\widehat{C^*(G)}$ onto $\hat{G}$. Thus we can transport the Fell topology $\mathcal{F}$ [3.1.4] on $\widehat{C^*(G)}$ onto $\hat{G}$ using this bijection. That is, if U open in $\widehat{C^*(G)}$ then $\Upsilon(U)$ is open in $\hat{G}$. We will again denote the Fell topology on $\hat{G}$ by $\mathcal{F}$. The topological space $(\hat{G}, \mathcal{F})$ is called the **spectrum** of the locally compact group G and we will denote it simpily by $\hat{G}$. Many authors also refer to our $\hat{G}$ as the dual space of G.

Definition 4.4.2 Let $\pi \in R(G)$ and $S \subset R(G)$. We say π is **weakly contained** in S if π, viewed as an element of $R(C^*(G))$ [4.3.7], is weakly contained [3.1.2] in S, where S is viewed as a subset of $R(C^*(G))$.

Theorem 4.4.3 (Equivalence Theorem II) *Let G be a locally compact group, $\pi \in R(G)$, and $S \subset R(A)$. Then the following are equivalent:*

i) π is weakly contained in S,

ii) Every $\phi \in \mathcal{P}_1(G)$ associated with π is the uniform limit over every compact set of sums of $\psi \in \mathcal{P}_1(G)$ associated with S.

Moreover when π has a cyclic vector ξ, the above condition are equivalent to:

ii') The function $x \to \phi(x) = (\pi(x)\xi|\xi)$ is the uniform limit over every compact set of nets $\{\psi_i\}$ in $\mathcal{P}(G)$ where each ψ_i is associated with S.

Proof: This is simpily a translation of the equivalence theorem 2.4.5 into a group context. We can identify $R(G)$ with $R(C^*(G))$ by 4.3.7. Hence we have $i) \Leftrightarrow \{2.4.5\ i)\}$ where the C*-algebra A is replaced by $C^*(G)$.

Our comment 4.3.10 describes the correspondence between the topological spaces $\mathcal{P}_1(G)$ paired with the relative topology of compact convergence and $S(C^*(G))$ paired with the relative weak*-topology. Therefore, it follows that $ii) \Leftrightarrow \{2.4.5\ iii)\}$ where again the C*-algebra A is replaced by $C^*(G)$. Hence applying the equivalence theorem 2.4.5 to $C^*(G)$ leads to the equivalence of $i)$, $ii)$ and $ii')$. □

Corollary 4.4.4 *Let $[\pi] \in \hat{G}$ and $S \subseteq \hat{G}$. Then the following are equivalent:*

i) $[\pi] \in \overline{S}^{\mathcal{F}}$

ii) Every $\phi \in \mathcal{P}(G)$ associated with $[\pi]$ is the uniform limit over every compact set of $\psi \in \mathcal{P}(G)$ associated with S.

Proof: Again, this is just the translation of corollary 3.2.2 into a group context and hence follows from the remarks made above in the proof of the equivalence theorem II. □

References: [5], [9], [10], [13].

4.5 The Reduced Dual

4.5.1 It is easily verified that the left and right regular representations, π_l and π_r in $R(G)$, each have a kernel equal to $\{1\}$, and hence are injective. As well, in comment 4.3.4 π_l as a non-degenerate representation of $L^1(G)$ is also injective. Since π_l is non-degenerate it extends to a representation of $C^*(G)$ [4.3.7]. However, π_l as representaion of $C^*(G)$ is not in general injective, ie. for non-amenable G [cf 4.5.6].

Definition 4.5.2 With notation as in 4.1.3 we again look at the left and right regular representations of G. Now consider the isomorphism $f \to \tilde{f}$, where $\tilde{f}(x) =$

$\Delta(x)^{-1/2}f(x^{-1})$, from $L^2(G)$ onto $L^2(G)$. For each $y \in G$ we have

$$\begin{aligned}\widetilde{L_y f}(x) &= \Delta(x)^{-1/2}(L_y f)(x^{-1}) = \Delta(x)^{-1/2}f(y^{-1}x^{-1}) \\ &= \Delta(y)^{1/2}\Delta(xy)^{-1/2}f((xy)^{-1}) = R_y\tilde{f}(x),\end{aligned}$$

hence $f \to \tilde{f}$ transforms π_l into π_r and it follows that π_l and π_r are equivalent. Thus we can call $[\pi_l]$ or equivalently $[\pi_r]$ the **regular representation** without indicating left or right.

Definition 4.5.3 Let π be a representation of a C*-algebra A. Then the **support** $supp(\pi)$ of π is the set of $[\rho] \in \hat{A}$ such that each $[\rho]$ is weakly contained in $[\pi]$. If π is a unitary representation of a locally compact group G then in light of definition 4.4.2 and with the notation used in comment 4.4.1 we have the support $supp(\pi)$ of π is the set

$$\left\{[\rho] \in \hat{G} : \ker\left[\Upsilon^{-1}([\rho])\right] \supset \ker\left[\Upsilon^{-1}([\pi])\right]\right\},$$

that is the set of $[\rho] \in \hat{G}$ such that each $[\rho]$ is weakly contained in $[\pi]$.

Definition 4.5.4 Considering $\pi_l \in R(L^1(G))$, the norm closure of $\pi_l(L^1(G))$ in $\mathcal{L}(L^2(G))$ is the **reduced group C*-algebra**, denoted $C_r^*(G)$. It follows directly from proposition 4.3.5 that $C_r^*(G) = \pi_l(C^*(G))$ where $\pi_l \in R(C^*(G))$ [4.3.7].

Definition 4.5.5 We define the **reduced dual** of G, denoted $\hat{G}_r$, to be the support of the regular representation of G. That is, with Υ as in 4.4.1 we have

$$\hat{G}_r = \left\{[\rho] \in \hat{G} : \ker\left[\Upsilon^{-1}([\rho])\right] \supset \ker\left[\Upsilon^{-1}([\pi_l])\right]\right\},$$

or in other words, $\hat{G}_r = \widehat{C_r^*(G)} = (\widehat{C^*(G)}/N)$ where $N = \ker(\pi_l)$.

4.5.6 Note, from the definition of $\hat{G}_r$, that if π_l as a representation of $C^*(G)$ [4.3.7] is injective then $\hat{G} = \hat{G}_r$. If π_l is injective then $N = 0$ and hence $\hat{G}_r = \widehat{C^*(G)} = \hat{G}$. In fact, this special case, when $\pi_l \in R(C^*(G))$ is injective, fully characterizes [18, theorem 7.3.9] the important class of *amenable groups*.

A locally compact group G is said to be *amenable* if there is a linear functional m on $L^\infty(G)$ satisfying the following conditions:

i) $m(1) = 1$,

ii) $m(L_x f) = m(f)$ for all $x \in G$ and $f \in L^\infty(G)$,

iii) $m(g) \geq 0$ if $g \geq 0$ in $L^\infty(G)$.

As in [8, 1.25] and [11, 3.5.2] the following are equivalent:

i) G is amenable;

ii) $\hat{G} = \hat{G}_r$;

iii) $C^*(G) = C^*_r(G)$;

iv) π_l as a representation of $C^*(G)$ is injective;

v) every $[\rho] \in \hat{G}$ is weakly contained in π_l, that is, $\ker[\pi_l] \subset \ker[\rho]$ for all $[\rho] \in \hat{G}$.

As well, it is well known that all abelian groups and all compact groups are amenable. However, not all groups are amenable. The free group on two generators with discrete topology is not amenable. Hence, for any non-amenable group G, π_l as a representation of $C^*(G)$ is not injective.

4.5.7 Suppose $[\rho'] \in \overline{\widehat{C^*(G)}_r}^{\mathcal{F}}$ so

$$\ker[\rho'] \supset \bigcap_{[\rho] \in \widehat{C^*(G)}_r} \ker[\rho] \supset \ker[\pi_l] \Rightarrow [\rho'] \in \widehat{C^*(G)}_r.$$

hence $\widehat{C^*(G)_r}$ is a closed subset of $\widehat{C^*(G)}$ from which it follows that $\hat{G}_r$ is a closed subset of $\hat{G}$.

References: [5], [6], [8], [13], [11], [18].

Chapter 5

Structure on a Locally Compact Group G as related to its spectrum $\hat{G}$

As mentioned in the introduction, the presentaion of this chapter differes from the other chapters. Many details are simiply glossed over and outside references are used frequently.

One of the remarkable attributes of the Fell topology on $\hat{G}$ is its characterization of properties of G by simple separation properties of $\hat{G}$. The aim of this chapter is to study a few of these characterizations. G will always denote a locally compact group.

5.1 The Topological Structure of $\hat{G}$

For a C*-algebra A, we can conclude, from comment 3.3.5 and proposition 3.3.10, that $\hat{A}$ is not in general a T_0-space. In this section the inherent topological structure on $\hat{A}$ for an arbitrary C*-algebra A is studied. Specifically, we show $\hat{A}$ to be a locally quasi-compact Baire space. Hence, $\hat{G}$ is a locally quasi-compact Baire space.

Definition 5.1.1 Let A be a topological space. A subset S of A is **rare** if the interior of the closure of S is empty, that is $\text{int}(\overline{S}) = \emptyset$. A subset F of A is **meager** if $F = \bigcup_1^\infty S_n$ with S_n rare for all n; otherwise F is said to be a **nonmeager** subset of A.

Definition 5.1.2 A **Baire space** is a topological space A which satisfies one and hence all of the following conditions which are equivalent [3, theorem 46.4]:

i) if S is any meager subet of A then $\overline{(A \sim S)} = A$,

ii) if U is a nonempty open subset of A then U is nonmeager,

iii) if $S = \bigcup_1^\infty S_n$, where the S_n are closed sets such that $\text{int}(S_n) = \emptyset$ for all n, then $\text{int}(S) = \emptyset$,

iv) if $S = \bigcap_1^\infty U_n$, where the U_n are dense open sets of A, then $\overline{S} = A$.

Proposition 5.1.3 *Let A be a C^*-algebra, then $\hat{A}$ is a Baire space.*

Proof: Recall the surjective map $\Phi : P(A) \to \hat{A}$ described in 3.2.1 and note that in the proof of proposition 3.2.3 we show Φ to be continuous and open. Let $(V_1, V_2, \ldots)$ be a decreasing sequence of dense open subsets of $\hat{A}$, and let $U_n = \Phi^{-1}(V_n)$, that is U_n is the inverse image of V_n in $P(A)$. Since Φ is continuous and open, each U_n is a dense open subset of $P(A)$. Due to G. Choquet [5, B 14, p. 395] it is known that $P(A)$ is a Baire space. Hence, $\overline{\bigcap U_n} = P(A)$ and therefore $\overline{\Phi(\bigcap U_n)} = \overline{\bigcap V_n} = \hat{A}$. □

Definition 5.1.4 Recall, a topological space is compact if and only if each family of closed sets which has the finite intersection property has a non-empty intersection. We say a topological space is **quasi-compact** if every decreasing filtering family of closed sets has a non-empty intersection. A topological space is said to be locally quasi-compact if each point has a base of quasi-compact neighourhoods.

Proposition 5.1.5 *Let A be a C*-algebra, $x \in A$ and $\alpha > 0$. Then $F = \{[\pi] \in \hat{A} : \|\pi(x)\| \geq \alpha\}$ is quasi-compact.*

Proof: Let $\{F_i\}$ be a decreasing filtering family of relatively closed sets of F. For each i, set $J_i = \bigcap_{[\pi] \in F_i} \ker[\pi]$. Then $\{J_i\}$ constitutes an increasing filtering family. Let $J = \overline{(\bigcup_i J_i)}$, then J is a closed two-sided ideal of A. For each i the canonical image of x modulo J_i is of norm $\geq \alpha$. In fact, by definition of the quotient norm of a normed space the canonical image of x modulo J is of norm $\geq \alpha$. Now, there exists a $[\rho] \in \hat{A}$ [5, lemma 3.3.6] such that $\|\rho(x)\| = \|x\|$ and $\ker[\rho] \supset J$. Thus $[\rho] \in F$. As well, $[\rho] \in \overline{F_i}^{\mathcal{F}}$ for each i and hence $[\rho] \in \bigcap_i F_i$. □

Corollary 5.1.6 *Let A be a C*-algebra, then $\hat{A}$ is locally quasi-compact.*

Proof: Let $\pi \in \hat{A}$, and let U be an open neighborhood of π in $\hat{A}$. Since $\hat{A} \sim U$ is closed, there is an $x \in A$ such that $\pi(x) \neq 0$ and $\rho(x) = 0$ for all $\rho \in \hat{A} \sim U$. Set $V = \{\rho \in \hat{A} : \|\rho(x)\| > \|\pi(x)\|/2\}$ and $W = \{\rho \in \hat{A} : \|\rho(x)\| \geq \|\pi(x)\|/2\}$. Since $\pi \to \|\pi(x)\|$ is lower semicontinuous on $\hat{A}$ [5, proposition 3.3.2], V is an open neighborhood of π. Therefore W is a neighborhood of π contained in U, and by proposition 5.1.5, W is quasi-compact. □

Theorem 5.1.7 *If G is a locally compact group then $\hat{G}$ is a quasi-compact Baire space.*

Proof: By propositions 5.1.3 and 5.1.4 $\widehat{C^*(G)}$ is a quasi-compact Baire space and our result follows immediately. □

References: [5, §3.3 and corollary 3.4.13].

5.2 The Spectrum of a Compact Group

Locally compact *Abelian* groups [and hence compact Abelian groups] possess many nice attributes, not the least of which is that all of their irreducible representations are unitary characters [complex-valued, multiplicative, not identically zero functions]. If $\pi \in r(G)$, G locally compact Abelian, then π is one dimensional. Hence, we can take $\mathcal{H}_\pi = \mathbf{C}$, so $\pi(x)(z) = \zeta(x)z$, where $z \in \mathbf{C}$ and ζ is a continuous morphism of G into the circle group. Moreover, for a locally compact Abelian group G, it turns out that $\hat{G}$ is the set of extreme points for the set of functions of positive type on G of norm 1 [10, theorem 3.25]. Thus we can give $\hat{G}$ the topology of compact convergence which in this case is exactly the relative weak*-topology on extreme points for the set of functions of positive type on G of norm 1.

Proposition 5.2.1 *Let G be a locally compact Abelian group. $\hat{G}$ is identified with the spectrum of $L^1(G)$ [set of non-zero multiplicative functionals on $L^1(G)$] via 5.1.*

Proof: ¿From proposition 4.1.4, each $\zeta \in \hat{G}$ determines a non-degenerate representation of $L^1(G)$ on $\mathbf{C}$ by

$$\zeta(f) = \int \zeta(x)f(x)dx. \tag{5.1}$$

Identifing $\mathcal{L}(\mathbf{C})$ with $\mathbf{C}$ all such representations are complex-valued, multiplicative, not identically zero functionals on $L^1(G)$. Conversely, $\Phi \in (L^1(G))^*$ is given by integration against some $\psi \in L^\infty(G)$. Choose $f \in L^1(G)$ with $\Phi(f) \neq 0$. Then for $g \in L^1(G)$,

$$\begin{aligned} \Phi(f)\int \psi(y)g(y)dy &= \Phi(f)\Phi(g) = \Phi(f * g) \\ &= \int\int \psi(x)f(xy^{-1})g(y)dydx \\ &= \int \Phi(L_y f)g(y)dy, \end{aligned}$$

so $\psi(y) = \Phi(L_y f)/\Phi(f)$ locally almost everywhere. We can redefine $\phi(y)$ such that $\phi(y) = \Phi(L_y f)/\Phi(f)$ for all y. Then ϕ is continuous and

$$\psi(xy)\Phi(f) = \Phi(L_{xy}f) = \Phi(L_x L_y f) = \psi(x)\psi(y)\Phi(f),$$

so $\psi(xy) = \psi(x)\psi(y)$. As well, $\psi(x^n) = \psi(x)^n$ for every n, and ψ is bounded, so $|\psi(x)| = 1$. That is, ψ maps G to the circle group. □

It is easily verified, that under pointwise multiplication, $\hat{G}$ is an Abelian group. It has the constant function 1 as its identity element and $\zeta^{-1}(x) = \zeta(x^{-1}) = \overline{\zeta(x)}$. As well it can be verified that $\hat{G} \bigcup \{0\}$ is a weak*-closed subset of the closed unit ball of $L^\infty(G)$. Hence as a consequence of Alaoglu's theorem, $\hat{G}$ is a locally compact Abelian group.

Definition 5.2.2 If G is a locally compact Abelian group then $\hat{G}$ endowed with the topology of compact convergence is called the **dual group**. For general non-commutative topological groups, $\hat{G}$ is *not* necessarily a group. Hence a dual group is merely a special case of the spectrum or dual space as defined in 4.4.1.

Proposition 5.2.3 *If G is an Abelian compact group with a normalized Haar measure, then $\hat{G}$ is an orthonormal set in $L^2(G)$.*

Proof: Note, if G is compact, then $\hat{G} \subset L^\infty(G) \subset L^p(G)$ for all $p \geq 1$. If $\zeta \in \hat{G}$ then $\int |\zeta(x)|^2 dx = \int dx = 1$, that is, $||\zeta||_2 = 1$. Further, let $\vartheta \in \hat{G}$ such that $\zeta \neq \vartheta$. Thus, there is a $z \in G$ such that $\zeta(z)\vartheta^{-1}(z) \neq 1$. Hence,

$$\begin{aligned} \int \zeta(x)\vartheta^{-1}(x)dx &= \zeta(z)\vartheta^{-1}(z) \int \zeta(z^{-1}x)\vartheta^{-1}(z^{-1}x)dx \\ &= \zeta(z)\vartheta^{-1}(z) \int \zeta(x)\vartheta^{-1}(x)dx, \end{aligned}$$

so $\int \zeta(x)\vartheta^{-1}(x)dx = 0$ and therefore $\int \zeta(x)\overline{\vartheta(x)}dx = 0$. □

Theorem 5.2.4 *Let G be a locally compact Abelian group. Then,*

i) if G is discrete, then $\hat{G}$ is compact;

ii) if G is compact, then $\hat{G}$ is discrete.

Proof: If G is discrete, then the point mass function which is equal to 1 at e and 0 elsewhere, is a unit for $L^1(G)$. Hence the spectrum of $L^1(G)$ is compact. Therefore, it follows directly from proposition 5.2.1 that $\hat{G}$ is compact.

If G is compact then the constant function 1 is in $L^1(G)$. Thus, the set $\{f \in L^\infty(G) : |\int f(x)dx| > \frac{1}{2}\}$ is weak*-open. It follows from proposition 5.2.3, if $\zeta \in \hat{G}$ then $\int \zeta(x)dx = 1$ if $\zeta = 1$, $\int \zeta(x)dx = 0$ if $\zeta \neq 1$. Hence, $\{1\}$ is open in $\hat{G}$ and $\hat{G}$ is discrete. □

We now look at general [non-abelian] compact groups. In the 728 page often quoted Abstract Harmonic Analysis II [the thicker one] by Hewitt and Ross [7], which is exclusively devoted to the study of non-abelian compact groups, the authors begin by lamenting the lack of detail in their presentation. We mention this to emphasize the sheer magnitude of theory that we cannot, and do not, do justice to in these few pages. In theorem 5.2.4 we saw that for compact abelian groups their spectrum is compact. The remainder of this section is devoted to generalizing this result to compact [non-Abelian] groups. Note that in Abelian case above $\hat{G}$ was an orthonormal set in $L^2(G)$. In the general case [non-Abelian] the corresponding set of functions is the set of matrix elements of unitary representations of G [cf. 5.2.5]. The following is a sketch of the celebrated Peter-Weyl Theorem as found in Folland's book [10]. Our aim here is show how the matrix elements of irreducible representations can be used to form an orthonormal basis for $L^2(G)$.

Definition 5.2.5 If π is a unitary representation of G, then the **matrix elements** of π are the functions,

$$\phi_{\xi,\nu}(x) = (\pi(x)\xi|\nu) \quad \xi, \nu \in \mathcal{H}_\pi.$$

If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$ then $\phi_{e_j,e_i}(x)$ is indeed one of the entries of the matrix $\pi(x)$ with respect to that basis, namely

$$\pi_{ij}(x) = \phi_{e_j,e_i}(x) = (\pi(x)e_j|e_i). \tag{5.2}$$

Let $\mathcal{E}_\pi$ denote the linear span of the matrix elements of π. Clearly $\mathcal{E}_\pi$ is a subspace of $C(G)$ the continuous functions on G and hence of $L^p(G)$ for all p. Further, let

$$\mathcal{E} = \text{the linear span of } \bigcup_{[\pi]\in\hat{G}} \mathcal{E}_\pi.$$

Finally, set $d_\pi = \dim\mathcal{H}_\pi$ and let tr$B$ denote the trace of a matrix B.

Theorem 5.2.6 (Peter-Weyl Theorem) *Let G be a compact group. Then*

i) $\mathcal{E}$ is dense in $C(G)$ in the uniform norm.

ii) $\mathcal{E}$ is dense in $L^p(G)$ in the L^p norm for $p < \infty$.

iii) $L^2(G) = \bigoplus_{[\pi]\in\hat{G}} \mathcal{E}_\pi$ and $\{\sqrt{d_\pi}\pi_{ij} : i, j = 1, \ldots, d_\pi, [\pi] \in \hat{G}\}$ is an orthonormal basis for $L^2(G)$.

<u>Proof:</u> Since $C(G)$ is dense in $L^p(G)$, it will suffice to show $\mathcal{E}$ is dense in $C(G)$ for both i) and ii) to hold. In fact, $\mathcal{E}$ satisfies the conditions of the Stone-Weierstrass theorem and the hence the result follows. By the Gelfand-Raikov theorem [6, theorem 22.12] $\mathcal{E}$ separates points. Since each representation has a contragredient, $\mathcal{E}$ is closed under conjugation. The existence of the trivial representation of G on $\mathbf{C}$ allows for the existence

of the constant fuctions. All that remains is to show $\mathcal{E}$ is an algebra. We refer the reader to [6, p. 23] where indeed $\mathcal{E}$ is shown to be an algebra. The basic idea is, if $[\pi], [\pi'] \in \hat{G}$ the we want to show $\pi_{ij}\pi'_{kl}$ is a matix element of some finite-dimensional representation of G. This is acheived by constructing the inner tensor product of π and π'.

Finally, *iii)* is a consequence of *ii)* and the following claim.

Claim: [Shur Orthogonality Relations] *If $[\pi], [\pi'] \in \hat{G}$ then: i) $[\pi] \neq [\pi']$ implies $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$, and ii) $\{\sqrt{d_\pi}\pi_{ij} : i, j = 1, \dots, d_\pi\}$ is an orthonormal basis for $\mathcal{E}_\pi$, where $\mathcal{E}_\pi$ and $\mathcal{E}_{\pi'}$ are considered as a subspaces of $L^2(G)$.*

Let T be any linear map from $\mathcal{H}_\pi$ to $\mathcal{H}_{\pi'}$, and define $\tilde{T}$ such that

$$\tilde{T} = \int \pi'(x^{-1})T\pi(x)dx.$$

So

$$\tilde{T}\pi(y) = \int \pi'(x^{-1})T\pi(xy)dx = \int \pi'(yx^{-1})T\pi(x)dx = \pi'(y)\tilde{T},$$

that is, $\tilde{T}$ is an intertwining operator for π and π'. Now, set $\nu \in \mathcal{H}_\pi$, $\nu' \in \mathcal{H}_{\pi'}$ and define T by $T\xi = (\xi|\nu)\nu'$. Thus for all $\xi \in \mathcal{H}_\pi$ and $\xi' \in \mathcal{H}_{\pi'}$ we have,

$$\begin{aligned}
(\tilde{T}\xi|\xi') &= \int (T\pi(x)\xi|\pi'(x)\xi')dx \\
&= \int (\pi(x)\xi|\nu)(\nu'|\pi'(x)\xi')dx \\
&= \int \phi_{\xi,\nu}(x)\overline{\phi_{\xi',\nu'}(x)}dx
\end{aligned}$$

It is a consequence of proposition A.5.5 that, if two irreducible unitary representations are *not* equivalent then the set of intertwining operators, for these two representations, is simply $\{0\}$, see [10, 3.5 p. 71]. So, if $[\pi] \neq [\pi']$ then $\tilde{T} = 0$, and therefore from our above equality we have $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$. This proves *i*). If $[\pi] = [\pi']$ then $\tilde{T} = cI$ [A.5.5]. So, if we take $\xi = e_i$, $\xi' = e_{i'}$, $\nu = e_j$ and $\nu' = e_{j'}$ then

$$\int \pi_{ij}(x)\overline{\pi_{i'j'}(x)}dx = c(e_i|e_{i'}).$$

But

$$cd_\pi = \text{tr}\tilde{T} = \int \text{tr}\left[\pi(x^{-1})T\pi(x)\right] dx = \text{tr}T,$$

and since $T\xi = (\xi|e_j)e_{j'}$ we have $\text{tr}T = (e_j|e_{j'})$. Hence

$$\int \pi_{ij}(x)\overline{\pi_{i'j'}(x)}dx = d_\pi^{-1}(e_i|e_{i'})(e_j|e_{j'}),$$

so $\{\sqrt{d_\pi}\pi_{ij}\}$ is an orthonormal set. Since it is known that $\dim\mathcal{E}_\pi \le d_\pi^2$, see [10, proposition 5.6], we can conclude that $\{\sqrt{d_\pi}\pi_{ij} : i, j = 1, \ldots, d_\pi\}$ is a basis. □

Proposition 5.2.7 *If G is compact then $\hat{G}$ is discrete.*

Proof: Let $[\pi] \in \hat{G}$. G is compact so π is finite-dimensional [10, theorem 5.2] and we can express the character χ_π of π by $\chi_\pi(x) = \text{tr}\ \pi(x)$. Note, since the matrix representation of equivalent representations have the same trace, χ_π depends only on the equivalence class of π. It follows from the Peter-Weyl theorem *iii)* that $\pi(\chi_\pi) = (d_\pi)^{-1}I$, where I is the identity operator on $\mathcal{H}_\pi$, and $\rho(\chi_\pi) = 0$ for $[\pi] \neq [\rho]$. Hence, $\chi_\pi \in \bigcap_{[\rho]\in\hat{G}\sim\{[\pi]\}} \ker[\rho]$ however $\chi_\pi \notin \ker[\pi]$. That is, $[\pi] \notin \overline{(\hat{G} \sim \{[\pi]\})}$, so $\{[\pi]\}$ is open. □

Baggett [1, theorem 3.4] proved that if G is separable and $\hat{G}$ is discrete then G is compact. As well, it has been proved by Wang [19, theorem 7.7] that if G is σ-compact and $\hat{G}$ or $\hat{G}_r$ is discrete then G is compact. Hence, the converse of proposition 5.2.7 is also true for these cases.

References: [5], [10], [7].

5.3 Some Remarks on the Separation Properties of $\hat{G}$

$\hat{G}$ is not in general a T_0-space. In fact, we have shown in proposition 3.3.10 that $\hat{G}$ is T_0, if and only if, for any $[\pi]$, $[\pi'] \in \hat{G}$, if $\ker[\pi] = \ker[\pi']$ then π is equivalent to π'. However, for a 2^{nd} countable G, $\hat{G}$ is T_0 if and only if G is Type I [GCR or postliminal], and $\hat{G}$ is T_1 if and only if G is CCR [or liminal], see [17]. For a connected G, $\hat{G}$ is T_2 if and only if G is a compact extension of an abelian group, see [2]. Liukkonen in [15] proved that, for a Type I [IN] group, $\hat{G}$ is T_2 if and only if G is $[FC]^-$.

Note: The following is a list of references where adequate descriptions of the above mentioned classes of C*-algebras and groups can be found.

i) For liminal[Type I or GCR] and postliminal[CCR] C*-algebras see Dixmier [5, Chapter 4]. A group is said to be GCR or CCR if $C^*(G)$ is GCR or CCR respectively.

ii) See [15] and references within for a description of [IN] groups.

iii) Finally, $[FC]^-$ groups are described in [17].

Appendix A

A.1 Structure on Topological Groups

The main purpose of this section is to show that it is essentially no restriction to assume a topological group is Hausdorff. We start by stating some basic facts about toopological groups.

Proposition A.1.1 *Let (G, τ) be a topological group:*

i) If $U \in \tau$ then for any $x \in G$ we have xU, Ux and U^{-1} are also in τ.

ii) For every neighborhood U of e there exists a symetric neighborhood V of e such that $VV \subset U$

iii) The τ closure of any subgroup of G is also a subgroup.

iv) If $A, B \subset G$ are comapct then so is AB.

Proof: These results are all a consequence of the continuity of the maps $(x, y) \to xy$ and $x \to x^{-1}$. □

Proposition A.1.2 *Let (G, τ) be a T_0 topological group. Then (G, τ) is regular and hence Hausdorff.*

Proof: Let U be an arbitrary neighborhood of e. Then from A.1.1 (ii) there exists a symetric neighborhood V of e such that $VV \subset U$. For $x \in \overline{V}$ the intersection $xV \bigcap V$ is not empty since xV is a neighborhood of x. Hence $xv_1 = v_2$ for $v_1, v_2 \in V$, so $x = v_2 v_1^{-1} \in VV^{-1} = VV \subset U$. We have therefore shown, that for any neighborhood U of e, there exists a closed neighborhood $\overline{V}$ of e such that $\overline{V} \subset U$. By proposition A.1.1 $i)$ we can transfer this regularity property from e to all $x \in G$. □

Proposition A.1.3 *Let (G, τ) be a topological group. If (G, τ) is not a T_0 space then $\overline{\{e\}}$ is a closed normal subgroup and $G/\overline{\{e\}}$ endowed with the quotient topology is regular topological group and hence Hausdorff. Moreover, if G is locally compact then so is $G/\overline{\{e\}}$.*

Proof: It follows from proposition A.1.1 $iii)$ that $\overline{\{e\}}$ is a subgroup. Since every subgroup of G must contain $\{e\}$, clearly $\overline{\{e\}}$ is the smallest closed subgroup of G. Thus $\overline{\{e\}}$ must be normal, otherwise we could intersect $\overline{\{e\}}$ with one of its conjugates to obtain a smaller subgroup. Therefore $\overline{\{e\}}$ is a closed normal subgroup.

We now show that $G/\overline{\{e\}}$ endowed with the quotient topology is a topological group. Since $\overline{\{e\}}$ is a normal subgroup, it is well known that the operation on the set of left cosets of $\overline{\{e\}}$ described by

$$x\overline{\{e\}}y\overline{\{e\}} = (xy)\overline{\{e\}},$$

is well defined. Let $q : G \to G/\overline{\{e\}}$ denote the canonical quotient map, let $x \in G$ and let U be a neighborhood of $q(x^{-1})$ in $G/\overline{\{e\}}$. By continuity of inversion in G at x we can find a neighborhood V of x such that $V \subset q^{-1}(U)$. Thus $q(V)$ is a neighborhood of $q(x)$ in $G/\overline{\{e\}}$ such that $q(V) \subset U$. Hence inversion is continuous with regard to the quotient topology on $G/\overline{\{e\}}$. A similar argument shows multiplication is also continuous. Therefore $G/\overline{\{e\}}$ endowed with the quotient topology is a topological group.

Moreover, if U is a compact neiborhood of e in G then q(Ux)is a compact neighborhood of $q(x)$ in $G/\overline{\{e\}}$. Hence if G is locally compact then so is $G/\overline{\{e\}}$.

Finally, we show $G/\overline{\{e\}}$ is a T_0 space and therefore by proposition A.1.2 is regular and hence Hausdorff. Let $q(x)$, $q(y) \in G/\overline{\{e\}}$ be distinct. Since $x\overline{\{e\}}y^{-1}$ does not contain e, it follows from A.1.1 *ii*) that we can find a symetric neighborhood U of e such that $U \bigcap x\overline{\{e\}}y^{-1} = \emptyset$ and by symmetry $U^{-1} \bigcap x\overline{\{e\}}y^{-1} = \emptyset$. Hence

$$e \notin Ux\overline{\{e\}}y^{-1} = (Ux\overline{\{e\}})(\overline{\{e\}}^{-1}y^{-1}) = (Ux\overline{\{e\}})(y\overline{\{e\}})^{-1},$$

so $(Ux\overline{\{e\}}) \bigcap (y\overline{\{e\}}) = \emptyset$ and therefore $q(Ux)$ is a neighborhood of $q(x)$ that does not contain $q(y)$. □

We have now accomplished the main objective of this section. In view of propositions A.1.2 and A.1.3 it is practically no restriction to assume a topological group is Hausdorff, for if not then we can just work with $G/\overline{\{e\}}$ instead. Henceforth we will always assume, in particular, that a locally compact group is Hausdorff.

References: [3], [6], [10].

A.2 The involutive algebra $L^1(G)$

A.2.1 Let G always denote a locally compact group. Since G is locally compact we know G possesses a left Haar measure λ [non-zero, left invarient, finite on compact sets, outer regular on Borel sets, inner regular on open sets, Borel measure] that is unique up to scalar multiplication [20, Sec 2.2]. We will fix once and for all the left Haar measure λ on G. We'll denote $d\lambda(x)$ by dx, $\int f d\lambda$ by $\int f$, and $|F|$ for $\lambda(F)$. The unit element of G will be denoted by e.

A.2.2 Let $M^1(G)$ be the space of bounded complex measures on G. If $\mu, \nu \in M^1(G)$ then we define the **convolution** of μ and ν as follows. The map $\phi \to \int\int \phi(xy)d\mu(x)d\nu(y)$ is a linear form on the space of continuous functions on G which vanish at infinity, and $|\int\int \phi(xy)d\mu(x)d\nu(y)| \leq \|\phi\|_\infty\|\mu\|\|\nu\|$. Hence this form is given by a measure $\mu * \nu \in M^1(G)$ with $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$. This measure, $\mu * \nu$ is called the convolution of μ and ν and we see that

$$\int \phi d(\mu * \nu) = \int\int \phi(xy)d\mu(x)d\nu(y).$$

It is readily checked $\int \phi d(\mu * \nu) = \int\int \phi(xy)d\mu(x)d\nu(y)$.reover convolution is commutative if and only if G is Abelian. $M^1(G)$ has a multiplicative identity δ_e, the point mass measure δ at e. $M^1(G)$ also has an **involution** $\mu \to \mu^*$ defined by $\mu^*(F) = \overline{\mu(F^{-1})}$ or $\int \phi(x)d\mu^* = \int \phi(x^{-1})d\overline{\mu}(x)$. Again this is readly checked to be an involution.

Definition A.2.3 Let $F \subset G$ and if we define $\lambda_x(F) = \lambda(Fx)$ then λ_x is again a left Haar measure and as such must be a scalar multiple of λ. Thus there is a scalar $\Delta(x) > 0$ such that $\lambda_x = \Delta(x)\lambda$. The mapping $\Delta : G \to (0, \infty)$ is called the **modular function**. Δ is said to be unimodular if $\Delta \equiv 1$. Obviously Abelian groups and discrete groups are unimodular.

Proposition A.2.4 *The modular function Δ is a continuous morphism of G to the multiplicative group of positive real numbers. Moreover, for any $f \in L^1(G)$,*

$$\int R_y f d\lambda = \Delta(y^{-1}) \int f d\lambda. \tag{A.1}$$

Proof: For any $x, y \in G$ and $E \subset G$,

$$\Delta(xy)\lambda(E) = \lambda(Exy) = \Delta(y)\lambda(Ex) = \Delta(y)\Delta(x)\lambda(E),$$

hence Δ is a morphism of G to the multiplicative group of positive real numbers.

Now to see equation (A.1) consider the characteristic function χ_E. Since $\chi_E(xy) = \chi_{Ey^{-1}}(x)$,

$$\int \chi_E(xy)d\lambda(x) = \lambda(Ey^{-1}) = \Delta(y^{-1})\lambda(E) = \Delta(y^{-1})\int \chi_E(x)d\lambda(x).$$

Thus we have shown (A.1) for $f = \chi_E$ so the (A.1) follows from the density of simple functions in $L^1(G)$.

Finally, the continuity of Δ follows from (A.1). Since each continuous function on G with compact support is right uniformly continuous [10, proposition 2.6] it follows immediately that $y \to \int R_y f d\lambda = \Delta(y^{-1}) \int f d\lambda$ is continuous from G to $\mathbf{C}$. □

Proposition A.2.5 *If G is compact then G is unimodular.*

Proof: We will in fact show the following more general result. If $K \subset G$ is any compact subgroup then the restriction $\Delta|_K$ of the modular function Δ to the subgroup K is equivalent to 1. Since Δ is a continuous morphism from G to the multiplicative group of positive real numbers [A.2.4?????], $\Delta|_K(G) = \Delta(K)$ must be a compact subgroup in this group of real numbers. Clearly $\Delta(G) = \{1\}$, that is $\Delta(x) = 1$ for all $x \in G$. □

A.2.6 If we identify each function $f \in L^1(G)$ with the measure $f(x)dx \in M^1(G)$ we can consider $L^1(G)$ as a subalgebra of $M^1(G)$. If $f, g \in L^1(G)$ the **convolution** of f and g is the function defined by

$$\begin{aligned} f * g(x) &= \int f(y)g(y^{-1}x)dy \\ &= \int f(xy)g(y^{-1})dy \\ &= \int f(y^{-1})g(yx)\Delta(y^{-1})dy \end{aligned}$$

$$\begin{aligned} &= \int f(xy^{-1})g(y)\Delta(y^{-1})dy \\ &= \int f(y)L_y g(x)dy \\ &= \int R_y f(x)g(y^{-1})dy. \end{aligned}$$

The restriction of the canonical involution on $M^1(G)$ to $L^1(G)$ is an **involution** defined by the relation $f^*(x)dx = \overline{f(x^{-1})}d(x^{-1})$ hence we have

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}.$$

A.2.7 We make use of the following application of the Lebesgue-Radon-Nikodym theorem. The theorem given here is just a special case of [6, Theorem 12.18] and we offer no proof here.

Theorem *$L^1(G)^* = L^\infty(G)$ in the sense that for every continuous linear functional ω on $L^1(G)$ there is a $g \in L^\infty(G)$ such that*

$$\omega(f) = \int fgd\lambda, \quad \forall f \in L^1(G),$$

and $\|\omega\| = \|g\|_\infty$.

Proposition A.2.8 *$L^1(G)$ has an approximate identity, [cf A.2.1] namely $\{e_U\}$.*

Proof: Let $\mathcal{U}$ be the family of compact symmetric neighborhoods U of $e \in G$ ordered by reverse inclusion. Then set $e_U = |U|^{-1}\chi_U$, where $|U| = \int_U dg$ and χ_U is the characteristic function of U. Clearly $\int e_U = e$ and since each U is symmetric $e_U(x^{-1}) = e_U(x)$. Thus

$$f * e_U(y) - f(y) = \int f(yx)e_U(x^{-1})dx - f(y)\int e_U(x)dx$$

$$= \int [R_x f(y) - f(y)] \, e_U(x) dx,$$

so by Minkowski's inequality

$$\|f * e_U - f\|_1 \leq \int \|R_x f - f\|_1 e_U(x) dx \leq \sup_{x \in U} \|R_x f - f\|_1.$$

Likewise we can show $\|e_U * f - f\| \leq \sup_{x \in U} \|L_x f - f\|_1$. Therefore we need only show that $\|R_x f - f\|_1 \to 0$ and $\|L_x f - f\|_1 \to 0$ as $x \to 1$.

First, consider $g \in C_0(G)$ and let $U_g = (\text{supp } g)U \bigcup U(\text{supp } g)$ where supp g is the support of g. Clearly U_g is compact, and $R_x g$ and $L_x g$ are supported in U_g when $x \in U$. Thus $\|R_x g - g\|_1 \leq |U_g| \|R_x g - g\|_\infty \to 0$ by uniform continuity. Similarly $\|L_x g - g\|_1 \to 0$. Now taking $f \in L^1(G)$, for $\epsilon \geq 0$, we can choose $g \in C_0(G)$ such that $\|f - g\|_1 \leq \epsilon$.

$$\begin{aligned} \|R_x f - f\|_1 &\leq \|R_x(f - g)\|_1 + \|R_x g - g\|_1 + \|g - f\|_1 \\ &\leq (\Delta(x)^{-1} + 1)\epsilon + \|R_x g - g\|_1, \end{aligned}$$

where $\|R_x g - g\|_1 \to 0$ as $x \to e$. Similarly $\|L_x f - f\|_1 \to 0$ as $x \to e$. □

References: [8], [10], [13].

A.3 Approximate identities

Definition A.3.1 Let A be a C*-algebra. An **approximate identity** of A is a net $\{e_i\}$ of elements in A such that:

$$\|e_i\| \leq 1, \ \forall i,$$
$$\|e_i x - x\| \to 0 \text{ and } \|x e_i - x\| \to 0, \ \forall x \in A.$$

We say $\{e_i\}$ is *increasing* if $e_i \geq 0$ and if $i \leq j$ implies $e_i \leq e_j$.

Corollary A.3.2 *Let A be a non-unital C*-algebra. Then there exists an increasing approximate identity $\{e_i\}$ for A.*

Proof: Since A is a self-adjoint two-sided ideal in A^1 this corollary is a direct result of the following Theorem. □

Theorem A.3.3 *Let A be a unital C*-algebra and I a left [right] ideal of A. Then there exists an increasing net $\{e_i\}$ of positive elements in $(I)_1$ such that $\|xe_i - x\| \to 0$ [$\|xe_i - x\| \to 0$] for every $x \in I$.*

Proof: Let Γ be the set of finite subsets of I ordered by inclusion. For $i = \{x_1, \cdots, x_n\} \in \Gamma$ let

$$\nu_i = x_1^* x_1 + \cdots + x_n^* x_n$$

and

$$e_i = \left(\frac{1}{n} + \nu_i\right)^{-1} \nu_i.$$

Since the function

$$f(t) = \left(\frac{1}{n} + t\right)^{-1} t, \qquad t \in \Re^+$$

only takes on values between 0 and 1, we have $0 \leq \nu_i \leq 1$. Furthermore,

$$\sum_{m=1}^{n}[(e_i - \mathbf{1})x_m]^*[(e_i - \mathbf{1})x_m] = (e_i - \mathbf{1})\nu_i(e_i - \mathbf{1}) = \frac{1}{n^2}\nu_i\left(\frac{1}{n} + \nu_i\right)^{-2}$$

and

$$\left(\frac{1}{n} + t\right)^{-2} t \leq \frac{n}{4}, \qquad t \in \Re^+.$$

Thus

$$\sum_{m=1}^{n}[(e_i - \mathbf{1})x_m][(e_i - \mathbf{1})x_m]^* \leq \frac{1}{4n}.$$

For $m = 1, \cdots, n$ we deduce that

$$[(e_i - \mathbf{1})x_m]^*[e_i - \mathbf{1})x_m] \leq \frac{1}{4n},$$

which implies

$$\|(e_i - \mathbf{1})x_m\|^2 \leq \frac{1}{4n}.$$

Hence $\|e_i x - x\| \to 0$ for every $x \in I$. So $\{e_i\}$ is a left approximate identity for I.

To see that $\{e_i\}$ is also a right approximate identity for I, that is $\|xe_i - x\| \to 0$ for every $x \in I$, consider the following. Let $I^* = \{x^* : x \in I\}$ and apply the first part of this proof.

Now let $\lambda, \ \eta \in \Gamma$ such that $\lambda \leq \eta$. We have $\lambda = \{x_1, \cdots, x_n\}, \ \eta = \{x_1, \cdots, x_m\}$ where $n \leq m$, so

$$\nu_\lambda \leq \nu_\eta, \ \left(\frac{1}{n} + \nu_\lambda\right)^{-1} \geq \left(\frac{1}{n} + \nu_\eta\right)^{-1}.$$

Since

$$\frac{1}{n}\left(\frac{1}{n} + t\right)^{-1} \geq \frac{1}{m}\left(\frac{1}{m} + t\right)^{-1}, \qquad t \geq 0,$$

we have

$$\frac{1}{n}\left(\frac{1}{n}+\nu_\eta\right)^{-1} \geq \frac{1}{m}\left(\frac{1}{m}+\nu_\eta\right)^{-1},$$

so

$$e_\lambda = 1-\frac{1}{n}\left(\frac{1}{n}+\nu_\lambda\right)^{-1} \leq 1-\frac{1}{n}\left(\frac{1}{n}+\nu_\eta\right)^{-1} \leq 1-\frac{1}{m}\left(\frac{1}{m}+\nu_\eta\right)^{-1} = e_\eta.$$

Hence $\{e_i\}$ is an increasing approximate identity. □

Proposition A.3.4 *Let A be a C^*-algebra. If $f \in A'^+$ and $\{e_i\}$ is an increasing approximate identity for A then*

$$\|f\| = \lim_{i\to\infty} f(e_i)$$

Proof: Without loss of generality we can assume $\|f\| = 1$. It is clear that $\{f(e_i)\}$ is an increasing net in $\Re$ which is bound above by 1. Therefore $\lim_i f(e_i) \leq 1$. Choosing $x \in A$ such that $\|x\| \leq 1$ we have

$$|f(e_i x)|^2 \leq f(e_i^* e_i) f(x^* x) \leq f(e_i) f(x^* x) \leq \lim_i f(e_i),$$

so $|f(x)|^2 \leq \lim_i f(e_i)$. Hence $1 \leq \lim_i f(e_i) \Rightarrow \lim_i f(e_i) = 1$. □

References: [5], [13], [16], [20].

A.4 Transitivity Theorem

The following is the so called *transitivity theorem* as found in [16].

Theorem A.4.1 *Let A be a C^*-algebra acting irreducibly on a Hilbert space $\mathcal{H}$, and let $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta$ be in $\mathcal{H}$ such that $\xi_1, \cdots, \xi_n$ are linearly independent. Then there* *exists an operator $T \in A$ such that $T(\xi_i) = \eta_i$ for $i = 1, \cdots, n$. If there is a hermitian operator T_h on $\mathcal{H}$ such that $T_h(\xi_i) = \eta_i$ for $i = 1, \cdots, n$ then we may choose T to be hermitian also. If A contains the identity element of $\mathcal{H}$ and there is a unitary U on $\mathcal{H}$ such that $U(\xi_i) = \eta_i$ for $i = 1, \cdots, n$ then we may choose T to be a unitary also.*

A proof of the above theorem can be found in many books outlining the theory of representations of C*-algebras. ie. [16, Thm 5.2.2], [5, Sec 2.8]. For our purposes here, we will only show the following special case of the transitivity theorem.

Theorem A.4.2 *Let A be a C*-algebra, $\pi \in r(A)$, and $\xi, \eta \in \mathcal{H}_\pi$ with ξ non-zero and $||\eta|| = 1$. Then there exists a $y \in A$ such that $\pi(y)\xi = \eta$.*

<u>*Proof:*</u> Let T map $\mathcal{H}_\pi$ onto $\mathcal{H}_\pi$ such that $T(\zeta) = (\zeta|\frac{\xi}{||\xi||^2})\eta$. Clearly $T \in \mathcal{L}(\mathcal{H}_\pi)$, $||T|| \leq ||\eta||$ and $T(\xi) = \eta$. Thus for arbitrary $\xi, \eta \in \mathcal{H}_\pi$ we can find a $T \in \mathcal{L}(\mathcal{H}_\pi)$ with $||T|| \leq ||\eta||$ such that $T(\xi) = \eta$.

So setting $\xi, \eta \in \mathcal{H}_\pi$ with ξ non-zero and $||\eta|| = 1$ we can find a $T_0 \in \mathcal{L}(\mathcal{H}_\pi)$ such that

$$T_0(\xi) = \eta, \quad ||T_0|| \leq ||\eta|| = 1.$$

By Kaplanski's density theorem, $(\pi(A))_1$ is strongly dense in $(\mathcal{L}(\mathcal{H}_\pi))_1$. So we can choose an $x_0 \in A$ such that

$$||\pi(x_0)\xi - T_0(\xi)|| \leq \frac{1}{2}, \quad ||\pi(x_0)|| \leq ||T_0|| \leq 1$$

Similarly there exists $T_1 \in \mathcal{L}(\mathcal{H}_\pi)$ such that

$$T_1(\xi) = \eta - \pi(x_0)\xi, \quad ||T_1|| \leq ||\eta - \pi(x_0)\xi|| \leq \frac{1}{2}.$$

By density, again we can choose $x_1 \in A$ such that

$$\|\pi(x_1)\xi - T_1(\xi)\| \leq \frac{1}{2}, \quad \|\pi(x_1)\| \leq \|T_1\| \leq \frac{1}{2}.$$

By induction we can construct sequences $\{T_k\}$ in $\mathcal{L}(\mathcal{H}_\pi)$ and $\{x_k\}$ in A such that

$$T_k(\xi) = \eta - \pi(x_0)\xi - \ldots - \pi(x_{k-1})\xi, \quad \|T_k\| \leq \frac{1}{2^k}.$$

and

$$\|\pi(x_k)\xi - T_k(\xi)\| \leq \frac{1}{2^k}, \quad \|\pi(x_k)\| \leq \|T_k\| \leq \frac{1}{2^k}.$$

Since $\sum_{k=0}^{\infty} \|\pi(x_k)\| \leq \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty$ we have $\sum_{k=0}^{\infty} \pi(x_k)$ is convergent in $\pi(A)$. As well since π is continuous $\sum_{k=0}^{\infty} x_k$ is convergent in A. Let $y = \sum_{k=0}^{\infty} x_k$ then

$$\|\pi(y)\xi - \eta\| = \lim_{m\to\infty} \|\sum_{k=0}^{m} \pi(x_k)\xi - \eta\| = \lim_{m\to\infty} \|\pi(x_m)\xi - T_m\xi\| \leq \lim_{m\to\infty} \frac{1}{2^m},$$

So $\pi(y)\xi = \eta$. □

References: [5], [16].

A.5 Von Neumann Algebras

Definition A.5.1 A C*-subalgebra of $\mathcal{L}(\mathcal{H})$ is called a **von Neumann algebra** if it is closed in the strong-operator topology.

A.5.2 It is well know that for any convex subset S of $\mathcal{L}(\mathcal{H})$ the weak-operator closure of S coincides with the strong-operator closure of S in $\mathcal{L}(\mathcal{H})$. [20, Thm 16.2]

Definition A.5.3 Let F be a subset of $\mathcal{L}(\mathcal{H})$. Then the set $\{T \in \mathcal{L}(\mathcal{H}) : TS = ST$ for all $S \in F\}$ is called the commutant of F.

Proposition A.5.4 *Let A be a C^*algebra and $\pi \in R(A)$ then the commutant Π of $\pi(A)$ is a von Neumann algebra.*

Proof: It is easily seen that Π is a self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$. Now suppose $T_i \to T$ where $T_i \in \Pi$ for each i. Then for any $x \in A$ and $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned}((T\pi(x) - \pi(x)T)\xi|\eta) &= (T\pi(x)\xi|\eta) - (T\xi|\pi(x)^*\eta) \\ &= \lim_i (T_i\pi(x)\xi|\eta) - (T_i\xi|\pi(x)^*\eta) \\ &= \lim_i ((T_i\pi(x) - \pi(x)T_i)\xi|\eta) \\ &= 0.\end{aligned}$$

Thus $T \in \Pi$ which implies Π is weak-operator closed in $\mathcal{L}(\mathcal{H})$. □

Proposition A.5.5 *Let A be a C^*algebra and $\pi \in R(A)$ then the commutant Π of $\pi(A)$ is equal to $\mathbf{C}I$ if and only if $\pi \in r(A)$.*

Proof: If $\pi \in r(A)$ and if P be a projection in $\mathcal{L}(\mathcal{H})$ then $P \in \Pi$ if and only if $P(\mathcal{H})$ is invarient for $\pi(A)$. So if $\pi \in r(A)$ the only projections in Π are the trivial ones. Since Π is a von Neumann algebra, proposition A.4.4 it is the closed linear span of its projections [20, theorem 20.3]. Therefore $\Pi = \mathbf{C}I$. Now suppose $\pi \in R(A)$ and $\Pi = \mathbf{C}I$. If K is any closed invariant subspace of $\mathcal{H}$ let P_K be the orthogonal projection of $\mathcal{H}$ onto K. Clearly $P_K\pi(A)P_K = \pi(A)P_K$ and $P_K\pi(A)^*P_K = \pi(A)^*P_K$. Thus we have

$$\pi(A)P_K = P_K\pi(A)P_K = (P_K\pi(A)^*P_K)^* = (\pi(A)^*P_K)^* = P_K\pi(A)$$

which implies $P_K \in \Pi \Rightarrow P_K = \lambda I \Rightarrow K = \mathcal{H}$ or $0 \Rightarrow \pi \in r(A)$. □

References: [3], [5], [13], [20].

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