AN EXPLICATIVE STUDY AND EXTENSIONS OF THE PEARSON SYSTEM OF FREQUENCY CURVES

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PREFACE

This study would not have been possible without the learned guidance of Dr. Lawrence Keith Roy, who proposed and directed this thesis. I am thankful to Dr. Roy on whose rich scholarship I have always drawn.

The cooperation of the Lakehead University Main Library and the Library of the Department of Mathematical Sciences have made this study possible. My thanks are due to the staff of the above mentioned libraries.

CONTENTS

ABSTRACT OF THE THESIS

This thesis studies the Pearson System of Frequency Curves from the following four standpoints: firstly, an introductory and simplified study of the derivations and characteristics of the curves; secondly, some interesting features of the curves; thirdly, classical polynomials connected with Pearson's Differential Equation, and fourthly, extensions of the system of curves.

In Chapter I, an introductory study of the curves is outlined. In Chapter II, a simplification in the study of the curves, as mentioned above, has been attempted by using the two parameters $\alpha_3(\alpha_3^2 = \beta_1, \alpha_4 = \beta_2$ in Pearson's notation) and $\delta = \frac{2\beta_2-3\beta_1-6}{\beta_2+3}$. In fact, the use of these two parameters outlines the whole of our discussion in Chapter II. The derivations and characterizations of the curves under the Pearson System are made in terms of these two parameters α_3 and δ. As a result, the various formulae that arise are relatively simple in form and easy to use.

In Chapter III, the bell-shaped Pearson curves are studied in detail. Also an alternative for the method of moments in the computation of the constants in the Pearson differential equation is studied. Interesting studies of Type II and Type III curves are also made.

Chapter IV is devoted to the study of the classical polynomials due to Hermite, Jacobi, Tschebycheff, Legendre and Laguerre in relation to the Pearson differential equation.

Chapter V is allocated to the study of two Extensions of the Pearson System of Frequency Curves. The differential equation

$$\frac{d \log f(x)}{dx} = \frac{\gamma_0+\gamma_1 x+\gamma_2 x^2}{\delta_0 x+\delta_1 x^2+\delta_2 x^3}$$

where $\gamma_0, \gamma_1, \gamma_2, \delta_0, \delta_1, \delta_2$ are real numbers, and its solution in the form

$$f(x) = Cx^{r_1}(a_1+a_2 x)^{r_2}(b_1+b_2 x)^{r_3}$$

where $C, r_1, a_1, a_2, r_2, b_1, b_2, r_3$ are real parameters, as also the differential equation

$$\frac{d \log f(x)}{dx} = \frac{x-a}{b_0+b_1 x+b_2 x^2+b_3 x^3}$$

are basically used for this purpose. As a matter of fact, the afore mentioned differential equation and its solution outline the major part of our discussion in Chapter V. Under certain conditions the solution of the differential equation in the given form is used to derive five curves whose parameters depend on the first seven moments. The Pearson curves are shown to be solutions of a special case of the above mentioned differential equation.

Ten extensions of the Pearson system as derived from the second differential equation are also studied.

SYNOPSIS AND SCOPE OF THE THESIS

As this thesis is primarily concerned with the study of frequency curves, it is quite relevant to begin with the definition of a frequency curve for a ready reference. Next, in logical importance, comes the study of historical background which led to the formulation of the Pearson curves. A brief analysis for the historical background in the formulation of Pearson System of Frequency Curves is thus called for. As the differential equation

$$\frac{d \log y}{dx} = \frac{1}{y}\frac{dy}{dx} = \frac{x+a}{b_0+b_1x+b_2x^2}$$

which is extended to the forms:

$$\frac{d \log y}{dx} = \frac{\gamma_0+\gamma_1x+\gamma_2x^2}{\delta_0x+\delta_1x^2+\delta_2x^3}$$

and

$$\frac{d \log y}{dx} = \frac{x-a}{b_0+b_1x+b_2x^2+b_3x^3}$$

later, constitutes the basis of our present discussion, it would be interesting to sketch a method which leads to the derivation of the above-mentioned differential equation named after Karl Pearson. Hence a brief discussion on the same ensues.

Thereafter follows the study of the solutions of the differential equation and specification of the curves accordingly, and this is the hub of our discussion in Chapters I and II.

The synopsis and scope of the thesis in Chapters III and IV are already indicated in the 'Abstract of the Thesis'. It will be relevant to observe in this section that the Bessel Function $J_n(x)$ with the differential equation

$$x^2J_n''(x)+xJ_n'(x)+(x^2-n^2)J_n(x) = 0$$

cannot be obtained as a special type of $P_n(k,x)$ [to be defined and discussed in Chapter IV].

The axle of our discussion of Chapter V rests on the second and third differential equations as mentioned above and their solutions. All the important continuous probability distributions are deduced from the solution of the second differential equation, as a by-product of the main discussion. Also, all the twelve types of Pearson curves are derived from the same. As an observation on the previous results, the connections between the Inverse Gaussian Distribution and the Pearson Type VI, and also Case 5 of extension are interesting to note. The discussion ends up with the Extensions of the Pearson System and related observations and results followed by some conclusions.

CHAPTER I

AN INTRODUCTORY STUDY OF THE PEARSON SYSTEM OF FREQUENCY CURVES

INTRODUCTION

In this chapter we give an introductory study of the Pearson System of Frequency Curves. We start with the definition of a frequency curve for a ready reference, then the historical importance of the Pearson system of frequency curves and a method of derivation of the Pearson differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0+a_1x}{b_0+b_1x+b_2x^2}$$

follow.

The chapter ends up with the discussion of the method of computing the constants involved in the afore mentioned differential equation in terms of the first four moments of the distribution, followed by the specification of the Pearson curves from Type I to Type XII. Criteria for the U-shaped, Bell-shaped and J-shaped curves for the Pearson system are also discussed, and the graphs of all the twelve types of Pearson curves are attached.

1. FREQUENCY DISTRIBUTIONS AND CURVES

If statistics are arranged so as to show the number of times, or frequency, with which an event happens in a particular way, then the arrangement is a frequency distribution.

It is necessary to have a name for the formula used to describe such distributions, and the term 'frequency curve' is adopted for the purpose.

2. THE SIGNIFICANCE AND IMPORTANCE OF FREQUENCY CURVES

Statistics tend towards a smooth series as the total number of cases is increased, and from this it can be seen how naturally practical statistics lead to the conception of a frequency-curve to describe the smooth distribution that would be obtained if an infinite supply of homogeneous material were available for investigation. In other words, such curves would give an approximation to the total 'population' of which the particular case investigated was a sample.

A frequency-curve can be interpreted to give a frequency corresponding to every value of the independent variable along the whole range of the distribution, and will not restrict us to a few more or less arbitrary groups as is necessary with actual statistics.

In the work in mathematical statistics a large number of the problems that require study involve data properly classified into groups and about which further information is sought. This data is often classified to form a frequency distribution. The frequency distribution when grouped may appear to lie on a certain curve. If it can be shown that this curve is a mathematical curve, i.e. one for which we are able to set up an equation, then this frequency distribution can be readily examined and studied.

3. HISTORICAL BACKGROUND IN THE FORMULATION OF PEARSON SYSTEM OF FREQUENCY CURVES

It is well recognized that the normal curve of error has played a prominent role in the development of the theory of Mathematical Statistics. Although it can describe more or less accurately many frequency distributions possessing a limited degree of skewness, there are many others in which it fails. In the decade from 1890 to 1900, it became well established experimentally that the normal probability function is inadequate to represent many frequency distributions which arise in biological data. To meet the situation it was clearly desirable either to devise methods for characterizing the most conspicuous departures from the normal distribution or to develop generalized frequency curves.

The problem of developing generalized frequency curves has been attacked from several different directions. Gram (1879), Thiele (1889), and Charlier (1905) in Scandinavian countries; Pearson (1895) and Edgeworth (1896) in England; and Fechner (1897) and Bruns (1897) in Germany have developed theories of generalized frequency curves from viewpoints which give very different degrees of prominence to the normal probability curve in the development of a more general theory.

Pearson's curves, named after the English biometrician Karl Pearson, which are widely different in general appearance, are so well known and so accessible that we shall take no time to comment on these as graduation curves for a great variety of frequency distributions, but we shall attempt to indicate the genesis of the curves with special reference to the methods by which they are grounded on or associated with underlying probabilities.

As already indicated, historically this sytem of curves was developed to generate frequency distributions of observational data which could not be represented by the Normal curve. Latterly, however, the system has been used increasingly to represent probability distributions whose moments are known but for which the mathematical equations are either undetermined or not expressible in simple form.

4. PEARSON'S DIFFERENTIAL EQUATION CONSTITUTING THE PEARSON SYSTEM OF FREQUENCY CURVES

Pearson* showed in a series of three articles from 1895-1916 how he obtained the equations of twelve distinct curves and this was done by considering the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0+a_1x}{b_0+b_1x+b_2x^2} \tag{1}$$

* Karl Pearson, "Mathematical contribution to the Theory of Evolution," Philosophical Transactions, A, Vol. 186 (1895), pp. 343-414; also "Supplement to a Memoir on Skew Variation," Phil. Trans. A, Vol. 197 (1901), pp. 443-456; also "Second Supplement to a Memoir on Skew Variation," Phil. Trans., A, Vol. 216 (1916), pp. 429-457.

and solving it, after assigning particular values to the parameters a_0, a_1, b_0, b_1, and b_2.

Equation (1) may also be put in the forms

$$\frac{d(\log y)}{dx} = \frac{x+a}{b_0+b_1x+b_2x^2} \text{ or } \frac{x-a}{b_0+b_1x+b_2x^2} \text{ or } \frac{a-x}{b_0+b_1x+b_2x^2} \tag{1a}$$

so that the independent parameters are a, b_0, b_1 and b_2.

5. DERIVATION OF PEARSON'S DIFFERENTIAL EQUATION

Considering the obvious characteristics of frequency distributions, we find they generally start at zero, rise to a maximum, and then fall sometimes at the same but often at a different rate. At the ends of the distribution there is often high contact. (Mathematically, high contact means that all the differential coefficients vanish at the point of contact.) This means, mathematically, that a series of equations $y = f(x)$, $y = \phi(x)$, etc., must be chosen, so that in each equation of the series $\frac{dy}{dx} = 0$ for certain values of x, viz at the maximum and at the end of the curve where there is contact with the axis of x.

The above suggests that $\frac{dy}{dx} = \frac{y(a_0+a_1x)}{F(x)}$; then, if $y = 0$, $\frac{dy}{dx} = 0$, and there is, therefore, contact at one end of the curve, while if $x = -\frac{a_0}{a_1}$, $\frac{dy}{dx} = 0$, and we have the maximum we require. So long as $F(x)$ is general the form assumed for $\frac{dy}{dx}$ is extremely general and includes cases when $\frac{dy}{dx}$ may not be zero when y is zero. If $F(x)$ is expanded by Maclaurin's theorem in ascending powers of x, we have

$$\frac{dy}{dx} = \frac{y(a_0+a_1x)}{b_0+b_1x+b_2x^2+\ldots} \qquad (2)$$

Thus, we see that a differential equation, viz (2), is derived, which is analogous to the differential equation (1) of the previous section.

It would be possible to obtain constants in the differential equation (2) by using a greater number of terms and retaining b_3, b_4, etc., but there are strong practical objections to such a course. Besides the increase in arithmetical work, the gain in introducing additional constants is small because the higher moments are untrustworthy. (Because the higher the moment the more liable it is to error when deduced from ungraduated observations; this is clear, when we remember that the ends of the experiences are multiplied by the highest numbers and their powers.) Karl Pearson has shown that "we might easily on a random sample reach a 7th or 8th moment having half or double the value it actually has in the general population. Constants based on these high moments will be practically idle. They may enable us to describe closely an individual random sample, but no safe argument can be drawn from this individual sample as to the general population at large, at any rate so far as the argument is based on the constants depending on these high moments."[1] In some acturial

[1] "Skew Correlation and Non-Linear Regression", Drapers' Co. Res. Mem. 1905, p. 9.

statistics where there are as many as 100,000 cases, it might be worth while to go as far as the next term of the series, but even here the value of the work is discounted because any other smaller body of statistics on the same subject could not be compared satisfactorily with the result. For practical purposes it is probable that the equation taken as far as b_2 will be sufficient, and we shall confine our attention to the forms thus obtained.

In this context, it will be interesting to see how equation (2) can be obtained up to the x^2 term in the denominator from the elementary propositions in the theory of probability.

The chance of getting r white balls from a bag containing np white and nq black balls (in the usual notation) in drawing s balls one at a time without replacements is given by

$$y_r = (sCr)\,\frac{(npPr)\,(nqPs-r)}{(nPs)} \qquad (3)$$

where (sCr) is the number of combinations of s things taken r at a time and (nPs) is the number of permutations of n things taken s at a time.

From (3), we write

$$y_r = \frac{s!}{r!\,(s-r)!}\;\frac{(np)!}{(np-r)!}\;\frac{(nq)!}{(nq-s+r)!}\;\frac{(n-s)!}{n!}$$

$$= \frac{(np)!\,(nq)!\,(n-s)!\,s!}{(np-r)!\,(nq-s+r)!\,n!\,r!\,(s-r)!}$$

$\therefore \frac{dy}{dx} \approx y_{r+1} - y_r$ (taking $D \approx \Delta$, where the symbols have their usual meanings)

$$= \frac{(np)!(nq)!)(n-s)!s!}{n!}\left[\frac{1}{(np-r-1)!(nq-s+r+1)!(r+1)!(s-r-1)!} - \frac{1}{(np-r)!(nq-s+r)!r!(s-r)!}\right]$$

$$= \frac{(np)!(nq)!(n-s)!s!}{n!(np-r-1)!(nq-s+r)!r!(s-r-1)!}\left[\frac{1}{(nq-s+r+1)(r+1)} - \frac{1}{(np-r)(s-r)}\right]$$

i.e. $$\frac{dy}{dx} \approx y_r \cdot \frac{s+nps-nq-1-r(n+2)}{(r+1)(r+1-nq-s)} \qquad (4)$$

The expression for y_r in (3) is a term of a hyper-geometric series. We represent the terms of this series as ordinates of a frequency polygon, and find the slope of a side of the frequency polygon. Thus, we make $r = 0,1,2,\ldots,s$ and obtain the $s+1$ ordinates $y_0, y_1, y_2, \ldots, y_s$ at unit intervals. At the middle point of the side joining the tops of ordinates y_r and y_{r+1}, we have

$$x = r + \frac{1}{2}, \; y = \frac{1}{2}(y_r + y_{r+1}) \qquad (5)$$

From (5), we can write

$$y = \frac{1}{2} y_r \left[\frac{(np-r)(s-r)}{(r+1)(r+1+nq-s)} + 1\right]$$

$$= \frac{1}{2} y_r \frac{nps+nq+1-s+r(nq+2-np-2s)+2r^2}{(r+1)(r+1+nq-s)} \qquad (6)$$

Taking (4)÷(6), and taking $r = x - \frac{1}{2}$ by means of (5), we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{2s+2nps-2nq-2-(2x-1)(n+2)}{nps+nq+1-s+\left(x - \frac{1}{2}\right)(nq+2-np-2s)+2\left(x - \frac{1}{2}\right)^2}$$

which may be put in the form

$$\frac{1}{y}\frac{dy}{dx} = \frac{a+x}{b_0+b_1x+b_2x^2} \qquad (7)$$

From (7), we observe that the slope of the frequency polygon, at the middle point of any side, divided by the ordinate at that point is equal to a fraction whose numerator is a linear function of x and whose denominator is a quadratic function of x.

The differential equation (1) gives a general statement of this property. It is more general than (7) in the sense that the constants of (7) are special values found from the law of probability involved in drawings from a limited supply without replacements. One of Pearson's generalizations therefore consists in admitting as frequency curves all those curves of which (1) is the differential equation without the limitations on the values of the constants involved in (7).

6. EVALUATION OF THE CONSTANTS OF PEARSON'S DIFFERENTIAL EQUATION: METHOD OF MOMENTS

Pearson's differential equation (1) is readily reducible to equation (7) of the previous section. So, the independent constants of Pearson's equation (7) are a, b_0, b_1, and b_2. In this section, we outline a method of determining these constants in terms of the first four moments of the distribution.

To do this, we have from equation (7),

$$(b_0+b_1x+b_2x^2)\ \frac{dy}{dx} = y(x+a)$$

$$\int x^n(b_0+b_1x+b_2x^2)\ \frac{dy}{dx}\ dx = \int y(x+a)x^n dx$$

$$x^n(b_0+b_1x+b_2x^2)y - \int\{nb_0x^{n-1}+(n+1)b_1x^n$$

$$+(n+2)b_2x^{n+1}\}ydx = \int yx^{n+1}dx + \int yax^n dx$$

integrating by parts.

If at the ends of the range of the curve

$$x^n(b_0+b_1x+b_2x^2)y$$

vanishes, we have, remembering $\mu'_n = \int yx^n dx$,

$$-nb_0\mu'_{n-1}-(n+1)b_1\mu'_n-(n+2)b_2\mu'_{n+1} = \mu'_{n+1}+a\mu'_n$$

$$a\mu'_n+nb_0\mu'_{n-1}+(n+1)b_1\mu'_n+(n+2)b_2\mu'_{n+1} = -\mu'_{n+1} \qquad (8)$$

Now putting $n = 0,1,2,3$, we get from (8),

$$
\begin{aligned}
a\mu_0'+b_1\mu_0'+2b_2\mu_1' &= -\mu_1' \\
a\mu_1'+b_0\mu_0'+2b_1\mu_1'+3b_2\mu_2' &= -\mu_2' \\
a\mu_2'+2b_0\mu_1'+3b_1\mu_2'+4b_2\mu_3' &= -\mu_3' \\
a\mu_3'+3b_0\mu_2'+4b_1\mu_3'+5b_2\mu_4' &= -\mu_4'
\end{aligned}
\qquad (9)
$$

Then bearing in mind that the result of making $\mu_1' = 0$ is to change the origin of the system to the mean of the distribution, and treating $\mu_0' = 1$, we have from (9),

$$
\begin{aligned}
a+b_1 &= 0 \\
b_0+3b_2\mu_2 &= -\mu_2 \\
a\mu_2+3b_1\mu_2+4b_2\mu_2 &= -\mu_3 \\
a\mu_3+3b_0\mu_2+4b_1\mu_3+5b_2\mu_4 &= -\mu_4
\end{aligned}
\qquad (10)
$$

From (10), the three equations in b_0, b_1, b_2 are

$$
\begin{aligned}
b_0+0b_1+3\mu_2 b_2 &= -\mu_2 \\
0b_0+2\mu_2 b_1+4\mu_3 b_2 &= -\mu_3 \\
3\mu_2 b_0+3\mu_3 b_1+5\mu_4 b_2 &= -\mu_4
\end{aligned}
\qquad (11)
$$

[replacing a by $-b_1$ from the first equation of (10)].

Using Cramer's method in (11), we have

$$
b_0 = \begin{vmatrix} -\mu_2 & 0 & 3\mu_2 \\ -\mu_3 & 2\mu_2 & 4\mu_3 \\ -\mu_4 & 3\mu_3 & 5\mu_4 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 0 & 3\mu_2 \\ 0 & 2\mu_2 & 4\mu_3 \\ 3\mu_2 & 3\mu_3 & 5\mu_4 \end{vmatrix}
$$

$$b_1 = \begin{vmatrix} 1 & -\mu_2 & 3\mu_2 \\ 0 & -\mu_3 & 4\mu_3 \\ 3\mu_2 & -\mu_4 & 5\mu_4 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 0 & 3\mu_2 \\ 0 & 2\mu_2 & 4\mu_3 \\ 3\mu_2 & 3\mu_3 & 5\mu_4 \end{vmatrix}$$

$$b_2 = \begin{vmatrix} 1 & 0 & -\mu_2 \\ 0 & 2\mu_2 & -\mu_3 \\ 3\mu_2 & 3\mu_3 & -\mu_4 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 0 & 3\mu_2 \\ 0 & 2\mu_2 & 4\mu_3 \\ 3\mu_2 & 3\mu_3 & 3\mu_4 \end{vmatrix} \qquad (12)$$

where $\begin{vmatrix} 1 & 0 & 3\mu_2 \\ 0 & 2\mu_2 & 4\mu_3 \\ 3\mu_2 & 3\mu_3 & 5\mu_4 \end{vmatrix} = 10\mu_2\mu_4 - 18\mu_2^3 - 12\mu_3^2 \neq 0$

From (12), we have

$$b_0 = -\frac{\mu_2(4\mu_2\mu_4 - 3\mu_3^2)}{10\mu_2\mu_4 - 18\mu_2^3 - 12\mu_3^2}$$

$$b_1 = -\frac{\mu_3(\mu_4 + 3\mu_2^2)}{10\mu_2\mu_4 - 18\mu_2^3 - 12\mu_3^2} \qquad (12a)$$

$$b_2 = -\frac{2\mu_2\mu_4 - 3\mu_3^2 - 6\mu_2^3}{10\mu_2\mu_4 - 18\mu_2^3 - 12\mu_3^2}$$

Since from the first equation of (1), $a = -b_1$, hence the constants a, b_0, b_1, b_2 of the differential equation (7) are determined in terms of the moments μ_2, μ_3, μ_4 as given by (12a).

If in (12a), we put

$$\beta_1 = \mu_3^2/\mu_2^3$$

$$\beta_2 = \mu_4/\mu_2^2 \qquad (13)$$

then we get from (12a)

$$b_0 = -\frac{\mu_2(4\mu_2\beta_2\mu_2^2-3\beta_1\mu_2^3)}{10\mu_2\beta_2\mu_2^2-18\mu_2^3-12\beta_1\mu_2^3} = -\frac{\mu_2(4\beta_2-3\beta_1)}{2(5\beta_2-6\beta_1-9)}$$

$$b_1 = -a = -\frac{\sqrt{\mu_2}\sqrt{\beta_1}(\beta_2+3)}{2(5\beta_2-6\beta_1-9)} \qquad (14)$$

$$b_2 = -\frac{2\beta_2-3\beta_1-6}{2(5\beta_2-6\beta_1-9)}$$

The expressions for a, b_0, b_1 and b_2 in (14) simplify our discussion in specifying the members of the Pearson system of frequency curves as we shall see in the next section.

7. SPECIFICATION OF THE PEARSON SYSTEM OF FREQUENCY CURVES

The specification of the Pearson's system of frequency curves, which are the solutions of the differential equation (7), depends naturally upon the nature of the roots of the equation

$$b_2x^2+b_1x+b_0 = 0 \qquad (15)$$

where $b_0 \neq 0$ (we shall prove this later).

Now the roots of (15) are

$$\frac{-b_1+\sqrt{b_1^2-4b_0b_2}}{2b_2} \quad \text{and} \quad \frac{-b_1-\sqrt{b_1^2-4b_0b_2}}{2b_2}$$

Evidently therefore the nature of the roots of (15) depends upon $\sqrt{b_1^2-4b_0b_2}$, or in other words, upon $b_1^2/(4b_0b_2)$.

Thus, $\frac{b_1^2}{4b_0b_2} < 0 \Rightarrow b_0$ and b_2 are of opposite signs. Under this condition, the roots can be shown to be real and of opposite signs as follows:

$$-b_1 + \sqrt{b_1^2 - 4b_0b_2} > 0$$

$$\Rightarrow \sqrt{b_1^2 - 4b_0b_2} > b_1$$

$$\Rightarrow b_1^2 - 4b_0b_2 > b_1^2$$

$$\Rightarrow -4b_0b_2 > 0$$

which gives the result.

$\therefore \quad \frac{b_1^2}{4b_0b_2} < 0 \Rightarrow$ the roots of (15) are real and of opposite signs.

This criterion, as we shall see later, gives one of the main types of curves - called Type I by Karl Pearson. Now

$$\frac{b_1^2}{4b_0b_2} > 0 \Rightarrow b_0 \text{ and } b_2 \text{ are of the same sign}$$

and $\quad \frac{b_1^2}{4b_0b_2} < 1 \Rightarrow b_1^2 - 4b_0b_2 < 0$ (by means of the above condition)

$\therefore \quad 0 < \frac{b_1^2}{4b_0b_2} < 1 \Rightarrow$ the roots of (15) are imaginary

These criteria lead to the second main type (Pearson Type IV) of curves.

Now, by means of the above arguments, it is easy to see that

$$\frac{b_1^2}{4b_0b_2} > 1$$

$\Rightarrow$ the roots are real and of the same sign. These conditions lead to the third main type (Pearson's Type VI) of curves.

This really covers the whole field, but in the limiting cases when one type changes into another we reach simpler forms of transition curves.

Thus, when $\frac{b_1^2}{4b_0b_2}$ is large (theoretically infinite) which means

$$\frac{b_1^2}{4b_0b_2} \to \infty \Rightarrow b_2 \to 0 \quad (\text{because } b_0 \neq 0)$$

Consequently $\left|\frac{-b_1-\sqrt{b_1^2-4b_0b_2}}{b_2}\right| \to \infty$ i.e. one root of (15) is ∞. This condition leads to Type III curves.

When $\frac{b_1^2}{4b_0b_2} = 1$, the roots of (15) are equal and we get Type V curve.

When $\frac{b_1^2}{4b_0b_2} = 0$, the roots of (15) are equal in magnitude but opposite in sign. Under this condition, we get Type II curve.

If in the last case $b_1 = b_2 = 0$, we reach what we shall call the 'normal curve of error': this name is open to some objection just as are the other names given to it. (e.g.

Probability curve, Gaussian curve, etc.). Then again the expression for $\frac{1}{y}\frac{dy}{dx} = \frac{d(\log y)}{dx}$ may be reducible to the form

$$a'/(b_0'+b_1'x)$$

and perhaps a straight line for the frequency curve (cf. Types VIII, IX and XI), while if the expression reduces to a constant the curve is the ordinary geometrical progression which we are pleased to find as a special case of a system of frequency curves because we are already familiar with it in the theory of probability in connection with sequences from coin tossing, etc.

From the above discussion, it is obvious that $\frac{b_1{}^2}{4b_0b_2}$ plays the fundamental role in characterizing the Pearson system of frequency curves. We shall call $b_1{}^2/(4b_0b_2)$ as 'the criterion' and denote it by κ. Before we express this

$$\kappa = \frac{b_1{}^2}{4b_0b_2}$$

in terms of the expression in (14) in the next section, it will be of great use to write down the types, equations and the differential equations of all the members of the Pearson system of frequency curves.

TYPE	EQUATION	DIFFERENTIAL EQUATION
I	$y = y_0\left(1 + \frac{x}{a_1}\right)^{m_1}\left(1 - \frac{x}{a_2}\right)^{m_2}$ where $\frac{m_1}{a_1} = \frac{m_2}{a_2}$ (=ν, say)	$\frac{dy}{dx} = \frac{\nu(a_1+a_2)}{(a_1+x)(a_2-x)}\, y$
II	$y = y_0\left(1 - \frac{x^2}{a^2}\right)^m$	$\frac{dy}{dx} = \frac{-2mx}{a^2-x^2}\, y$
III	$y = y_0 e^{-\gamma x}\left(1 + \frac{x}{a}\right)^{\gamma a}$	$\frac{dy}{dx} = \frac{-\gamma x}{a+x}\, y$
IV	$y = y_0\left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-\nu \arctan\frac{x}{a}}$	$\frac{dy}{dx} = \frac{-2mx-\nu a}{a^2+x^2}\, y$
V	$y = y_0 x^{-p} e^{\frac{-\gamma}{x}}$	$\frac{dy}{dx} = \frac{\gamma-px}{x^2}\, y$
VI	$y = y_0 (x-a)^{q_2} x^{-q_1}$	$\frac{dy}{dx} = \frac{q_1 a+(q_2-q_1)x}{x^2-ax}\, y$
VII	$y = y_0 e^{-\frac{x^2}{2\sigma^2}}$	$\frac{dy}{dx} = -\frac{x}{\sigma^2}\, y$
VIII	$y = y_0\left(1 + \frac{x}{a}\right)^{-m}$	$\frac{dy}{dx} = \frac{-m}{a+x}\, y$
IX	$y = y_0\left(1 + \frac{x}{a}\right)^{m}$	$\frac{dy}{dx} = \frac{m}{a+x}\, y$
X	$y = \frac{1}{\sigma} e^{\pm\frac{x}{\sigma}}$	$\frac{dy}{dx} = \pm\frac{1}{\sigma}\, y$
XI	$y = y_0 x^{-m}$	$\frac{dy}{dx} = -\frac{m}{x}\, y$
XII	$y = y_0\left(\frac{a_1+x}{a_2-x}\right)^p$	$\frac{dy}{dx} = \frac{p(a_1+a_2+2x)}{(a_1+x)(a_2-x)}\, y$

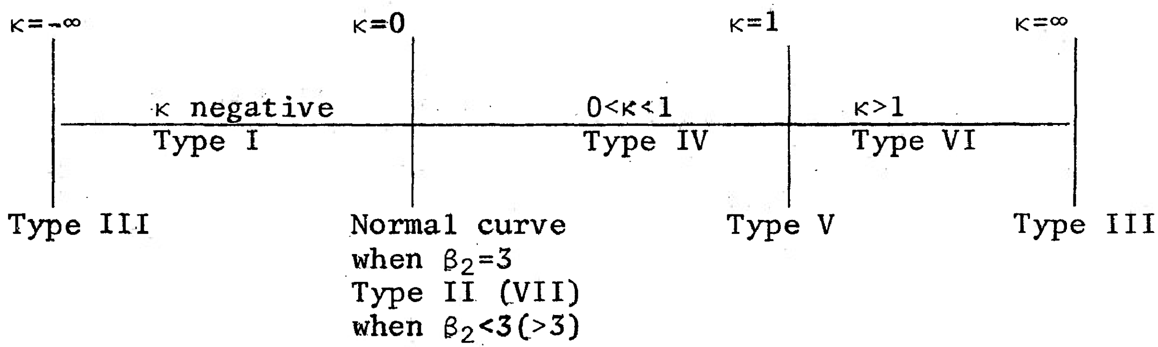
Table I Pearson Curves

8. AN EXPRESSION FOR THE CRITERION κ

Using the expressions for b_0, b_1 and b_2 in (14), we get

$$\kappa = \frac{b_1^2}{4b_0 b_2} = \frac{\beta_1(\beta_2+3)^2}{4(2\beta_2-3\beta_1-6)(4\beta_2-3\beta_1)} \qquad (16)$$

This may have any value from $-\infty$ to ∞. The following diagram summarizes the discussion in section 7.



9. CRITERIA FOR U-SHAPED, BELL-SHAPED AND J-SHAPED CURVES

The Type I curve will be:

(i) U-shaped if $m_1 < 0, m_2 < 0$

(ii) bell-shaped if $m_1 > 0, m_2 > 0$ (17)

(iii) J-shaped if either $m_1 < 0, m_2 > 0$ or $m_1 > 0, m_2 < 0$

If m_1 is negative and m_2 is positive the curve is J-shaped; it starts at an infinite ordinate, falls rapidly and runs out at $x = a_2$. And conversely, if $m_1 > 0$ and $m_2 < 0$ we have a reversed J-shaped curve.

If both m_1 and m_2 are negative, the curve is U-shaped, starting and ending with infinite ordinates.

In the J- and U-shaped curves, though the limiting ordinate is infinite, the area is finite.

The Type II curve would be U-shaped when the exponent is negative and the Type III curve would be J-shaped if γa were negative. Otherwise, Type III is usually bell-shaped.

In Type VI curve, $q_1 > q_2$ (to be proved later). If q_2 is negative, the curve is J-shaped.

The attached graphs are rough illustrations of the Pearson curves for particular positive values of the parameters.

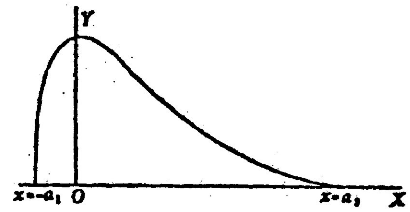
PEARSON FREQUENCY CURVES

Type I

$$y=y_0\left(1+\frac{x}{a_1}\right)^{m_1}\left(1-\frac{x}{a_2}\right)^{m_2},$$

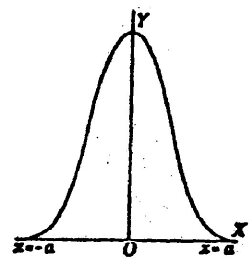
where

$$\frac{m_1}{a_1}=\frac{m_2}{a_2}.$$



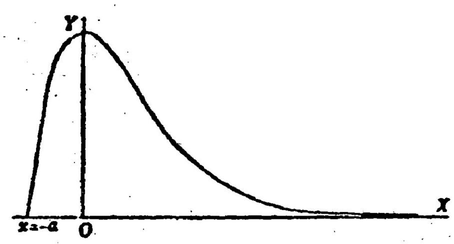
Type II

$$y=y_0\left(1-\frac{x^2}{a^2}\right)^m$$



Type III

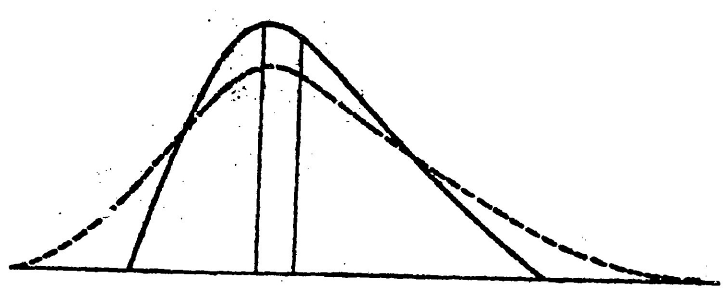
$$y=y_0e^{-\gamma x}\left(1+\frac{x}{a}\right)^{\gamma a}.$$



Type IV

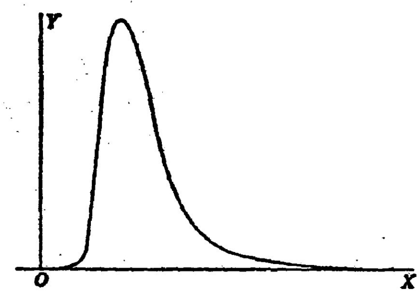
$$y=y_0\left(1+\frac{x^2}{a^2}\right)^{-m}e^{-\nu \arctan \frac{x}{a}}.$$

A skew curve of unlimited range at both ends, roughly described in general appearance as a slightly deformed normal curve



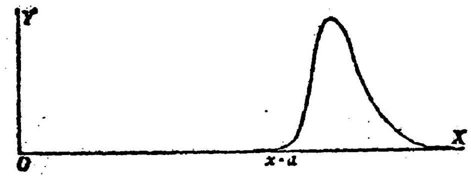
TYPE V

$$y=y_0x^{-p}e^{-\frac{\gamma}{x}}.$$



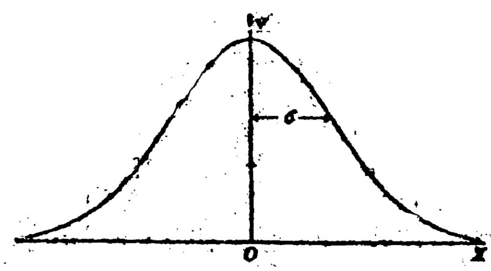
TYPE VI

$$y=y_0(x-a)^{q_2}x^{-q_1}.$$



TYPE VII

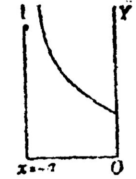
$$y=y_0e^{-\frac{x^2}{2\sigma^2}}.$$



The normal frequency curve.

TYPE VIII

$$y=y_0\left(1+\frac{x}{a}\right)^{-m}.$$

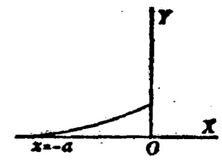


This type degenerates into an equilateral hyperbola when $m=1$.

PEARSON FREQUENCY CURVES

Type IX

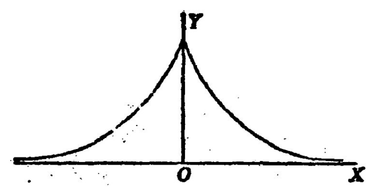
$$y=y_0\left(1+\frac{x}{a}\right)^m.$$



This type degenerates into a straight line when $m=1$.

Type X

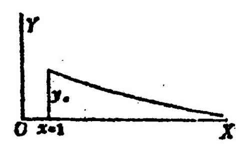
$$y=\frac{n}{\sigma}e^{\pm\frac{x}{\sigma}}.$$



This type is Laplace's first frequency curve while the normal curve is sometimes called his second frequency curve. The curve is shown for negative values of $\pm x/\sigma$.

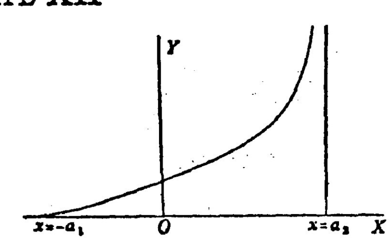
Type XI

$$y=y_0x^{-m}.$$



Type XII

$$y=y_0\left(\frac{a_1+x}{a_2-x}\right)^p$$



CHAPTER II

CRAIG'S TREATMENT OF THE PEARSON SYSTEM OF FREQUENCY CURVES

INTRODUCTION

This chapter gives a study of the Pearson curves in terms of the two parameters α_3 and δ, as indicated by C. C. Craig. The parameters α_3 and δ are defined, and the properties and specifications of all the twelve types of Pearson curves are studied in full detail in terms of these two parameters only.

With the progress of dicussion, and also in conclusion, we have opportunity to observe how the study of the well-known and time-honoured Pearson curves is simplified strikingly when characterized and specified in terms of the two parameters.

10. CRAIG'S NEW EXPOSITION AND TREATMENT FOR THE PEARSON SYSTEM OF FREQUENCY CURVES

In a paper in the Annals of Mathematical Statistics [6], Cecil C. Craig has expanded the treatment of the Pearson system of frequency functions by making the two parameters

$$\alpha_3 (\alpha_3{}^2 = \beta_1 ,\ \alpha_4 = \beta_2 \quad \text{in Pearson's notation})$$

and

$$\delta = \frac{2\alpha_4 - 3\alpha_3{}^2 - 6}{\alpha_4 + 3} \left(= \frac{2\beta_2 - 3\beta_1 - 6}{\beta_2 + 3} \right) \qquad (18)$$

fundamental in the discussion. The criteria for the different members of the system of functions are expressed very simply in terms of α_3 and δ.

(A) A CLUE FOR THE EXPRESSION FOR δ IN (18)

In fact, the clue lies in the expression (16) of section 8. If we use

$$\delta = \frac{2\beta_2-3\beta_1-6}{\beta_2+3}$$

which is contained in the expression in (16), we find that

$$\kappa = \frac{b_1^2}{4b_0b_2} = \frac{\beta_1(\beta_2+3)}{4\delta(4\beta_2-3\beta_1)} = \frac{\beta_1(\beta_2+3)}{4\delta(2\beta_2-3\beta_1-6+2\beta_2+6)}$$

$$= \frac{\beta_1(\beta_2+3)}{4\delta[(\beta_2+3)\delta+2(\beta_2+3)]} = \frac{\beta_1}{4\delta(\delta+2)}$$

$$\text{i.e.} \qquad \kappa = \frac{b_1^2}{4b_0b_2} = \frac{\alpha_3^2}{4\delta(\delta+2)} \tag{19}$$

Thus we see that the expression in (16) is strikingly simplified in (19) by means of the expression for δ in (18).

(B) DETERMINATION OF THE CONSTANTS OF PEARSON'S DIFFERENTIAL EQUATION (1)

Craig uses the differential equation (1) in the form

$$\frac{1}{y}\frac{dy}{dt} = \frac{a-t}{b_0+b_1t+b_2t^2} \tag{20}$$

For this differential equation, the recursion formula for moments similar to (8) is obtained from

$$[y(b_0+b_1t+b_2t^2)t^n]_{-\ell_1}^{\ell_2}$$

$$-\int_{-\ell_1}^{\ell_2} y[nb_0t^{n-1}+(n+1)b_1t^n+(n+2)b_2t^{n+1}]dt = \int_{-\ell_1}^{\ell_2}(ayt^n-yt^{n+1})dt \tag{21}$$

which can be written in the form

$$a\alpha_n+nb_0\alpha_{n-1}+(n+1)b_1\alpha_n+(n+2)b_2\alpha_{n+1} = \alpha_{n+1} \tag{22}$$

assuming

(i) range of variation of t is $[-\ell_1,\ell_2]$

(ii) the first expression in (21) vanishes

(iii) the first $(n+1)$ moments over the range exist.

Also, using the same procedure as in section (6)

$$\int_{-\ell_1}^{\ell_2} t^n y dt = \frac{\mu_n}{\sigma^n} = \alpha_n \tag{23}$$

where $\sigma^2 = \mu_2$. We clearly see from (23) that

$$\alpha_0 = 1,\ \alpha_1 = 0,\ \text{and } \alpha_2 = 1 \tag{24}$$

Equation (22) together with (24) lead to the determination of a, b_0, b_1, and b_2 of equation (20) as follows:

$$b_0 = \frac{4\beta_2-3\beta_1}{2(5\beta_2-6\beta_1-9)}$$

$$b_1 = \frac{\sqrt{\beta_1}\,(\beta_2+3)}{2(5\beta_2-6\beta_1-9)} = -a \tag{25}$$

$$b_2 = \frac{2\beta_2-3\beta_1-6}{2(5\beta_2-6\beta_1-9)}$$

These are obtained from (14) of section 6 with a slight rearrangement of sign.

Now to express (25) in terms of α_3 and δ we have, using (18).

$$b_0 = \frac{2\beta_2-3\beta_1-6+2\beta_2+6}{2[2(2\beta_2-3\beta_1-6)+\beta_2+3]}$$

$$= \frac{\delta(\beta_2+3)+2(\beta_2+3)}{2[2\delta(\beta_2+3)(\beta_2+3)}$$

$$= \frac{\delta+2}{2(1+2\delta)}$$

Similarly,

$$b_1 = \frac{\alpha_3}{2(1+2\delta)} \tag{26}$$

$$b_2 = \frac{\delta}{2(1+2\delta)}$$

$$a = -\frac{\alpha_3}{2(1+2\delta)}$$

For (26) to be valid $\delta \neq -\frac{1}{2}$. The case in which

$\delta = -\frac{1}{2}$ will be included in the discussion of the transitional types of functions.

(C) RANGE OF THE ADMISSIBLE VALUES OF δ

In this section, it is interesting to see that

$$-2 < \delta < 2. \qquad (27)$$

Proof: We have,

$$\int_{-\ell_1}^{\ell_2} f(t)(t^2+\lambda t)^2 dt$$

$$= \int_{-\ell_1}^{\ell_2} f(t)[t^4+2\lambda t^3+\lambda^2 t^2]dt$$

$$= \alpha_4+2\lambda\alpha_3+\lambda^2\alpha_2$$

$$= \alpha_4+2\lambda\alpha_3+\lambda^2 \qquad \text{[by (24)]}$$

$$\nless 0$$

$$\because f(t) \geq 0 \quad \text{for} \quad -\ell_1 \leq t \leq \ell_2 \qquad \lambda \in \mathbb{R}$$

$$\Rightarrow (\lambda+\alpha_3)^2+(\alpha_4-\alpha_3^2) \nless 0$$

$$\Rightarrow \alpha_3^2 \leq \alpha_4$$

Now choose $$k_1 = \frac{4\alpha_4-3\alpha_3^2}{\alpha_4+3} = \frac{4(\alpha_4-\alpha_3^2)+\alpha_3^2}{\alpha_4+3} > 0$$

and $$k_2 = \frac{3(\alpha_3^2+4)}{\alpha_4+3} > 0$$

Then,

$$-2+k_1 = \frac{4\alpha_4-3\alpha_3^2}{\alpha_4+3} - 2 = \frac{2\alpha_4-3\alpha_3^2}{\alpha_4+3} = \delta$$

$$2-k_2 = 2 - \frac{3(\alpha_3^2+4)}{\alpha_4+3} = \frac{2\alpha_4-3\alpha_3^2-6}{\alpha_4+3} = \delta$$

$$\Rightarrow -2 < \delta < 2$$

$$\Rightarrow b_0 \neq 0 \quad [\text{by } (26)]$$

for any Pearson frequency function possessing moments of the fourth order.

(D) INTEGRATION OF (20) AND DEVELOPMENT OF THE VARIOUS FORMS OF $y = f(t)$

To do this, we make use of the following assumptions:

(i) Over the range of variation of t, we must have $f(t) \geq 0$

(ii) The area under the curve $y = f(t)$ over the range of variation must be finite. This being true then we always determine the constant of integration so that this area is unity.

(iii) The range in each case is taken as the maximum one for which (20) and (22) may be secured, and which contains the point $t = 0$.

(iv) It is sufficient throughout to take $\alpha_3 \geq 0$ since the curve for $\alpha_3 = -k(k > 0)$ is only a reflection of that for $\alpha_3 = k$ through the line $t = 0$.

We will use the relations (26) as definitions of a, b_0

b_1 and b_2 in terms of α_3 and δ. Using the values of a and the b's given by any choice of α_3 and δ, we solve (20). If the solution is such that for it (22) may be derived, then the relations (26) are valid when α_3 and δ have their usual meanings.

For convenience, let us denote:

the conditions for the validity of (22) (α)

(a) THE NORMAL FREQUENCY FUNCTION (TRANSITIONAL TYPE VII)

$$\alpha_3 = \delta = 0 \qquad a = b_1 = b_2 = 0,\ b_0 = 1 \text{ [by (26)]}$$

$\therefore$ Equation (20) becomes

$$\frac{1}{y}\frac{dy}{dt} = -t$$

$$\text{or, } y = ce^{-\frac{t^2}{2}}$$

The range of $y = f(t)$ is $(-\infty,\infty)$ and so

$$\int_{-\infty}^{\infty} ce^{-\frac{t^2}{2}} dt = 1$$

$$\text{or, } 2c\int_0^{\infty} ce^{-\frac{t^2}{2}} dt = 1 \quad (\because \text{ the function is an even function})$$

$$\text{or, } 2c\int_0^{\infty} e^{-z^2} dz\sqrt{2} = 1$$

or, $2c\sqrt{2}\ \frac{\sqrt{\pi}}{2} = 1$

or, $c = \frac{1}{\sqrt{2\pi}}$

$\therefore\ f(t) = \frac{1}{\sqrt{2\pi}}\ e^{-\frac{t^2}{2}}$ Type VII

conditions (α) are satisfied for above $f(t)$. [Check: $b_0 t^n f(t)\Big|_{-\infty}^{\infty} = \frac{1}{\sqrt{2\pi}}\, t^n e^{-\frac{t^2}{2}}\Big|_{-\infty}^{\infty} = 0$; $\alpha_{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{n+1} e^{-\frac{t^2}{2}}\, dt = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} t^{n+1} e^{-\frac{t^2}{2}}\, dt$ exists.]

(b) TRANSITIONAL TYPES III AND X

The conditions are,

for Type III, $\alpha_3 \neq 0,\ \delta = 0$

for Type X, $\alpha_3^2 = 4,\ \delta = 0$

Now $\delta = 0 \Rightarrow b_0 = 1,\ b_1 = 0,\ b_2 = -a = \frac{\alpha_3}{2}$ [by (26)]

$\therefore$ For Type III, equation (20) becomes

$$\frac{1}{y}\frac{dy}{dt} = \frac{-\frac{\alpha_3}{2} - t}{1 + \frac{\alpha_3}{2}t}$$

i.e. $$\frac{1}{y}\frac{dy}{dt} = \frac{-\left(\frac{1}{A} + t\right)}{1 + \frac{1}{A}t}\ [A = 2/\alpha_3] = -\frac{1+At}{A+t}$$

or, $$\frac{dy}{y} = -\frac{1}{A+t}\,dt - A\left[\frac{t}{A+t}\right]dt$$

$$\text{or,}\quad \log cy = -\log(A+t) - A\int\left(1 - \frac{A}{A+t}\right)dt$$

$$= \log(A+t)^{-1} - At + A^2\log(A+t)$$

$$= \log[(A+t)^{A^2-1}e^{-At}]$$

or, $y = f(t) = k(A+t)^{A^2-1}e^{-At}$, k = constant of integration to be determined.

Range of $f(t)$ is $(-A,\infty)$ and so

$$\int_{-A}^{\infty} f(t)dt = 1 \quad \text{gives}$$

$$k\int_{-A}^{\infty}(A+t)^{A^2-1}e^{-At}dt = 1$$

Put $x = A+t \Rightarrow t = x-A$, $dx = dt$, the limits are $0, \infty$.

So from above,

$$k\int_{0}^{\infty} x^{A^2-1}e^{-A(x-A)}dx = 1$$

$$\text{or,}\quad ke^{A^2}\int_{0}^{\infty} x^{A^2-1}e^{-Ax}dx = 1$$

Put $Ax = z$ $\quad dx = \frac{1}{A}dz$

$$\therefore\ ke^{A^2}\int_{0}^{\infty}\left(\frac{z}{A}\right)^{A^2-1}e^{-z}dz\ \frac{1}{A}dz = 1$$

or, $$\frac{keA^2}{A^{A^2}}\int_0^\infty a^{A^2-1}e^{-z}dz = 1$$

or, $$\frac{keA^2}{A^{A^2}}\Gamma(A^2) = 1$$

or, $$= \frac{A^{A^2}e^{-A^2}}{\Gamma(A^2)}$$

$$\therefore \quad f(t) = \frac{A^{A^2}e^{-A^2}}{\Gamma(A^2)}(A+t)^{A^2-1}e^{-At} \qquad \text{Type III}$$

Since $A^2-1 > -1$, conditions (α) are satisfied. [Check: $k(A+t)^{A^2-1}e^{-At}\Big|_{-A}^{\infty} = 0$; $\alpha_{n+1} = k\int_{-A}^{\infty}(A+t)^{A^2-1}e^{-At}dt$ exists.]

SHAPE OF THE CURVE $f(t)$

$A^2 > 1 \Rightarrow \frac{4}{\alpha_3{}^2} > 1 \Rightarrow \alpha_3{}^2 < 4 \Rightarrow f(t)$ is bell-shaped

$A^2 < 1 \Rightarrow \alpha_3{}^2 > 4 \Rightarrow f(t)$ is J-shaped with an infinite ordinate at $t = -A$ [By conditions (ii) and (iii) in (17).]

For the bell-shaped curve, the mode (position of the maximum ordinate) is obtained from

$$f'(t) = 0$$

$$\Rightarrow (A^2-1)(A+t)^{A^2-2}e^{-At}+(A+t)^{A^2-1}(-A)e^{-At} = 0$$

$$\frac{A^2-1}{A+t} = A$$

$$t = -\frac{1}{A}$$

Now,

$$f''(t)\Big|_{t=-\frac{1}{A}} < 0 = \frac{d}{dt}\left[k(A+t)^{A^2-1}e^{-At}\left(\frac{A^2-1}{A+t} - A\right)\right]\Big|_{t=-\frac{1}{A}}$$

$$= -k\,\frac{d}{dt}\left[(A+t)^{A^2-1}e^{-At}\,\frac{At+1}{A+t}\right]\Big|_{t=-\frac{1}{A}}$$

$$= -k\,\frac{d}{dt}\,[(A+t)^{A^2-2}e^{-At}(At+1)]\Big|_{t=-\frac{1}{A}}$$

$$= -\frac{d}{dt}\left[f(t)\left(\frac{At+1}{A+t}\right)\right]\Big|_{t=-\frac{1}{A}}$$

$$= -f'(t)\left(\frac{At+1}{A+t}\right)\Big|_{t=-\frac{1}{A}} - f(t)\,\frac{(A+t)A-At-1}{(A+t)^2}\Big|_{t=-\frac{1}{A}}$$

$$= 0-k\left(A-\frac{1}{A}\right)^{A^2-1} e\,\frac{A^2-1}{\left(A-\frac{1}{A}\right)^2}$$

$$= -ke(A^2-1)\left(A-\frac{1}{A}\right)^{A^2-3}$$

$$< 0$$

$\Rightarrow f(t)$ is maximum at $t = -\frac{1}{A}$.

The mean of $f(t)$ is given by

$$m = \int_{-A}^{\infty} tf(t)dt \Big/ \int_{-A}^{\infty} f(t)dt$$

$$= \int_{-A}^{\infty} tf(t)dt \quad \text{[Since denominator = 1]}$$

$$= \int_{-A}^{\infty} kt(A+t)^{A^2-1} e^{-At} dt$$

$$= ke^{A^2} \int_0^{\infty} (x-A) x^{A^2-1} e^{-Ax} dx, \ [x=A+t]$$

$$= ke^{A^2} \left[\int_0^{\infty} x^{A^2} e^{-Ax} dx - A \int_0^{\infty} x^{A^2-1} e^{-Ax} dx \right]$$

$$= ke^{A^2} \left[\left. \frac{x^{A^2} e^{-Ax}}{-A} \right|_0^{\infty} + \int_0^{\infty} \frac{e^{-Ax}}{A} A^2 x^{A^2-1} dx - A \int_0^{\infty} x^{A^2-1} e^{-Ax} dx \right]$$

integrating the first integral by parts.

Therefore from above, $m = 0$

$$\therefore \quad \text{Mean} - \text{Mode} = 0 - \left(-\frac{1}{A} \right) = \frac{1}{A} = \frac{\alpha_3}{2}$$

For $A^2 = 1 \Rightarrow \alpha_3^{\,2} = 4$,

$$f(t) = \frac{1 \cdot e^{-1} e^{-t}}{\Gamma(1)} = \frac{e^{-t}}{e} \qquad \text{Type X}$$

whcih represents a J-shaped curve with the range $(-1, \infty)$ [By condition (iii) in (17)]

For $A^2 \neq 1 \Rightarrow \alpha_3^{\,2} \neq 4$, the function has been designated type III.

(c) DISCUSSION OF THE THREE MAIN TYPES (TYPES I, IV AND VI)

$$\delta \neq 0 \Rightarrow b_2 \neq 0 \quad [\text{by } (26)]$$

Consequently

$$b_0+b_1t+b_2t^2 \equiv b_2(t-r_1)(t-r_2), \; r_1 \neq 0, \; r_2 \neq 0$$

$(b_0 \neq 0)$ where

$$r_1 = \frac{-b_1+\sqrt{b_1^2-4b_0b_2}}{2b_2}$$

$$\text{and} \quad r_2 = \frac{-b-\sqrt{b_1^2-4b_0b_2}}{2b_2}$$

or, in terms of α_3 and δ,

$$r_1 = \frac{-\alpha_3+\sqrt{D}}{2\delta}$$

$$r_2 = \frac{-\alpha_3-\sqrt{D}}{2\delta} \qquad (28)$$

$$D = \alpha_3^2-4\delta(\delta+2)$$

[By means of (26)]

Also $r_1 \neq r_2$ in general; $r_1 = r_2$ is a special case.

Now we have

$$\frac{a-t}{b_0+b_1t+b_2t^2} = \frac{a-t}{b_2(t-r_1)(t-r_2)} = \frac{1}{b_2}\left[\frac{A}{t-r_1}+\frac{B}{t-r_2}\right] \quad \text{(say)}$$

So $$a-t = A(t-r_2)+B(t-r_1)$$

$$t = r_1 \Rightarrow a-r_1 = A(r_1-r_2) \Rightarrow A = \frac{a-r_1}{r_1-r_2}$$

$$t = r_2 \Rightarrow a-r_2 = A(r_2-r_1) \Rightarrow B = \frac{a-r_2}{r_2-r_1}$$

$$\therefore \quad \frac{a-t}{b_0+b_1t+b_2t^2} = \frac{a-r_1}{b_2(r_1-r_2)}\frac{1}{t-r_1} + \frac{a-r_2}{b_2(r_2-r_1)}\frac{1}{t-r_1}$$

$$= \frac{m_1}{t-r_1} + \frac{m_2}{t-r_2}$$

where

$$m_1 = \frac{a-r_1}{b_2(r_1-r_2)}$$

$$\text{and } m_2 = \frac{a-r_2}{b_2(r_2-r_1)} \qquad (29)$$

Then the solution of (20) can be put in the form

$$y = f(t) = c(t-r_1)^{m_1}(t-r_2)^{m_2} \qquad (30$$

c = integrating constant to be determined

Expressing m_1 and m_2 in (29) in terms of α_3 and δ, by means of (26) and (28), we have

$$m_1 = \frac{1+\delta}{\delta}\frac{\alpha_3}{\sqrt{D}} - \frac{1+2\delta}{\delta}$$

$$m_2 = -\frac{1+\delta}{\delta}\frac{\alpha_3}{\sqrt{D}} - \frac{1+2\delta}{\delta} \qquad (31)$$

Then, by (28)

(i) for $\delta < 0$, the r's are real and opposite in sign

(ii) for $\delta > 0$ and $\alpha_3{}^2 < 4\delta(\delta+2)$, the r's are complex

(iii) for $\delta > 0$ and $\alpha_3{}^2 > 4\delta(\delta+2)$, the r's are real and of the same sign.

Thus we can see how the discussion in section 7 is simplified by Craig.

The three conditions together with the additional condition that $\alpha_3 \neq 0$ give rise respectively to the main types of frequency functions designated types I, IV and VI.

MAIN TYPE I:

Craig's conditions for this curve are

$$\alpha_3 \neq 0$$

$$-1 < \delta < 0, \quad \delta \neq -\frac{1}{2}$$

$$(2+3\delta)\alpha_3{}^2 \neq 4(1+2\delta)^2(2+\delta)$$

Now, for $\alpha_3 > 0$ [By hypothesis (iv) of section 10(D)] we see that

$$r_1 < 0,\ r_2 > 0 \qquad r_1 < 0 < r_2 \qquad \text{[By (28)]}$$
$$\text{and that } |r_1| < |r_2|$$

The range is (r_1, r_2) and equation (30) is written

$$y = f(t) = c(t-r_1)^{m_1}(r_2-t)^{m_2} \qquad \text{Type I}$$

The area under the curve over this interval, i.e.

$$\int_{r_1}^{r_2} (t-r_1)^{m_1}(r_2-t)^{m_2}dt$$

is finite only when $m_1+1 > 0$, $m_2+1 > 0$, and in these cases, moments of all order exist. Conditions (α) are also satisfied accordingly.

Now

$$m_1+1 = -\frac{1+\delta}{\delta}\left(1 - \frac{\alpha_3}{\sqrt{D}}\right)$$

$$m_2+1 = -\frac{1+\delta}{\delta}\left(1 + \frac{\alpha_3}{\sqrt{D}}\right) \qquad \text{[By (31)]}$$

Therefore, in the present case, since

$$(m_1+1)(m_2+1) > 0$$

then
$$\left(\frac{1+\delta}{\delta}\right)^2 \left(1 - \frac{\alpha_3^{\,2}}{D}\right) > 0$$

This implies that $1 - \frac{\alpha_3^{\,2}}{D} > 0$.

So that $\qquad 1 \pm \frac{\alpha_3}{\sqrt{D}} > 0$.

Now

$$m_1 = -\frac{1+\delta}{\delta}\left(1 - \frac{\alpha_3}{\sqrt{D}}\right) - 1$$

So $-1 < \delta < -\frac{1}{2} \Rightarrow 0 < 1+\delta \Rightarrow 0 < \frac{1+\delta}{\delta}$ $(\because -\delta > 0) \Rightarrow 0 < -\frac{1+\delta}{\delta}$

$$\Rightarrow 1+\delta < -\delta \Rightarrow -\frac{1+\delta}{\delta} < 1$$

$$\Rightarrow 0 < -\frac{1+\delta}{\delta} < 1$$

Therefore $m_1 < 1 - \frac{\alpha_3}{\sqrt{D}} - 1 = -\frac{\alpha_3}{\sqrt{D}} < 0$ (since $\alpha_3 > 0$) for the above interval of δ.

For $-\frac{1}{2} < \delta < 0$,

$$m_1 > 0 \Rightarrow -\frac{1+\delta}{\delta}\left(1 - \frac{\alpha_3}{\sqrt{D}}\right) > 1$$

$$-\frac{1}{2} < \delta \Rightarrow -1 < 2\delta \Rightarrow -(1+\delta) < \delta \Rightarrow -\frac{1+\delta}{\delta} > 0$$

(since $\delta < 0$).

Therefore $-\frac{1+\delta}{\delta}\left(1 - \frac{\alpha_3}{\sqrt{D}}\right) > 1$

$$- 1 - \frac{\alpha_3}{\sqrt{D}} > -\frac{\delta}{1+\delta} \quad \left(\text{since } -\frac{1+\delta}{\delta} > 0\right.$$

$$\Rightarrow -\frac{\alpha_3}{\sqrt{D}} > 1 - \frac{\delta}{1+\delta}$$

$$\Rightarrow \frac{\alpha_3}{\sqrt{D}} < 1 + \frac{\delta}{1+\delta} = \frac{1+2\delta}{1+\delta}$$

$$\Rightarrow (1+\delta)\alpha_3 < (1+2\delta)\sqrt{D}$$

$$\Rightarrow (1+\delta)^2\alpha_3^2 < (1+2\delta)^2 D$$

$$\Rightarrow (1+\delta)^2\alpha_3^2 < (1+2\delta)^2[\alpha_3^2-4\delta(\delta+2)] \quad [\text{By } (28)]$$

$$\Rightarrow [(1+\delta)^2-(1+2\delta)^2]\alpha_3^2 < -4\delta(\delta+2)(1+2\delta)^2$$

$$\Rightarrow (2+3\delta)(-\delta)\alpha_3^2 < -4\delta(2+\delta)(1+2\delta)^2$$

$$\Rightarrow (2+3\delta)^2\alpha_3^2 < 4(2+\delta)(1+2\delta)^2 \quad (\text{since } -\delta > 0)$$

That is $m_1 > 0$ for $-\frac{1}{2} < \delta < 0$

$$\Rightarrow (2+3\delta)^2\alpha_3^2 < 4(2+\delta)(1+2\delta)^2$$

Similarly, $m_2 > 0$ for $-\frac{1}{2} < \delta < 0$

$$\Rightarrow (2+3\delta)^2\alpha_3^2 > 4(2+\delta)(1+2\delta)^2$$

Note: $\delta < -\frac{2}{3} \Rightarrow 3\delta < -2 \Rightarrow 2(1+\delta)+\delta < 0$

$$\Rightarrow 2(1+\delta) < -\delta \Rightarrow 1+\delta < \frac{1}{2}(-\delta)$$

$$\Rightarrow -\frac{1+\delta}{\delta} < \frac{1}{2} \quad [\text{since } -\delta > 0]$$

$$\Rightarrow 1+2\delta < -(1+\delta) \Rightarrow -\frac{1+2\delta}{\delta} < \frac{1+\delta}{\delta}$$

Next we want to show that for $\delta < -\frac{2}{3}$

$$m_1 < -\frac{1+2\delta}{\delta}\left(1 + \frac{\alpha_3}{\sqrt{D}}\right) \Rightarrow m_1 < 0$$

Proof: To determine whether $-\frac{1+2\delta}{\delta}$ > or < 0 for $\delta < -\frac{2}{3}$,

note that $-\frac{1+2\delta}{\delta} > 0 \Rightarrow 1+2\delta > 0 \Rightarrow \delta > -\frac{1}{2}$

$$\Rightarrow -\delta < \frac{1}{2} = \frac{3}{6}$$

This is a contradiction since $-\delta > \frac{2}{3} = \frac{4}{6}$.

So $-\frac{1+2\delta}{\delta} < 0$ for $\delta < -\frac{2}{3}$. That is $m_1 < 0$ for $\delta < -\frac{2}{3}$.

Similarly, $m_2 < 0$ for $\delta < -\frac{2}{3}$, and this implies that a Type I curve is U-shaped. [By condition (i) of (17).]

Next, note that

$r_2 - r_1 > 0$ and $b_2 \gtrless 0$ according as $\delta \lessgtr -\frac{1}{2}$ [By (26)].

So $r_1 < a < r_2$ for all U or bell-shaped curves only. [By (29) and (17).]

The sign of a in the differential equation (20) is always opposite to that of α_3 for curves with a mode. This is important for graphs, and interesting to note.

Finally, to determine the constant C, we have, by hypothesis (ii) of section 10(D),

$$C \int_{r_1}^{r_2} (t-r_1)^{m_1}(r_2-t)^{m_2} dt = 1$$

To integrate, let $t - r_1 = z$ so that the limits become 0 to $r_2 - r_1$, and

$$\int_{r_1}^{r_2} (t-r_1)^{m_1}(r_2-t)^{m_2} dt$$

$$= \int_{0}^{r_2-r_1} z^{m_1}[(r_2-r_1)-z]^{m_2} dz$$

$$= (r_2-r_1)^{m_2} \int_{0}^{r_2-r_1} z^{m_1} \left(1 - \frac{z}{r_2-r_1}\right)^{m_2} dz$$

$$= (r_2-r_1)^{m_2} \int_{0}^{1} (r_2-r_1)^{m_1} y^{m_1} (1-y)^{m_2} (r_2-r_1) dy$$

$$= (r_2-r_1)^{m_1+m_2+1} \int_{0}^{1} y^{m_1}(1-y)^{m_2} dy$$

$$= (r_2-r_1)^{m_1+m_2+1}\beta(m_1+1,m_2+1)$$

Therefore $$C = \frac{1}{\beta(m_1+1,\ m_2+1)(r_2-r_1)^{m_1+m_2+1}}$$

MAIN TYPE IV

Craig's conditions are

$$\alpha_3 \neq 0$$

$$\delta > 0$$

$$\alpha_3^2 < 4\delta(\delta+2)$$

In this case, by (28),

$$r_1 = -\frac{\alpha_3}{2\delta} + \frac{i\sqrt{-D}}{2\delta}$$

$$= -r+is$$

$$r_2 = -r-is$$

where $r = -\frac{\alpha_3}{2\delta}$, $s = \frac{\sqrt{-D}}{2\delta}$, and $-D > 0$. By (31),

$$m_1 = \frac{1+\delta}{\delta}\frac{\alpha_3}{i\sqrt{-D}} - \frac{1+2\delta}{\delta}$$

$$= -\frac{1+\delta}{\delta}\frac{i\alpha_3}{\sqrt{-D}} - \frac{1+2\delta}{\delta} = \frac{\nu i}{2} - m,$$

and similarly $m_2 = -\frac{\nu i}{2} - m$, where $\nu = \frac{-2(1+\delta)}{\delta}\frac{\alpha_3}{\sqrt{-D}}$ and $m = \frac{1+2\delta}{\delta}$. By (30), the solution $y = f(t)$ becomes:

$$y = C\left[\{(t+r)-is\}^{\frac{\nu i}{2}-m}\{(t+r)+is\}^{-\frac{\nu i}{2}-m}\right]$$

$$= C\left(\frac{t+r-is}{t+r+is}\right)^{\frac{\nu i}{2}}[(t+r)^2+s^2]^{-m}.$$

Now setting $t+r = R\cos\theta$, $s = R\sin\theta$, we get

$$\left(\frac{t+r-is}{t+r+is}\right)^{\frac{\nu i}{2}} = \left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right)^{\frac{\nu i}{2}} = \left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^{\frac{\nu i}{2}}$$

$$= e^{\nu\theta} = e^{\left(\nu\tan^{-1}\frac{s}{t+r}\right)} = e^{\frac{\nu\pi}{2}}\, e^{\left(\nu\tan^{-1}\frac{t+r}{s}\right)}$$

∴ The above solution becomes

$$y = f(t) = ce^{\frac{\nu\pi}{2}}[(t+r)^2+s^2]^{-m}e^{-\left(\nu\tan^{-1}\frac{t+r}{s}\right)} \qquad \text{Type IV}$$

Now $m = \frac{1+2\delta}{\delta} > 0$ (for $\delta > 0$), ν and α_3 are of opposite signs by our hypothesis, and

$$e^{\frac{\nu\pi}{2}} < e^{\left(-\nu\tan^{-1}\frac{t+r}{s}\right)}\; e^{\frac{-\nu\pi}{2}} \qquad (\nu < 0)$$

and the range of $f(t)$ is $(-\infty,\infty)$.

In the previously discussed cases in which $\delta \leq 0$, if the area under the curve was finite, moments of all order existed. In the present case, the area and the first four moments are always finite, but this may fail to be true of moments of higher orders.

The integral $\int_{-\infty}^{\infty} t^n f(t)dt$, which represents the n-th moment for the distribution given by $f(t)$, is

$$k \int_{-\infty}^{\infty} \frac{t^n}{[(t+r)^2+s^2]^m} e^{-\nu \tan^{-1} \frac{t+r}{s}} dt, \quad k = ce^{\frac{\nu\pi}{2}}$$

This integral exists if

$$2m > n+1 \tag{32}$$

Now $\delta \geq 1 \Rightarrow 1+2\delta \geq 3 \Rightarrow \frac{1+2\delta}{\delta} \leq 3 \Rightarrow m \leq 3$

So, by (32), $n+1 < 2m$.

$$\Rightarrow n+1 < 6 \qquad n < 5$$

$\Rightarrow n \leq 4$ for which moments are finite, and for $n = 5$ the moments are infinite.

By (32), in order that the n-th moment exists, we must have

$$\frac{2+4\delta}{\delta} > n+1$$

$$2+(3-n)\delta > 0$$

$$2 > (n-3)\delta$$

$$\delta < \frac{2}{n-3} \tag{33}$$

Pearson designated as 'heterotypic' those members of his system of frequency curves for which the eighth moment failed to exist. (In such a case the standard deviation of the fourth moment in samples would be infinite.)

It was apparent that conditions (α) were satisfied for $-1 < \delta < 0$. (It will appear later that the case in whcih $\delta = -\frac{1}{2}$ is no exception.) For $\delta > 0$, it will be seen that it is generally true, as in the present case, that the formulae (22) and (26) can be derived if α_{n+2} exists, that is if

$$\delta < \frac{2}{n-1} \qquad \text{[By (33)]}$$

To determine the constant C, we have

$$C\int_{-\infty}^{\infty} \frac{e^{\nu\left(\frac{\pi}{2} - \tan^{-1}\frac{t+r}{s}\right)}}{[(t+r)^2+s^2]^m}\, dt = 1$$

Putting $\phi = \frac{\pi}{2} - \tan^{-1}\frac{t+r}{s}$ so that the limits become π to 0 and

$$\tan^{-1}\frac{t+r}{s} = \frac{\pi}{2} - \phi \Rightarrow \cot\phi = \frac{t+r}{2}\,.$$

Then $(t+r) + s^2 = s^2[1+\cot^2\phi] = s^2\operatorname{cosec}^2\phi$

Also $t+r = s\cot\phi$

$dt = -s\operatorname{cosec}^2\phi\, d\phi$

We get ultimately

$$C\int_{\pi}^{0} \frac{e^{\nu\phi}}{s^{2m}\operatorname{cosec}^{2m}}(-s\operatorname{cosec}^2\phi)\,d\phi = 1$$

$$\text{or} \quad \frac{C}{s^{2m-1}}\int_{0}^{\pi} e^{\nu\phi}\sin^{2m-2}\phi\, d\phi = 1$$

$$\text{or} \quad \frac{C}{s^{2m-1}}\, G(2m-2,\nu) = 1$$

$$\text{or,} \quad C = \frac{s^{2m-1}}{G(2m-2,\nu)}$$

where $$G(2m-2,\nu) = \int_0^{\pi} \sin^{2m-2}\phi\, e^{\nu\phi}\, d\phi .$$

MAIN TYPE VI

The conditions of Craig are

$$\alpha_3 \neq 0$$

$$\delta > 0$$

$$\alpha_3^2 > 4\delta(\delta+2)$$

$$(2+3\delta)\alpha_3^2 \neq 4(1+2\delta)(2+\delta)$$

Note:

The last condition for a Type VI curve due to Craig is not, as a matter of fact, an additional constraint as shown by L. K. Roy [26]. Its significance is also clarified by him in the same.

The equation of the frequency curve is

$$y = C(t-r_1)^{m_1}(t-r_2)^{m_2}$$

PROPERTIES OF THE CURVE:

(i) $r_1 < 0 \Rightarrow -\alpha_3+\sqrt{D} < 0 \Rightarrow -\alpha_3 < -\sqrt{D} \Rightarrow \alpha_3 > \sqrt{D} \Rightarrow \alpha_3+\sqrt{D} > 2\sqrt{D}$

$$\Rightarrow -(\alpha_3+\sqrt{D}) < -2\sqrt{D}$$

$$\Rightarrow -(\alpha_3+\sqrt{D}) < 0$$

$$\Rightarrow r_2 < 0 \quad [\text{Since } \delta > 0]$$

Note also that since $\alpha_3 > \sqrt{D}$, then $\alpha_3 > 0$. Thus α_3 is opposite in sign to r_1 and r_2.

(ii) $|r_2| > |r_1|$

(iii) $m_2 = -\frac{1+\delta}{\delta}\frac{\alpha_3}{\sqrt{D}} - \frac{1+2\delta}{\delta} < 0$

(iv) $m_1 \gtrless 0$ according as $(1+\delta)\frac{\alpha_3}{\sqrt{D}} \gtrless 1+2\delta$ or according as

$(2+3\delta)\alpha_3{}^2 \lessgtr 4(2+\delta)(1+2\delta)^2$ [As in Main Type I.]

(v) $a-r_2 = b_2(r_2-r_1)m_2 > 0$ [Since $b_2 > 0$, $r_2-r_1 < 0$, $m_2 < 0$.]

(vi) $a-r_1 = b_2(r_1-r_2)m_1$ has the same sign as m_1.

(vii) $a < 0$ (Since $\alpha_3 > 0$, $\delta > 0$)

(viii) The range of $f(t)$ is (r_1,∞).

(ix) $m_1 > 0 \Rightarrow$ the curve is bell-shaped [Since $m_2 < 0$]

(x) $m_1 < 0 \Rightarrow$ the curve is J-shaped

$\Rightarrow t = a$ is to the left of $t = r_1$.

(xi) $m_1+m_2 = \frac{-2(1+2\delta)}{\delta} = -4 - \frac{2}{\delta}$

(xii) $\alpha_n = C\int_{r_1}^{\infty} t^n(t-r_1)^{m_1}(t-r_2)^{m_2}dt$

exists if

$$-(m_1+m_2) > n+1.$$

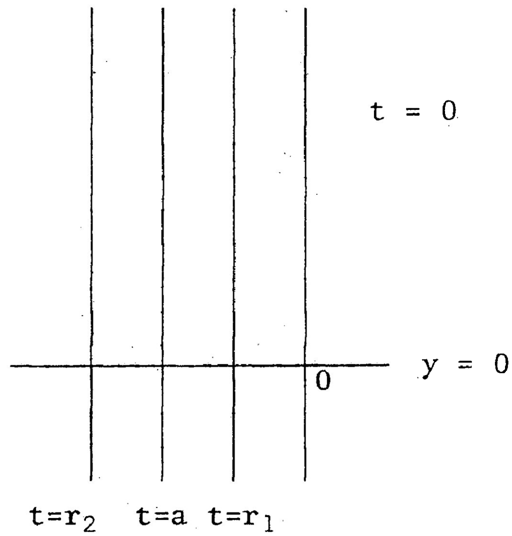
This implies that $4 + \frac{2}{\delta} > n+1$

$$\Rightarrow 3 + \frac{2}{\delta} > n$$

$$\Rightarrow n < 3 + \frac{2}{\delta}$$

$$\Rightarrow n \leq 4 \quad [\text{by } (27)]$$

Thus the first four moments always exist.



If the origin is shifted to the point $t = r$, we have, writing

$$t-r_2 = z, \quad r_1-r_2 = \alpha,$$

for the type VI function

$$y = f(z) = cz^{m_2}(z-\alpha)^{m_1} \qquad \text{Type VI}$$

with the range (α,∞).

Finally we have,

$$C\int_{\alpha}^{\infty} z^{m_2}(z-\alpha)^{m_1}dz = 1$$

$$C\int_{\alpha}^{\infty} z^{-k}(z-\alpha)^{m_1}dz = 1 \quad \text{where } k = -m_2 > 0$$

$$C \int_{\alpha}^{\infty} \frac{(z-\alpha)^{m_1}}{z^k} dz = 1$$

$$C \int_{\alpha}^{\infty} \frac{(z-\alpha)^{m_1}}{z^{m_1} z^{k-m_1}} dz = 1$$

$$C \int_{\alpha}^{\infty} z^{m_1-k} \left(1 - \frac{\alpha}{z}\right)^{m_1} dz = 1$$

$$C \int_{\alpha}^{\infty} z^{m_1+m_2} \left(1 - \frac{\alpha}{z}\right)^{m_1} dz = 1$$

Put $\frac{\alpha}{z} = x$, then

$$C \int_{1}^{0} \left(\frac{\alpha}{x}\right)^{m_1+m_2} (1-x)^{m_1} \left(-\frac{\alpha}{x^2}\right) dx = 1$$

$$C\alpha^{m_1+m_2+1} \int_{0}^{1} x^{-(m_1+m_2+2)} (1-x)^{m_1} dx = 1$$

$$C\alpha^{m_1+m_2+1} \beta(m_1+1, -m_1-m_2-1) = 1 \quad [\beta(m,n) = \beta(n,m)]$$

$$C = \frac{1}{\alpha^{m_1+m_2+1} \beta(m_1+1, -m_1-m_2-1)}$$

(d) TRANSITIONAL TYPE II

The conditions are

$$\alpha_3 = 0$$

$$-1 < \delta < 0$$

$$\delta \neq -\frac{1}{2}$$

In this case, by (28),

$$r_1 = \frac{\sqrt{-\delta(\delta+2)}}{\delta} < 0$$

$$r_2 = \frac{-\sqrt{-\delta(\delta+2)}}{\delta} > 0$$

So $r_1 = -r_2$.

By (31),

$$m_1 = m_2 = -\frac{1+2\delta}{\delta} \gtrless \text{ according as } \delta \gtrless -\frac{1}{2}.$$

The frequency function in this case is evidently a special case of Type I.

Setting $r_1 = -r_2 = s$

and $m_1 = m_2 = M$,

the frequency function takes the form

$$y = C(s^2-t^2)^M \qquad \text{Type II}$$

PROPERTIES:

(i) It is symmetrical about the axis of t.

(ii) The range is $(-s,s)$.

(iii) $-1 < \delta < -\frac{1}{2} \Rightarrow m_1 = m_2 < 0 \Rightarrow$ the curve is U-shaped.

(iv) $-\frac{1}{2} < \delta < 0 \Rightarrow m_1 = m_2 > 0 \Rightarrow$ the curve is bell-shaped.

(v) As in Type I, the area and moments do not exist for $\delta \leq -1$.

Finally, to evaluate C, we have

$$C\int_{-s}^{s}(s^2-t^2)^M dt = 1$$

$$2C\int_{0}^{s}(s^2-t^2)^M dt = 1$$

$$2C\int_{0}^{s}(s+t)^M(s-t)^M dt = 1$$

put $s-t = z \Rightarrow s+t = 2s-z.$

Then $2C\int_{s}^{0}(2s-z)^M z^M(-dz) = 1$

$$2C\int_{0}^{s} z^M(2s)^M\left(1-\frac{z}{2s}\right)^M dz = 1$$

$$C(2s)^M 2\int_{0}^{s} z^M\left(1-\frac{z}{2s}\right)^M dz = 1$$

$$C(2s)^M\int_{0}^{2s} z^M\left(1-\frac{z}{2s}\right)^M dz = 1$$

$$C(2s)^M\int_{0}^{1}(2s)^M x^M(1-x)^M(2s)dx = 1$$

$$C(2s)^{2M+1}\beta(M+1,M+1) = 1$$

$$C = \frac{1}{(2s)^{2M+1}\beta(M+1,M+1)}$$

(e) TRANSITIONAL TYPE VII (DUE TO CRAIG)

Craig's conditions for this curve are

$$\alpha_3 = 0$$

$$\delta > 0$$

The frequency function in this case is obviously a special case of type IV, with

$$r = 0,\ s = \frac{\sqrt{-D}}{2\delta} = \frac{\sqrt{\delta(\delta+2)}}{\delta} > 0,\ \nu = 0,\ m = \frac{1+2\delta}{\delta} > 0.$$

The equation is

$$y = C(s^2+t^2)^{-m} \qquad \text{Type VII}$$

This equation could also be derived from the Type II function by noting that $s = is$ and $M = -m$.

The range of the curve is $(-\infty,\infty)$. For $\delta \geq 2/5$, the curve is heterotypic [By (33)].

To determine C, we have

$$C\int_{-\infty}^{\infty}(s^2+t^2)^{-m}dt = 1$$

$$2C\int_{0}^{\infty}(s^2+t^2)^{-m}dt = 1$$

$$2C\,\frac{1}{s^{2m-1}}\int_{0}^{\pi 2}\cos^{2m-2}\theta\, d\theta\ [t = s\tan\theta]$$

$$C \frac{2}{s^{2m-1}} I_{2m-2} = 1 \quad \left[I_{2m-2} = \int_0^{\pi/2} \cos^{2m-2}\theta d\theta \right.$$

$$C = \frac{s^{2m-1}}{2I_{2m-2}}$$

or, in another way, we can write

$$y = cs^{-2m} \left(1 + \frac{t^2}{s^2} \right)^{-m}$$

Put $\quad 1 + \frac{t^2}{s^2} = z^{-1}$

$$\frac{2tdt}{s^2} = -z^{-2}dz$$

$$dt = -\frac{s^2}{2t}\frac{dz}{z^2} = -\frac{s^2}{2\left(\frac{1}{z} - 1\right)^{1/2} s}\frac{dz}{z^2}$$

$$= -\frac{s}{2(1-z)^{1/2} z^{3/2}} dz$$

So the required integral equals

$$2Cs^{-2m} \int_0^\infty (1+t^2/s^2)dt$$

$$= -\frac{2C}{s^{2m}} \int_1^0 z^{m-3/2}/(1-z)^{-1/2} \frac{s}{2} dz$$

$$= \frac{C}{s^{2m-1}} \int_0^1 z^{m-3/2}(1-z)^{-1/2} dz$$

$$= \frac{C}{s^{2m-1}} \beta \left(m - \frac{1}{2}, \frac{1}{2} \right)$$

$$= \frac{C}{s^{2m-1}} \frac{\Gamma\left(m - \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m)}$$

$$= \frac{C}{s^{2m-1}} \frac{\Gamma\left(m - \frac{1}{2}\right)\sqrt{\pi}}{\Gamma(m)} = 1$$

$$C = \frac{s^{2m-1}}{\sqrt{\pi}} \frac{\Gamma(m)}{\Gamma\left(\frac{2m-1}{2}\right)}$$

(f) TRANSITIONAL TYPE V

Craig's conditions are

$$\alpha_3 \neq 0$$

$$\delta > 0$$

$$\alpha_3{}^2 = 4\delta(\delta+2)$$

$$\Rightarrow D = 0$$

The type V function can be derived as a limiting form of type VI function.

Here $r_1 = r_2 = -r$ (say)

Hence from type VI function we get

$$y = C(t+r)^{m_2}(t+r-\alpha)^{m_1} \quad [z = t-r_2 = t+r]$$

$$= C(t+r)^{m_1+m_2}\left(1 - \frac{\alpha}{t+r}\right)^{m_1}$$

$$= C(t+r)^{-2m}\left(1 - \frac{\frac{\sqrt{D}}{\delta}}{t+r}\right)^{\frac{1+\delta}{\delta}\frac{\alpha_3}{\sqrt{D}} - \frac{1+2\delta}{\delta}} \quad \left[\alpha = r_1 - r_2 = \frac{\sqrt{D}}{\delta}\right]$$

Now we proceed to find

$$\lim_{D\to 0}\left(1-\frac{\frac{\sqrt{D}}{\delta}}{t+r}\right)^{\frac{1+\delta}{\delta}\frac{\alpha_3}{\sqrt{D}}-\frac{1+2\delta}{\delta}}$$

$$=\lim_{D\to 0}\left(1-\frac{\frac{\sqrt{D}}{\delta}}{t+r}\right)^{\frac{1+\delta}{\delta}\frac{\alpha_3}{\sqrt{D}}}$$

$$=\lim_{D\to 0}\left[\left(1-\frac{\frac{1}{\delta}\sqrt{D}}{t+r}\right)^{\frac{1}{\sqrt{D}}}\right]^{\frac{1+\delta}{\delta}\alpha_3}$$

$$=\left[\lim_{D\to 0}\left(1-\frac{\frac{1}{\delta}\sqrt{D}}{t+r}\right)^{\frac{1}{\sqrt{D}}}\right]^{\frac{1+\delta}{\delta}\alpha_3}$$

$$=\left[e^{-\frac{1}{\delta(t+r)}}\right]^{\frac{1+\delta}{\delta}\alpha_3}$$

$$=e^{-\frac{(1+\delta)}{\delta^2(t+r)}}$$

$$=e^{-\frac{(m-1)}{t+r}\frac{\alpha_3}{\delta}}\qquad\left[\frac{1+\delta}{\delta}=m-1\right]$$

$$=e^{-\frac{(m-1)}{t+r}2r}\qquad\left[r=\frac{\alpha_3}{2\delta}\right]$$

$$=e^{-\frac{2r(m-1)}{t+r}}$$

∴ The frequency function for type V becomes

$$y=C(t+r)^{-2m}\,e^{-\frac{2r(m-1)}{t+r}}\qquad\text{Type V}$$

We could also get the same equation by solving the differential equation

$$\frac{1}{y}\frac{dy}{dt} = \frac{a-t}{b_2(t+r)^2}$$

for this case.

PROPERTIES OF THE CURVE

(i) r has the same sign as α_3 $(\alpha_3 > 0)$.

(ii) $m = 2 + \frac{1}{\delta}$

(iii) The range is $(-r,\infty)$.

(iv) The curve is always bell-shaped.

(v) For the existence of the n-th moment, we must have

$$4+2/\delta > n+1$$

which leads to the same conclusions as in the Type IV or VI.

To evaluate C,

$$C\int_{-r}^{\infty} (t+r)^{-2m}\, e^{-\frac{2r(m-1)}{t+r}}\, dt = 1$$

Put $\frac{2r(m-1)}{t+r} = z$. The limits become ∞ to 0..

Also $\frac{-2r(m-1)}{(t+r)^2}\,dt = dz$

$$dt = -\frac{(t+r)^2}{2r(m-1)}\,dz = -\frac{[2r(m-1)]^2 dz}{z^2\, 2r(m-1)} = -2r(m-1)\,\frac{dz}{z^2}$$

So we have,

$$C\int_0^{\infty} 2r(m-1)\left[\frac{2r(m-1)}{z}\right]^{-2m} e^{-z}\,\frac{dz}{z^2} = 1$$

$$C[2r(m-1)]^{1-2m}\int_0^{\infty} z^{2m-2}e^{-z}dz = 1$$

$$C[2r(m-1)]^{1-2m}\Gamma(2m-1) = 1$$

$$C = \frac{[2r(m-1)]^{2m-1}}{\Gamma(2m-1)} \qquad \text{where} \quad 2m-1 > 0.$$

(g) TRANSITIONAL TYPE VIII

The conditions are

$$\alpha_3 \neq 0$$

$$\delta < -\frac{1}{2}$$

$$(2+3\delta)\alpha_3^2 = 4(1+2\delta)^2(2+\delta)$$

The frequency function in this case is a special case of type I in which $m_1 < 0$ and $m_2 = 0$.

By type I, $\delta < -\frac{1}{2}$

$$\Rightarrow m_1 < 0.$$

Now $m_2 = 0$ $\quad m_1 = \frac{-2(1+2\delta)}{\delta} = -2m$, so that the frequency function becomes

$$y = C(t-r_1)^{-2m} \qquad \text{Type VIII}$$

PROPERTIES:

(i) The range is (r_1, r_2).

(ii) The curve is J-shaped with an infinite ordinate at $t = r_1$ and a finite one at $t = r_2$.

To determine C, we have from type I,

$$C = \frac{1}{\beta(1-2m,1)(r_2-r_1)^{1-2m}}$$

$$= \frac{\Gamma(2-2m)}{\Gamma(1-2m)\Gamma(1)} \cdot \frac{1}{(r_2-r_1)^{1-2m}}$$

$$= \frac{(1-2m)!}{(-2m)!} \quad \frac{1}{(r_2-r_1)^{1-2m}}$$

$$= \frac{1-2m}{(r_2-r_1)^{1-2m}}$$

(h) TRANSITIONAL TYPE IX

Craig's conditions for this type of curve are

$$\alpha_3 \neq 0$$

$$-\frac{1}{2} < \delta < 0$$

$$(2+3\delta)\alpha_3^2 = 4(1+2\delta)^2(2+\delta)$$

This is also a special Type I function where $m_1 = 0$ and $m_2 = -2m > 0$.

The curve is

$$y = C(r_2-t)^{-2m} \qquad \text{Type IX}$$

PROPERTIES:

(i) The range is (r_1,r_2)

(ii) The curve is J-shaped with a finite ordinate at $t = r_2$

($\because$ $-2m > 0$).

(iii) C has the same value as in Type VIII, that is

$$C = \frac{1-2m}{(r_2-r_1)^{1-2m}} .$$

(i) TRANSITIONAL TYPE XI

Craig's conditions are

$$\alpha_3 \neq 0$$

$$0 < \delta < 2/5$$

$$(2+3\delta)\alpha_3^{\,2} = 4(1+2\delta)^2(2+\delta)$$

The function is a special type VI where

$$m_1 = 0 \quad \text{and} \quad m_2 = -2m < 0 \quad (\text{for} \quad 0 < \delta < \tfrac{2}{5}).$$

The frequency function is obviously

$$y = C(t-r_2)^{-2m} \qquad \text{Type XI}$$

PROPERTIES

(i) The range is (r_1,∞).

(ii) The curve is J-shaped with a finite ordinate at $t = r_1$ and an infinite one at $t = r_2$.

(iii) The C is found from the C for Type VI with $m_1 = 0$, $m_2 = -2m$.

So $$C = \frac{1}{(r_1-r_2)^{1-2m}\ (1,2m-1)} = \frac{\Gamma(2m)}{\Gamma(1)\Gamma(2m-1)(r_1-r_2)^{1-2m}}$$

$$= \frac{2m-1}{(r_1-r_2)^{1-2m}}$$

(j) TRANSITIONAL TYPE XII

Craig's condition is

$$\delta = -\frac{1}{2}$$

Now if $\delta = -\frac{1}{2}$, the values of a, b_0, b_1, b_2 become indeterminate as we see from (26). In this case, we set the values of a, b_0, b_1, b_2 as obtained in (26) in the differential equation (20) and from its limiting form as $\delta \to -\frac{1}{2}$, we derive the function appropriate to this case in the following way.

We get from substituting (26) into (20)

$$\frac{1}{y}\frac{dy}{dt} = \frac{-\frac{\alpha_3}{2(1+2\delta)} - t}{\frac{2+\delta}{2(1+2\delta)} + \frac{\alpha_3}{2(1+2\delta)}t + \frac{\delta}{2(1+2\delta)}t^2}$$

$$= -\frac{\alpha_3+2(1+2\delta)t}{(2+\delta)+\alpha_3 t+\delta t^2}$$

Now $\delta = -\frac{1}{2}$ gives:

$$\frac{1}{y}\frac{dy}{dt} = -\frac{\alpha_3+0}{\left(2-\frac{1}{2}\right)+\alpha_3 t-\frac{1}{2}t^2} = \frac{2\alpha_3}{t^2-2\alpha_3 t-3}$$

$$= \frac{2\alpha_3}{(t-r_1)(t-r_2)}$$

where $$r_1 = \frac{2\alpha_3-\sqrt{4\alpha_3^2+12}}{2} = \alpha_3-\sqrt{\alpha_3^2+3}$$

$$r_2 = \alpha_3 + \sqrt{\alpha_3^2+3}$$

Then the above equation can be written as

$$\frac{1}{y}\frac{dy}{dt} = 2\alpha_3\left[\frac{A}{t-r_1} + \frac{B}{t-r_2}\right]$$

where $A = \frac{1}{r_1-r_2}$ and $B = \frac{1}{r_2-r_1}$.

so that we get

$$\frac{dy}{y} = \frac{2\alpha_3}{r_2-r_1}\left[\frac{1}{t-r_2} - \frac{1}{t-r_1}\right]$$

$$= \frac{\alpha_3}{\sqrt{\alpha_3^2+3}}\left[\frac{1}{t-r_2} - \frac{1}{t-r_1}\right]$$

Integrating, $y = C'(t-r_1)^{m_1}(t-r_2)^{m_2}$.

where $$m_2 = -m_1 = \frac{\alpha_3}{\sqrt{\alpha_3^2+3}}$$

Since $\alpha_3 > 0$, then $r_2 > 0$.

Now $r_1 = \alpha_3 - \sqrt{\alpha_3^2+3} < 0$.

So $r_2 > 0 > r_1$.

Thus $|r_2| > |r_1|$.

Also $m_2 = -m_1 > 0$.

Therefore ultimately the frequency function becomes

$$y = C'(t-r_1)^{-m_2}(t-r_2)^{m_2}$$

$$= C'\left(\frac{t-r_2}{t-r_1}\right)^{m_2}$$

$$\text{or,} \quad y = C\left(\frac{r_2-t}{t-r_1}\right)^{m_2} \qquad \text{Type XII}$$

the range being (r_1,r_2).

This curve is J-shaped. The frequency function in this case could also be derived as a special case of the Type I function in which $\delta = -\frac{1}{2}$. Hence

$$C = \frac{1}{(r_2-r_1)\beta(1-m_2,1+m_2)}, \quad \text{using in type I,} \quad m_1 = -m_2.$$

(k) Finally, we note that for $\alpha_3 = 0$, the Type XII curve reduces to

$$y = C \quad [\text{Since } m_2 = 0 \text{ for } \alpha_3 = 0]$$

thus including the rectangular distribution function among the Pearson system.

(ℓ) A SYNOPSIS OF THE STUDY IN CHAPTER II

$\delta = 0$; $\alpha_3 = 0 \Rightarrow$ Normal frequency function

$\alpha_3 \neq 0 \Rightarrow$ Type III

$\alpha_3^2 = 4 \Rightarrow$ Type X

$$\left.\begin{array}{ll} -1 < \delta < 0,\ \delta \neq -\frac{1}{2};\ \alpha_3 \neq 0 & \Rightarrow \text{Type I} \\ \delta > 0;\ \alpha_3^2 > 4\delta(\delta+2) & \Rightarrow \text{Type VI} \end{array}\right\} (2+3\delta)\alpha_3^2 \neq 4(1+2\delta)^2(2+\delta)$$

$$\alpha_3^2 < 4\delta(\delta+2) \qquad \Rightarrow \text{Type IV}$$

$$-1 < \delta < 0,\ \delta \neq -\frac{1}{2};\ \alpha_3 = 0 \qquad \Rightarrow \text{Type II}$$

$$\delta > 0,\ \alpha_3 = 0 \qquad \Rightarrow \text{Type VII}$$

$$\alpha_3^2 = 4\delta(\delta+2) \qquad \Rightarrow \text{Type V}$$

$$\left.\begin{array}{ll} \delta < -\frac{1}{2};\ \alpha_3 \neq 0 & \Rightarrow \text{Type VIII} \\ -\frac{1}{2} < \delta < 0;\ \alpha_3 \neq 0 & \Rightarrow \text{Type IX} \\ 0 < \delta < \frac{2}{5};\ \alpha_3 \neq 0 & \Rightarrow \text{Type XI} \\ \delta = -\frac{1}{2};\ \alpha_3 \neq 0 & \Rightarrow \text{Type XII} \end{array}\right\} (2+3\delta)\alpha_3^2 = 4(1+2\delta)^2(2+\delta)$$

$$\alpha_3 = 0 \qquad \Rightarrow \text{Rectangular distribution}$$

(m) SOME CONCLUSIONS

The scheme in the previous section shows how the new exposition for the Pearson system of frequency curves by C. C. Craig possesses marked advantages in unity, clarity and elegance in specifying the curves in terms of the two parameters δ and α_3. The definition of δ is given by (18) in terms of α_3 and α_4, while the definitions for $\alpha_3(=\sqrt{\beta_1})$ and $\alpha_4(=\beta_2)$ are obtained from (23).

In the above mentioned specification of the curves as outlined in the previous section, we see that

$$\delta \in (-1, 2/5).$$

By (33), it is inferred that for all twelve types of Pearson curves

$$\delta < \frac{2}{5} \text{ [using } n = 8 \text{ in (33)]}$$

ensures the existence of eighth moment.

J-SHAPED, U-SHAPED, AND "COCKED-HAT" CURVES

The following page contains a diagram showing transition from two separated blocks of frequency through U-shaped and J-shpaed curves to the more common "cocked hat" shapes.

(1) represents two separted blocks of frequency developing into U-shaped curve in (2). The horizontal line of (3) is, as it were, the bottom piece of the U-curve and the Type VIII curve of (4) is like part of the U-curve. (5) and (6) are limits when straight lines are reached. (7) is Type IX and (8) is the exponential. The next two curves (9) and (10) are J-shaped curves of Types III, and I, and (11) is Type (XII). From this we proceed to Types, I, III, IV, V and VI, curves of the "cocked hat" shape, three examples being given in (12), (13) and (14).

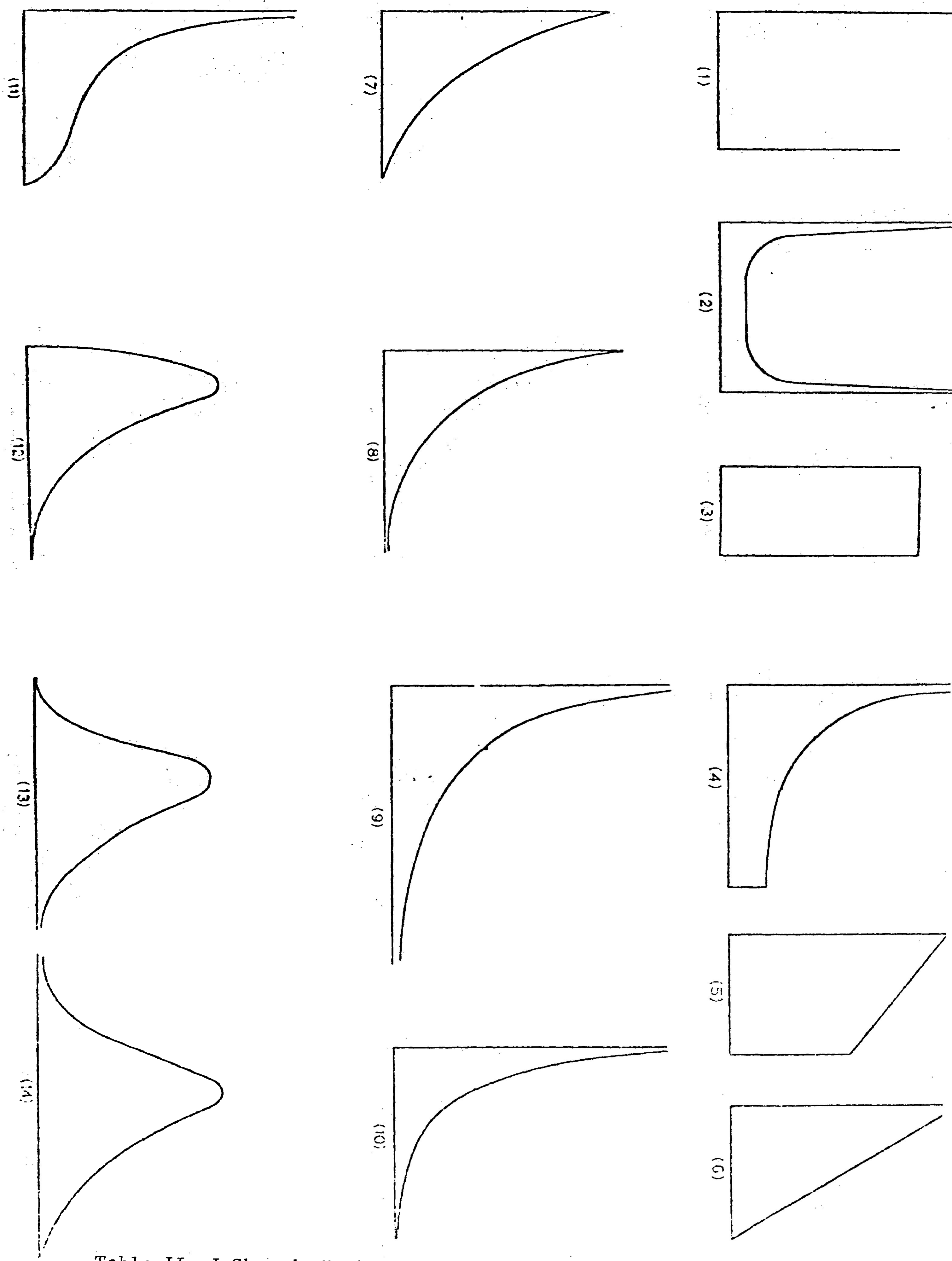


Table II. J-Shaped, U-Shaped, and "Cocked Hat" Curves

CHAPTER III

SOME INTERESTING FEATURES OF PEARSON CURVES

INTRODUCTION

In this Chapter, we have a detailed study of the bell-shaped Pearson curves. Another method than the method of moments for computing the constants in the Pearson differential equation is developed.

A study of Type II curves by the method of maximum likelihood is outlined [4], while it is interesting to see how Pearson's Type II curve occurs in Thompson's criterion for the rejection of outlying observations. This indicates a practical application of Pearson's Type II curve.

The Chapter ends up with the development of Pearson's Type III curve from Bernoulli distribution, also indicating how the normal curve of error can be obtained as a limiting case of the Type III curve as the skewness approaches zero.

11. DEFINITION OF A BELL-SHAPED CURVE

A bell-shaped curve is a continuous curve which starts at zero (or zero as a limit), rises to a single maximum, at which maximum point the first derivative is zero, and then falls to zero (or zero as a limit).

Or, analytically, $y = G(x)$ is a bell-shaped curve if

$G(x_1) = G(x_2) = 0$ and if $G'(P) = 0$ and $G''(P) < 0$ where $G(x)$ is continuous and does not vanish in the interval from x_1 to x_2 and P is a unique point in this interval.

12. POINTS OF INFLEXION OF BELL-SHAPED CURVES

A discussion of the points of inflexion beings out the limitations of the bell-shaped Pearson curves in the most striking manner.

Consider the differential equation (20) of Chapter II in the form

$$\frac{dy}{dx} = \frac{y(x-A)}{b_2x^2+b_1x+b_0} \tag{34}$$

If we put $x-A = X$ i.e. shift the mode to $X = 0$, we get

$$\frac{dy}{dX} = \frac{yX}{\pm B_2X^2 \pm B_1X+B_0} \tag{35}$$

the + or - sign is taken according to the type of the curve. We shall show that $B_0 < 0$ later.

Since in the Type III curve $B_2 = 0$ and in the "Normal Curve" both $B_2 = 0$ and $B_1 = 0$ it would be of advantage to consider the general case

$$\frac{dy}{dX} = \frac{yX}{F(X)} \tag{36}$$

in place of (35), where $F(X)$ is a rational integral function of degree n, and then consider $n = 2$ for Pearson curves.

By (36), $$\frac{d^2y}{dX^2} = \frac{y}{[F(X)]^2}[X^2+F(X)-XF'(X)] \qquad (37)$$

For points of inflexion of y,

$$\frac{d^2y}{dX^2} = 0$$

$$X +F(X)-XF'(X) = 0 \qquad (38)$$

[By (37)]

The equation (38) is always of the same degree as $F(X)$ except when $F(X)$ is linear or constant. Thus we get the following theorem:

Theorem I: If $y = G(X)$ is a solution of (36), then the number of points of inflexion of $y[=G(x)]$ cannot exceed the degree of $F(X)$ when $F(X)$ is of degree greater than one.

Hence for Pearson's bell-shaped curves the maximum number of points of inflexion is two, that is, a Pearson bell-shaped curve has at most two points of inflexion, and no more.

Now $F(X)$ can be written in the form

$$F(X) = B_nX^n+B_{n-1}X^{n-1}+\ldots+B_2X^2+B_1X+B_0 \qquad (39)$$

By (39), equation (38) becomes

$$(1-n)B_nX^n+(2-n)B_{n-1}X^{n-1}+(3-n)B_{n-2}X^{n-2}+\dots$$

$$+(r-n)B_{n-r+1}X^{n-r+1}+\dots-3B_4X^4-2B_3X^3+(1-B_2)X^2+B_0 = 0 \qquad (40)$$

Thus from (40), we have the following theorem:

Theorem II: The coefficient of the linear term of X in the equation of the points of inflexion is zero.

Now for the "Normal Curve" and Type III, we have

$$B_2 = B_3 = B_4 = \dots = B_n = 0$$

Hence the points of inflexion of these two types of curves, as is easily seen from (40), are given by

$$X = \pm\sqrt{-B_0} \qquad (41)$$

Again for types I and II, $B_2 > 0$ and

$$B_3 = B_4 = \dots = B_n = 0$$

and the points of inflexion are

$$X = \pm\sqrt{\frac{-B_0}{1-B_2}} \qquad (42)$$

[By (40)]

And for types IV, V, VI and VII, $B_2 < 0$ and

$$B_3 = B_4 = \dots = B_n = 0$$

and the points of inflexion are at

$$X = \pm\sqrt{\frac{-B_0}{1+|B_2|}} \tag{43}$$

[an obvious modification of (42)]

13. SOME OBSERVATIONS

(i) In some of these types of curves it may happen that the abscissae of the points of inflexion though real will be beyond the range of the curve.

(ii) Thus types III and VI may have 1 or 2 points of inflexion, the single point of inflexion occuring when

$$\left|\sqrt{\frac{-B_0}{1+B_2}}\right| >$$ range of the curve in the direction that the range is limited.

(iii) Type II may have 0 or 2 points of inflexion, as there will be no real point of inflexion when $B_2 \geq 1$.

(iv) Type I may have 0, 1 or 2 points of inflexion.

(v) Types IV, V, VII and the "Normal Curve" always have 2 and only two points of inflexion.

(vi) By (41), $B_0 < 0$.

14. LIMITATIONS OF THE BELL-SHAPED PEARSON CURVES

Consider the three hypothetical histograms as given in the attached diagram. All these are bell-shaped yet none of them will be closely fitted by any of the Pearson curves.

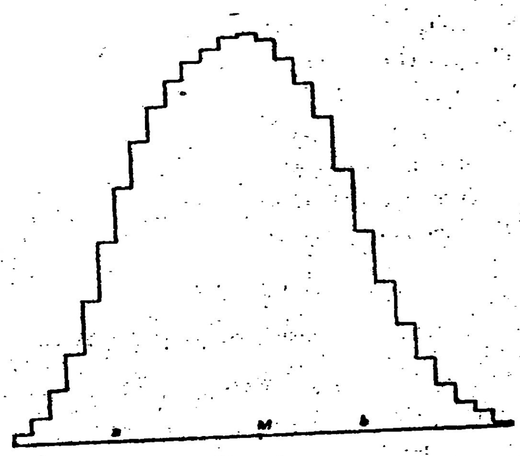


FIG. 1

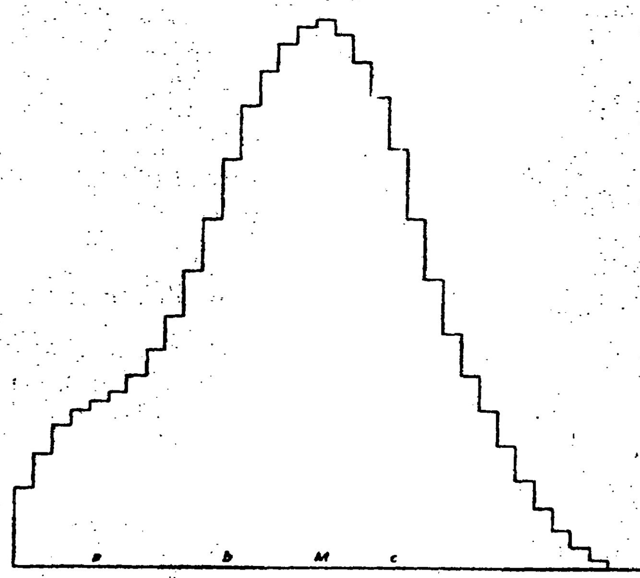


FIG. 2

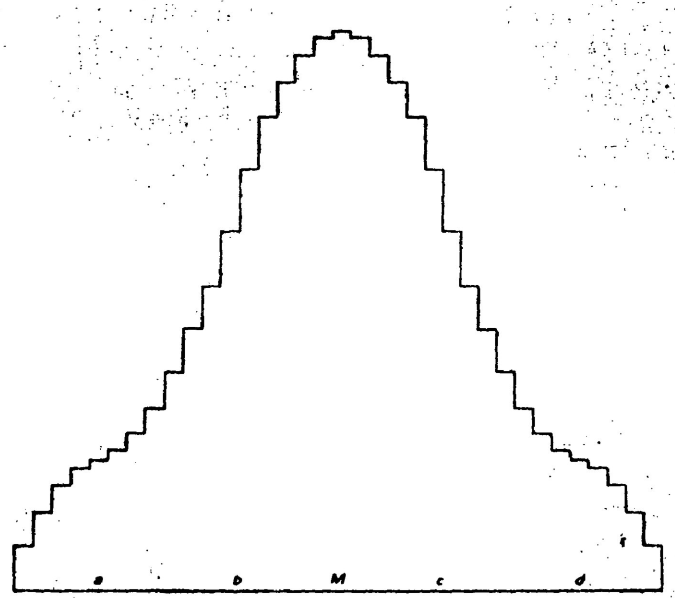


FIG. 3

In section 12, we have seen that when one of the eight bell-shaped Pearson curves (types I, II, III, IV, V, VI, VII and the "Normal Curve") has two points of inflexion then the abscissae of these two points of inflexion are equidistant from the abicissa of the mode.

In figure 1, a point of inflexion will be at abscissa b and another at abscissa a, M is the abscissa of the mode. Now since $(b-M) \neq (M-a)$ none of the Pearson curves will fit this histogram closely.

In figure 2, points of inflexion occur at abscissae a,b, and c. Since a Pearson curve can have at most two points of inflexion, no Pearson curve will fit this histograms closely.

In figure 3, there are four points of inflexion and no Pearson curve will fit this histograms closely.

15. DEFINITIONS OF RANGE FOR BELL-SHAPED CURVES

Definition 1: If a bell-shaped curve has the value of zero at two finite points, one on each side of the maximum (mode), it is said to be of limited range in both directions, or briefly, of limited range.

Definition 2: If a bell-shaped curve has the value of zero at only one finite point it is said to be of limited range in one direction, or, of unlimited range in one direction.

Definition 3: If a bell-shaped curve has the value of zero only at $\pm\infty$, that is, at no finite points, it is said to be of unlimited range in both directions, or simply, of unlimited range.

16. ZOCH'S THEOREM

In this section, we state the theorem due to Richmond T. Zoch [32] (with a corollary) without proof. The usefulness of this theorem in relation to our present disucssion will be brought out in the next section.

THEOREM: If $F(x)$ is a polynomial with real coefficients and $y = G(x)$ is a bell-shaped curve which is a solution of the differential equation

$$\frac{dy}{dx} = \frac{y(x-A)}{F(x)} \qquad \text{[Equation (36) of section 12]}$$

then the necessary and sufficient condition: (i) that $G(x)$ is of unlimited range in both directions is that $F(x) = 0$ has no real roots; (ii) that $G(x)$ is of limited range in one direction is that all the real roots of $F(x) = 0$ lie on the same side of A; (iii) that $G(x)$ is of limited range in both directions is that at least one real root of $F(x) = 0$ lie on one side of A and one on the other.

Corollary: $F(x) < 0$ throughout the range of y.

17. IMPORTANCE OF ZOCH'S THEOREM

Suppose we have some statistics which we wish to graduate and the statistics are of such nature that we would expect a bell-shaped curve, rather than a J- or U-shaped curve, and we desire the best fit.

If we use a curve which is a solution of the differential euqation (34) [the Pearson curves being special cases] to fit the statistics and if in computing the constants for the curve one of the following cases arises:

(a) $b_0 \nless 0$ when this constant is computed

or (b) $B_0 \nless 0$ when the origin is shifted to the mode

or (c) a root is located within the range of the statistics

then it means that:

(i) A mistake may have been made in the computation; thus Zoch's theorem provides a rough check on the work of computation.

(ii) If no mistake has been made in the computation it may indicate that the bell-shaped Pearson curves will not closely fit the statistics and that some other graduation curves be used, e.g. the Gram-Charlier Types A or B might be tried.

(iii) If no mistake has been made in the computation it may happen that one of the bell-shaped Pearson curves will give an excellent fit but a different method than or a modification of the Method of Moments (as in Chapter I) should be used to compute the constants.

18. COMPUTATION OF THE CONSTANTS: A MODIFICATION OF THE METHOD OF MOMENTS

In the differential equation (34), put

$$x-A = X \qquad dx = dX, \quad x = X+A \quad \text{and we get}$$

$$\frac{dy}{dX} = \frac{yX}{b_2(X+A)^2+b_1(X+A)+b_0}$$

$$= \frac{yX}{b_2X^2+(2Ab_2+b_1)X+(A^2b_2+Ab_1+b_0)} \tag{44}$$

Now

$$\begin{aligned} b_2 &= B_2 \\ 2Ab_2+b_1 &= B_1 \\ A^2b_2+Ab_1+b_0 &= B_0 \end{aligned} \tag{45}$$

$$\Rightarrow \frac{dy}{dX} = \frac{yX}{B_2X^2+B_1X+B_0} \tag{46}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x-A)}{B_2(x-A)^2+B_1(x-A)+B_0} \tag{47}$$

It is seen from (34) and (47) that for a particular curve B_2, B_1 and B_0 are constants, that is, their values do not change with a change of the origin, but the values of b_1 and

b_0 do change with a change of origin.

From (47)

$$y(x-A)dx = [B_2(x-A)^2+B_1(x-A)+B_0]dy$$

$$\Rightarrow \int_{x_1}^{x_2} e^{\eta x} y(x-A)dx = \int_{x_1}^{x_2} e^{\eta x}[B_2(x-A)^2+B_1(x-A)+B_0]dy \qquad (48)$$

If $y = G(x)$ is a solution of (47) such that $G(x\ ) = G(x\ ) = 0$, then (48) becomes:

$$\int_{x_1}^{x_2} e^{\eta x} y(x-A)G(x)dx$$

$$= B_2 \int_{x_1}^{x_2} e^{\eta x}[x^2-2Ax+A^2]G'(x)dx+B_1\int_{x_1}^{x_2} e^{\eta x}(x-A)G'(x)dx$$

$$+ B_0 \int_{x_1}^{x_2} e^{\eta x}G'(x)dx \qquad (49)$$

Noting that

$$\int_{x_1}^{x_2} e^{\eta x}x^2G'(x)dx = e^{\eta x}x^2G(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} G(x)[2xe^{\eta x}+x^2\eta e^{\eta x}]dx,$$

etc., we get from (49), remembering $G(x_1) = G(x_2) = 0$,

$$\int_{x_1}^{x_2} xe^{\eta x}G(x)dx - A\int_{x_1}^{x_2} e^{\eta x}G(x)dx = B_2\Bigg[-2\int_{x_1}^{x_2} xe^{\eta x}G(x)dx$$

$$+ \eta \int_{x_1}^{x_2} x^2 e^{\eta x} G(x)dx + 2A \int_{x_1}^{x_2} e^{\eta x} G(x)dx - A^2 \eta \int_{x_1}^{x_2} e^{\eta x} G(x)dx \Big]$$

$$+ B_1 \left[- \int_{x_1}^{x_2} e^{\eta x} G(x)dx - \eta \int_{x_1}^{x_2} x e^{\eta x} G(x)dx + A\eta \int_{x_1}^{x_2} e^{\eta x} G(x)dx \right]$$

$$- B_0 \eta \int_{x_1}^{x_2} e^{\eta x} G(x)dx \tag{50}$$

Now if we put

$$I = \int_{x_1}^{x_2} e^{\eta x} G(x)dx = e^{f(\eta)} \tag{51}$$

where $$f(\eta) = \lambda_1 \eta + \lambda_2 \frac{\eta^2}{2!} + \lambda_3 \frac{\eta^3}{3!} + \ldots \tag{52}$$

then (50) becomes

$$\frac{dI}{d\eta} - AI = B_2 \left[(\eta - 2) \frac{dI}{d\eta} - \eta \frac{d^2 I}{d\eta^2} + (2A - \eta) I \right]$$

$$+ B_1 \left[A\eta I - \left(I + \eta \frac{dI}{d\eta} \right) \right] - B_0 \eta I \tag{53}$$

Again, by means of (51) and (52), (53) becomes

$$f'(\eta) - A = B_2 [(\eta - 2) f'(\eta) - \eta \{ f'^2(\eta) + f''(\eta) + (2A - \eta)]$$

$$+ B_1 [(A\eta - 1) - f'(\eta)] - B_0 \eta \tag{54}$$

Finally equating coefficients of η^0, η^1, η^2 and η^3 from both sides of (54), we get

$$\lambda_1 - A + B_1 - 2AB_2 + 2B_2\lambda_1 = 0,$$

$$\lambda_2 + B_0 - AB_1 + A^2B_2 + B_1\lambda_1 - 2AB_2\lambda_1 + 3B_2\lambda_2 + B_2\lambda_1^2 = 0, \tag{55}$$

$$\lambda_3 + 2\lambda_2 B_1 - 4AB_2\lambda_2 + 4B_2\lambda_3 + 4B_2\lambda_1\lambda_2 = 0,$$

$$\lambda_4 + 3B_1\lambda_3 - 6AB_2\lambda_3 + 5B_2\lambda_4 + 6B_2\lambda_2^2 + 6B_2\lambda_1\lambda_3 = 0$$

Since we can compute the moments from the raw statistics and the semi-invariants λ_i from the moments, we may regard λ_2, λ_3 and λ_4 in these equations as knowns and the B_0, B_1, B_2, A and λ_1 as unknowns. But the origin has not yet been specified. Let the origin be placed at the mean where $\mu_1 = \lambda_1 = 0$. As $\lambda_2, \lambda_3, \lambda_4, B_0, B_1$ and B_2 are unchanged by a change of origin, we have:

$$B_1 - A_0 - 2A_0B_2 = 0,$$

$$\lambda_2 + B_0 - A_0B_1 + A_0^2B_2 + 3B_2\lambda_2 = 0, \tag{56}$$

$$\lambda_3 + 2B_1\lambda_2 - 4A_0B_2\lambda_2 + 4B_2\lambda_3 = 0,$$

$$\lambda_4 + 3B_1\lambda_3 - 6A_0B_2\lambda_3 + 5B_2\lambda_4 + 6B_2\lambda_2^2 = 0$$

Now define

$$b_0^1 = B_0 - A_0B_1 + A_0^2B_2,$$

$$b_1^1 = B_1 - 2A_0B_2, \tag{57}$$

$$b_2^1 = B_2$$

$\Rightarrow$ from (56)

$$\begin{aligned}
&b_1{}^1 - A_0 = 0, \\
&\lambda_2 + b_0{}^1 + 3b_2{}^1\lambda_2 = 0, \\
&\lambda_3 + 2b_1{}^1\lambda_2 + 4b_2{}^1\lambda_3 = 0, \\
&\lambda_4 + 3b_1{}^1\lambda_3 + 5b_2{}^1\lambda_4 + 6b_2{}^1\lambda_2{}^2 = 0
\end{aligned} \tag{58}$$

Reversed transformation of (57) $\Rightarrow$

$$\begin{aligned}
B_2 &= b_2{}^1, \\
B_1 &= b_1{}^1 + 2A_0 b_2{}^1, \\
B_0 &= b_0{}^1 + A_0(b_1{}^1 + A_0 b_2{}^1).
\end{aligned} \tag{59}$$

The above theory suggests the following procedure for computing the constants of a frequency curve:

First the moments are computed about an arbitrary origin, then the semi-invariants (or alternatively the moments about the mean), then the equations (58) are solved, and then by means of equations (59), the B_2, B_1 and B_0 are computed.

Next we solve the quadratic equation

$$B_2X^2 + B_1X + B_0 = 0$$

The character of the roots of this equation indicates which type to use and it is unnecessary to compute the criterion. The constants of the frequency curve are simple functions of the roots of the above quadratic equation and can be readily found by integrating the differential equation (47), being careful to write the solution as a function of

$$X = x-A.$$

19. A STUDY OF TYPE II CURVES: METHOD OF MAXIMUM LIKELIHOOD

The object of this section is to study the distribution of the estimates of the parameter of location (m) for Pearson's type II curve, estimated by the method of maximum likelihood from small samples.

Before entering into the topic under discussion it will be useful to review some elementary facts regarding the curve. We shall first take the general equation for the curve in the form

$$y = y_0 \left\{ 1 - \frac{(x-m)^2}{a^2} \right\}^p \qquad (60)$$

and determine the effect of variation of the constants. The significance and definition of m will be given later.

(i) $p = 0 \Rightarrow y = y_0$, a straight line.

(ii) $p = 1 \Rightarrow y = y_0 \left\{ 1 - \frac{(x-m)^2}{a^2} \right\}$, a parabola with $y = y_0$ at the point $x = m$, and the intercepts with the axis of x are at the points $x = m \pm a$.

(iii) $p = 2 \Rightarrow y = y_0 \left\{ 1 - \frac{(x-m)^2}{a^2} \right\}^2$, a fourth degree curve in x with double intersection with the axis of x at the points $x = m \pm a$.

And so on.

Since (60) is a probability curve,

$$\int_{m-a}^{m+a} y\,dx = 1$$

$$\Rightarrow y_0 \int_{m-a}^{m+a} \left\{1 - \frac{(x-m)^2}{a^2}\right\}^p dx$$

$$= y_0 \int_{-a}^{a} \left(1 - \frac{z^2}{a^2}\right)^p dx \quad (z = x-m)$$

$$= 2y_0 \int_{0}^{a} \left(1 - \frac{z^2}{a^2}\right)^p dz$$

$$= 2ay_0 \int_{0}^{\pi/2} \cos^{2p+1}\theta\,d\theta \quad (z = a\sin\theta)$$

$$= 1$$

Now, $$\int_{0}^{\pi/2} \cos^n\theta\,d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \tag{61}$$

Combining the last two results, we have

$$2ay_0 \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(p+1)}{\Gamma\left(P + \frac{3}{2}\right)} = 1$$

$$\Rightarrow y_0 = \frac{\Gamma\left(p + \frac{3}{2}\right)}{a\sqrt{\pi}\,\Gamma(p+1)}$$

and hence equation (54) becomes

$$y = \frac{\Gamma(p+3/2)}{a\sqrt{\pi}\Gamma(p+1)} \left[1 - \frac{(x-m)^2}{a^2} \right]^p \tag{62}$$

Next we define the parameters m, a and p in (62) as follows:

m is parameter of location,

a is parameter of scaling,

p is parameter of shape.

Then we consider the likelihood function L together with its first and second partial derivatives with respect to m. We have from (62)

$$L = n \log \frac{\Gamma(p+3/2)}{a\sqrt{\pi}\Gamma(p+1)} + p \sum_{i=1}^{n} \log \left[1 - \frac{(x_i-m)^2}{a^2} \right] \tag{63}$$

and so

$$\frac{\delta L}{\delta m} = 2p \sum_{i=1}^{n} \frac{x_i-m}{a^2-(x_i-m)^2} = 0 \tag{64}$$

$$\frac{\delta^2 L}{\delta m^2} = -2p \sum_{i=1}^{n} \frac{a^2+(x_i-m)^2}{[a^2-(x_i-m)^2]} \tag{65}$$

Now we consider the effect of variation of the parameters a and p upon our estimate $\hat{m}$ of m. From (64) it is easily seen that the estimate $\hat{m}$ will be independent of p, for any particular sample. This is not the case when we consider a, however, for any change in a allows a change in the variance of $\hat{m}$ for the particular sample.

The variance of the distribution is by (62)

$$\delta^2 = \frac{2\Gamma(p+3/2)}{a\sqrt{\pi}\Gamma(p+1)} \int_0^a x^2 \left(1 - \frac{x^2}{a^2}\right)^p dx$$

$$= \frac{2a^2\Gamma(p+3/2)}{\sqrt{\pi}\Gamma(p+1)} \int_0^{\pi/2} [\cos^{2p+1}\theta - \cos^{2p+3}\theta]d\theta$$

$$= \frac{a^2}{2p+3} \qquad \text{[By (61)]}$$

Hence

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = \frac{a^2}{n(2p+3)},$$ $\bar{x}$ and n having usual meanings.

We shall now calculate the limiting form of the variance of $\hat{m}$. Fisher [9] has proved that if the distribution of optimum statistics is normal the variance of an optimum statistic is equal to the negative reciprocal of the mathematical expectation of the second partial derivative of the logarithm of the likelihood with respect to the parameter in question.

We may therefore write

$$-\frac{1}{\sigma_{\hat{m}}^2} = \frac{-4np\Gamma(p+3/2)}{a\sqrt{\pi}\Gamma(p+1)} \int_0^a \frac{(a^2+x^2)}{(a^2-x^2)} \left[1 - \frac{x^2}{a^2}\right]^p dx$$

$$= \frac{-4np\Gamma(p+3/2)}{a\sqrt{\pi}\Gamma(p+1)} \int_0^{\pi/2} (2\cos^{2p-3}\theta - \cos^{2p-1}\theta)d\theta$$

$$[x = a \sin \theta]$$

$$= -\frac{2np\ p + \frac{1}{2}}{a^2(p-1)} \qquad \text{[By (61)]}$$

$$\sigma_{\hat{m}}^{\ 2} = \frac{a^2(p-1)}{np(2p+1)} \tag{66}$$

The efficiency of the mean is then

$$E = \frac{\sigma_{\hat{m}}^{\ 2}}{\sigma_{\bar{x}}^{\ 2}} = \frac{(p-1)(2p+3)}{p(2p+1)} \tag{67}$$

20. OCCURENCE OF PEARSON'S TYPE II DISTRIBUTION IN THOMPSON'S CRITERION FOR THE REJECTION OF OUTLYING OBSERVATIONS

In an interesting paper published in the Annals of Mathematical Statistics, William R. Thompson [29] has suggested a new criterion for the rejection of outlying observations.

If $x_1, x_2, \ldots, x_N$ represent a series of observed values of a variable x, and

$$\bar{x} = \sum_{i=1}^{N} (x_i)/N, \quad s^2 = \sum_{i=1}^{N} (x_i-\bar{x})^2/N$$

then Thompson writes

$$\tau_i = (x_i-\bar{x})/s.$$

He then shows that if x_i is an observation arbitarily selected from a random sample of size N from an infinite normal population, then the elementary probability distribution of τ is

$$p(\tau) = \frac{\Gamma\left(\frac{N-1}{2}\right)}{\sqrt{(N-1)\pi}\,\Gamma\left(\frac{N-2}{2}\right)} \left(1 - \frac{\tau^2}{N-1}\right)^{\frac{N-4}{2}} \qquad (68)$$

This is a symmetrical limited range distribution of Pearson's Type II.

21. DEVELOPMENT OF PEARSON'S TYPE III FUNCTION BY BERNOULLI'S SERIES

Consider the Bernoulli series

$$(q+p)^r = \sum_{x=0}^{r} \binom{r}{x} q^{r-x} p^x \qquad (69)$$

Where p is the probability that an event will happen in a single trial, and $q = 1-p$ is the probability that it will fail to happen.

Representing by

$$y_x = \binom{r}{x} q^{r-x} p^x$$

the ordinate corresponding to x successes $(x = 0,1,2,\ldots r)$, we may plot the $(r+1)$ points (x,y_x). Through these $(r+1)$ points we may imagine a curve that can be represented by an analytic function.

Since

$$y_x = \binom{r}{x} q^{r-x} p^x$$

then $$y_{x+1} = \binom{r}{x+1} q^{r-x-1} p^{x+1}$$

so that $$\frac{y_{x+1}}{y_x} = \frac{rp-px}{qx+q} \tag{70}$$

This is the difference equation of the continuous curve.

Now, from (70)

$$\frac{y_{x+1}-y_x}{y_{x+1}+y_x} = \frac{rp-q-x}{rp+q+(q-p)x} \tag{71}$$

The mean of any two consecutive ordinates $(y_x$ and $y_{x+1})$ will be considered as approximately equal to the ordinate $\left(y_{x+\frac{1}{2}}\right)$ midway between them. The slope of the line joining any two points (x, y_x) and $(x+1, y_{x+1})$ is also approximately equal to the tangent at the point midway between these two on the continuous curve and the error resulting from this approximation would be zero if the curve were a parabola. Under these two assumptions, equation (71) may be written as

$$\frac{D_x y_{x+1/2}}{y_{x+\frac{1}{2}}} = \frac{2(rp-q-x)}{rp+q+(q-p)x} \tag{72}$$

The right side of (72) is the derivative of log y at the point $(x+\frac{1}{2}, y_{x+1/2})$. At any point (x, y_x) this derivative is

$$\frac{d}{dx}(\log y) = \frac{2[rp-q-(x-1/2)]}{rp+q+(q-p)(x-1/2)}$$

so that $$\frac{d}{dx}(\log y) = \frac{\frac{q-p}{2} + (x-rp)}{rpq+\frac{1}{4} + \frac{(x-rp)(q-p)}{2}} \tag{73}$$

Now setting $$\frac{q-p}{\sqrt{rpq}} = \alpha_3,$$

$$\frac{x-rp}{\sqrt{rpq}} = t$$

then by (73),

$$\frac{d}{dt}(\log y) = \frac{-\frac{\alpha_3}{2} + t}{1 + \frac{\alpha_3}{2}t + \frac{1}{4pqr}} \tag{74}$$

Now if rpq is so large that $\frac{1}{4pqr}$ is relatively insignificant and may consequently be neglected, equation (74) becomes

$$\frac{d}{dt}(\log y) = \frac{-\frac{\alpha_3}{2} + t}{1 + \frac{\alpha_3}{2}t} \tag{75}$$

Next, integrating (75) we get

$$y = y_0\left(1 + \frac{\alpha_3 t}{2}\right)^{\frac{4}{\alpha_3^2}-1} e^{-\frac{2}{\alpha_3}t} \tag{76}$$

which is Pearson's Type III frequency curve.

Thus, equation (76) has been obtained without resorting to the method of moments.

We conclude this chapter by showing that the normal curve of error is the limit of Type III curve as the skewness approaches zero.

By (76),

$$\log_e y = \log_e y_0 + \left(\frac{4}{\alpha_3^2} - 1\right) \log_e \left(1 + \frac{\alpha_3}{2} t\right) - \frac{2}{\alpha_3} t$$

$$\Rightarrow \log_e(y/y_0) = \left(\frac{4}{\alpha_3^2} - 1\right)\left(\frac{\alpha_3}{2} t - \frac{1}{2} \frac{\alpha_3^2 t^2}{4} + \frac{1}{3} \cdot \frac{\alpha_3^3 t^3}{8} - \ldots\right) - \frac{2}{\alpha_3} t$$

$$= -\frac{1}{2} t^2 + f(\alpha_3 t)$$

$$\Rightarrow \lim_{\alpha_3 \to 0} \log\left(\frac{y}{y_0}\right) = -\frac{1}{2} t^2$$

$$\Rightarrow y = y_0 e^{-\frac{1}{2} t^2}$$

which is the normal curve of error, t being expressed in standard units.

CHAPTER IV

CLASSICAL ORTHOGONAL POLYNOMIALS ASSOCIATED WITH PEARSON'S DIFFERENTIAL EQUATION

22. INTRODUCTION

In a paper in the Annals of Mathematical Statistics, E.H. Hildebrandt [11] has established the existence of a general system of polynomials $P_n(k,x)$ associated with the solutions of Pearson's Differential Equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0+a_1x}{b_0+b_1x+b_2x^2} \equiv \frac{N}{D} \tag{77}$$

N and D being polynomials in x of degrees not exceeding one and two repsectively with no factor in common.

It is shown that the polynomials $P_n(k,x) \equiv P_n$ themselves satisfy certain differential equations and a recurrence relation. The classical polynomials of Hermite, Tschebycheff, Legendre, Laguerre, and Jacobi are special types of $P_n(k,x)$.

The classical polynomials are employed extensively in statistical theory, and this chapter is devoted to the study of the aforementioned polynomials.

23. A REVIEW OF THE CLASSICAL POLYNOMIALS

Before entering into the main topic of discussion, it will be helpful to have a review of the classical polynomials.

(i) HERMITE POLYNOMIALS $H_n(x)$:

The Hermite polynomials $H_n(x)$ have the following properties

(a) $\frac{dHn(x)}{dx} = 2nH_{n-1}(x)$

(b) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$ (78)

(c) $H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$

(ii) JACOBI POLYNOMIALS $J_n(x,\alpha,\beta)$:

The Jacobi polynomials $J_n(x,\alpha,\beta)$ are defined as follows:

$$J_n(x,\alpha,\beta) = x^{1-\alpha}(1-x)^{1-\beta} \frac{d^n}{dx^n} [x^{n+\alpha-1}(1-x)^{n+\beta-1}] \qquad (79)$$

α,β real.

The nth Jacobi polynomial satisfies the second order differential equation

$$x(1-x)J''_n(x,\alpha,\beta)+[\beta-(\alpha+1)x]J'_n(x,\alpha,\beta)+(\alpha+n)nJ_n(x,\alpha,\beta) = 0 \qquad (80)$$

(iii) LAGUERRE POLYNOMIALS $L_n(x)$:

The Laguerre polynomials $L_n(x)$ are defined by

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}) \qquad (81)$$

The recurrence relation and the differential equation for the Laguerre polynomials $L_n(x)$ are respectively

$$L_{n+1}(x)-(2n+1-x)L_n(x)+n^2L_{n-1}(x) = 0 \tag{82}$$

and

$$L_n'(x)-nL_{n-1}'(x)+nL_{n-1}(x) = 0 \tag{83}$$

(iv) TSCHEBYCHEFF POLYNOMIALS $P_n(n,x)$:

The Tschebycheff polynomials are developed from the differential equation

$$\frac{dy}{dx} = \frac{x}{1-x^2}\,y \tag{84}$$

and the differential equation involving the nth Tschebycheff polynomials $P_n(n,x)$ is

$$(1-x^2)P_n''(n,x)-xP_n'(n,x)+n^2P_n(n,x) = 0 \tag{85}$$

(v) LEGENDRE POLYNOMIALS $P_n(n,x)$:

The Legendre polynomials $P_n(n,x)$ are defined by

$$P_n(n,x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{86}$$

The corresponding differential equation is

$$(1-x^2)y''-2xy'+n(n+1)y = 0 \tag{87}$$

n real. [or $\frac{dy}{dx} = \frac{0}{x^2-1} y$ when expressed in first (88)
order differential equation, as we shall see later]
and the differential equation involving $P_n(n,x)$ is evidently

$$(x^2-1)P_n''(n,x)+2xP_n'(n,x)-n(n+1)P_n(n,x) = 0 \tag{89}$$

24. ROMANOVSKY'S INVESTIGATIONS

Romanovsky [25] gave a study of the relations between some of the Pearson curves and the above mentioned polynomials as outlined below:

(i) The normal curve of error requires the use of Hermite polynomials.

(ii) Type I and Type II (which is a special case of type I) curves involve the Jacobi polynomials.

(iii) Type III curves involve the Laguerre polynomials.

(iv) The generalization of the Type IV curve gives the polynomial

$$P_n(m,x) = (a^2+x^2)^m e^{\nu\theta} \frac{d^n}{dx^n} [(a^2+x^2)^{-m+n} e^{\nu\theta}] \tag{90}$$

where $\theta = \arctan \frac{x}{a}$.

These polynomials have properties similar to the other polynomials as discussed above, that is

$$P_{n+1}(n+1,x) = [2(n+1-m)x-\nu a]P_n(n,x)+2n[n+1-m](a^2+x^2)P_n(n,x) \tag{91}$$

and

$$(a^2+x^2)P_n''(n,x)+[2(1-m)x-\nu a]P_n'(n,x)-n(n+1-2m)P_n(n,x) = 0 \qquad (92)$$

(v) For the Type V curve Romanovsky finds the polynomials

$$P_n(p,x) = x^p e^{-\gamma/x} \frac{d^n}{dx^n} (x^{-p+2n} e^{-\gamma/x})$$

with

$$P_{n+1}(n+1,x) = [(2n+2-p)x+\gamma]P_n(n,x)+n(2n+2-p)x^2P_{n-1}(n,x)$$

$$(93)$$

and $x^2P_n''(n,x)+[x(2-p)+\gamma]P_n'(n,x)-n(n+1-p)P_n(n,x) = 0$

(vi) Finally for the Type VI curve Romanovsky gets the polynomials:

$$P_n(-q_1,q_2,x) = (x-a)^{-q_2}x^{q_1} \frac{d^n}{dx^n} [(x-a)^{q_2+n}x^{-q_1+n}]$$

and the relations

$$P_{n+1}(n+1,x) = [(-q_1+1)(x-a)+(q+1)x]P_n(n,x)+x(x-a)P_n'(n,x)$$

$$x(x-a)P_n''(n,x)+[(-q_1+1)(x-a)+(q_2+1)x]P_n'(n,x)-n(n+1+q_2-q_1)P_n(n,x) = 0 \qquad (94)$$

25. DEFINITION OF THE POLYNOMIALS $P_n(x)$ AND $P_n(k,x)$

The polynomials $P_n(x)$ and $P_n(k,x)$ are defined by Hildebrandt as follows:

$$P_n(x) = \frac{D^n}{y}\frac{d^n y}{dx^n},$$

and (95)

$$P_n(k,x) = \frac{1}{y} D^{n-k} \frac{d^n}{dx^n} D^k y,$$

where y is a non-identically vanishing solution of the differential equation (77), that is

$$\frac{1}{y}\frac{dy}{dx} = \frac{N}{D} \Rightarrow \frac{dy}{dx} = \frac{N}{D} y$$

26. SOME POLYNOMIAL THEOREMS CONCERNING THE SOLUTION OF THE DIFFERENTIAL EQUATION (77)

THEOREM I: If y is a non-identically zero solution of (77), then $P_n(x)$ is a polynomial of degree at most n.

Proof: By Induction.

It is obvious that the theorem holds for $n = 1$, $P_1(x)$ being N, by definition.

Now, from (77)

$$D \frac{dy}{dx} = Ny$$

$$\Rightarrow D \frac{d^2y}{dx^2} + D' \frac{dy}{dx} = N \frac{dy}{dx} + N'y$$

$$\Rightarrow D^2 \frac{d^2y}{dx^2} + DD' \frac{dy}{dx} = ND \frac{dy}{dx} + N'Dy$$

$$\Rightarrow \quad D^2 \frac{d^2y}{dx^2} = (N^2 - ND' + N'D)y \qquad \text{[By (77)]}$$

Now since D' is linear and N' is a constant, obviously the right side of

$$\frac{D^2}{y}\frac{d^2y}{dx^2} = N^2 - ND' + N'D$$

is at most of degree 2.

Assume now that the theorem is true for $m = n$, and we have

$$D^n \frac{d^n y}{dx^n} = P_n(x)y \tag{96}$$

Differentiating once, we get

$$nD^{n-1}D' \frac{d^n y}{dx^n} + D^n \frac{d^{n+1}y}{dx^{n+1}} = P_n(x)\frac{dy}{dx} + \frac{dP_n(x)}{dx}y$$

$$\Rightarrow \quad D^{n+1}\frac{d^{n+1}y}{dx^{n+1}} = DP_n(x)\frac{dy}{dx} - nD^nD' \frac{d^n y}{dx^n} + D\frac{dP_n(x)}{dx}y$$

$\Rightarrow$ By (77) and (96),

$$P_{n+1}(x)y = NP_n(x)y - nD'P_n(x)y + D\frac{dP_n(x)}{dx}y$$

$$\Rightarrow \quad \frac{D^{n+1}}{y}\frac{d^{n+1}y}{dx} = NP_n(x) - nD'P_n(x) + D\frac{dP_n(x)}{dx}$$

So the right side of the above equation is obviously a polynomial of degree at most $n+1$. This proves the theorem.

From above, we have the relation

$$P_{n+1}(x) = P_n(x)(N-nD') + D\frac{dP_n(x)}{dx} \tag{97}$$

Relation (97) enables us to write down the successive polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$, ... etc. as follows:

$$P_1(x) = N,$$

$$P_2(x) = (N-D')P_1(x)+D\frac{dP_1(x)}{dx}$$

$$= N^2-ND'+N'D,$$

$$P_3(x) = (N-2D')P_2(x)+D\frac{dP_2(x)}{dx}$$

$$= N^3-3N^2D'+3NN'D+2ND'^2-2N'D'D-NDD'',$$

and so on.

More generally we have:

THEOREM II: If y is a non-identically zero solution of (77), then $P_n(k,x)$ is a polynomial in x of degree at most n.

PROOF: The proof can be attained exactly in a similar way as adopted in the proof of Theorem I. But the following lemma simplifies the proof in this case.

LEMMA: If y satisfies the differential equation (77), then $D^k y$, where k is any real number, satisfies a differential equation of the same type, namely

$$\frac{d}{dx}(D^k y) = \frac{N+kD'}{D}(D^k y)$$

PROOF: Let $u = D^k y$

$$\Rightarrow \log u = k \log D + \log y$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = k \cdot \frac{1}{D} \cdot D' + \frac{1}{y}\frac{dy}{dx} = \frac{N+kD'}{D} \qquad \text{[By (77)]}$$

Now it follows from the lemma that any result which we derive concerning the polynomials $P_n(x)$ is immediately extensible to the polynomials $P_n(k,x)$, by replacing N by $N+kD'$.

In particular, relation (97) becomes:

$$P_{n+1}(k+1,x) = [N+(k-n+1)D']P_n(k+1,x)+D\frac{dP_n(k+1,x)}{dx} \tag{98}$$

For $k = n$, (98) $\Rightarrow$

$$P_{n+1}(n+1,x) = (N+D')P_n(n+1,x)+D\frac{dP_n(n+1,x)}{dx} \tag{99}$$

27. RECURRENCE RELATIONS INVOLVING $P_{n+1}(x)$, $P_n(x)$ AND $P_{n-1}(x)$ AND THEIR FIRST DERIVATIVES

From (77) $\qquad D\frac{dy}{dx} = Ny$

Taking the nth derivative of both sides by Leibnitz's theorem and noting that $\frac{d^3D}{dx^3} = 0$, we get

$$D \frac{d^{n+1}y}{dx^{n+1}} + nD' \frac{d^n y}{dx^n} + \frac{n(n-1)}{2!} D'' \frac{d^{n-1}y}{dx^{n-1}} = N \frac{d^n y}{dx^n} + nN' \frac{d^{n-1}y}{dx^{n-1}}$$

Multiplying throughout by D^n, we get

$$D^{n+1} \frac{d^{n+1}y}{dx^{n+1}} + D^n(nD'-N) \frac{d^n y}{dx^n} + D^n \left[\frac{n(n-1)}{2} D''-nN'\right] \frac{d^{n-1}y}{dx^{n-1}} = 0$$

By definition, the above formula becomes:

$$P_{n+1}(x)y+(nD'-N)P_n(x)y+n \left[\frac{n-1}{2} D''-N'\right] DP_{n-1}(x)y = 0$$

$$P_{n+1}(x)+(nD'-N)P_n(x)+n \left[\frac{n-1}{2} D''-N'\right] DP_{n-1}(x) = 0 \qquad (100)$$

$\Rightarrow$ By (97),

$$\frac{dP_n(x)}{dx} = n \left(N' - \frac{n-1}{2} D'' \right) P_{n-1}(x) \qquad (101)$$

or, replacing n by $n+1$,

$$\frac{dP_{n+1}(x)}{dx} = (n+1)\left(N' - \frac{n}{2} D''\right)P_n(x)$$

$$= (n+1)(a - nb_2)P_n(x) \quad [\text{By } (77)] \qquad (102)$$

28. DERIVATION OF THE CLASSICAL POLYNOMIALS

(i) HERMITE POLYNOMIALS:

The equation (102) is the generalized form of the one for the Hermite polynomials, namely

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x)$$

Relations (97) and (101) may now be used to obtain a second order differential equation.

Differentiating (97),

$$P'_{n+1}(x)+(nD''-N')P_n(x)+(nD'-N)P'_n(x)-D'P'_n(x)-DP''_n(x) = 0$$

$\Rightarrow$ By (101), using $n+1$ for n,

$$DP''_n(x)+[N-(n-1)D']P'_n(x)-n\left[N' - \frac{(n-1)D''}{2}\right]P_n(x) = 0 \qquad (103)$$

Now it is readily seen that the relation for the Hermite polynomials

$$H''_n(x)-2xH'_n(x)+2nH_n(x) = 0$$

is a special case of (103).

(ii) JACOBI POLYNOMIALS:

By definitions in section 25, and section 23 (ii), it follows that the Jacobi polynomials $J_n(x,\alpha,\beta)$ is a special type of $P_n(k,x)$ with

$$N \equiv (-\beta-\alpha)x+\alpha,$$

$$D \equiv x(1-x),$$

$$n = k+1.$$

Proof: $J_n(x,\alpha,\beta) = x^{1-\alpha}(1-x)^{1-\beta}\frac{d^n}{dx^n}[x^{n+\alpha-1}(1-x)^{n+\beta-1}]$

$$\Rightarrow J_{k+1}(x,\alpha,\beta) = \frac{x(1-x)}{x^\alpha(1-x)^\beta}\frac{d^{k+1}}{dx^{k+1}}[x^\alpha(1-x)^\beta\{x(1-x)\}^k] \qquad (104)$$

$$= \frac{D}{x^\alpha(1-x)^\beta}\frac{d^{k+1}}{dx^{k+1}}[D^k x^\alpha(1-x)^\beta]$$

Now, put

$$y = x^\alpha(1-x)^\beta$$

$$\therefore \frac{dy}{dx} = -x^\alpha\beta(1-x)^{\beta-1}+\alpha x^{\alpha-1}(1-x)^\beta$$

$$= x^\alpha(1-x)^\beta\left[-\frac{\beta}{1-x}+\frac{\alpha}{x}\right]$$

$$= y\left[\frac{(-\beta-\alpha)x+\alpha}{x(1-x)}\right] = y\frac{N}{D}$$

That is $y = x^\alpha(1-x)^\beta$ satisfies (77).

Therefore by (104),

$$J_{k+1}(x,\alpha,\beta) = \frac{D}{y}\frac{d^{k+1}}{dx^{k+1}}(D^{k+1}y)$$

$$= \frac{D^{(k+1)-k}}{y}\frac{d^{k+1}}{dx^{k+1}}(D^{k+1}y)$$

$$= P_{k+1}(k,x) \qquad (105)$$

(iii) TSCHEBYCHEFF AND LEGENDRE POLYNOMIALS:

Using the lemma and replacing N by N+KD', we can write (103) for the polynomials $P_n(k,x)$ and $P_n(n,x)$ as:

$$DP''_n(k,x)+[N-(n-k-1)D']P'_n(k,x)-n\left[N' - \frac{(n-2k-1)}{2}D''\right]P_n(k,x) = 0 \tag{106}$$

and

$$DP''_n(n,x)+(N+D')P'_n(n,x)-n\left[N' + \frac{n+1}{2}D''\right]P_n(n,x) = 0 \tag{107}$$

Now we can easily recognize (92), the last equation of (93), and the last equation of (94) of section 24, and also (80) of section 23 (ii) as special cases of (107).

Some further illustrations of (107) are the Tschebycheff and Legendre polynomials.

For the Tschebycheff polynomials, the differential equation (84) implies (85), [By (107)] and for the Legendre polynomials (86), (87) or (88) implies (89) [By (107)].

(iv) LAGUERRE POLYNOMIALS:

Just as in formula (100) giving a recurrence relation for the polynomials $P_n(x)$, we now obtain the same for the polynomials $P_n(n,x)$.

We have $\frac{d}{dx}(D^{k+1}y) = (k+1)D'(D^k y)+D^{k+1}y'$

$$= [N+(k+1)D']D^k y \qquad \text{[By (97)]}$$

Taking nth derivative of both sides

$$\frac{d^{n+1}}{dx^{n+1}}(D^{k+1}y) = [N+(k+1)D']\frac{d^n}{dx^n}(D^k y)$$

$$+ n[N'+(k+1)D'']\frac{d^{n-1}}{dx^{n-1}}(D^k y)$$

Multiplying both sides by D^{n-k} and writing $D^{n-k}\frac{d^n}{dx^n}(D^k y)$ $= P_n(k,x)y$, by definition, we get

$$P_{n+1}(k+1,x) = [N+(k+1)D']P_n(k,x)+n[N'+(n+1)D'']D\cdot P_{n-1}(k,x) \qquad (108)$$

Letting $k = n$ gives

$$P_{n+1}(n+1,x) = [N+(n+1)D']P_n(n,x)+n[N'+(n+1)D'']D\cdot P_{n-1}(n,x) \qquad (109)$$

a recurrence relation similar to (100).

Equation (109) can be written again in a form similar to (97).

Putting $N+kD'$ for N in (101), we get

$$\frac{dP_n(n,x)}{dx} = n\left[N' + \frac{(n+1)}{2}D''\right]P_{n-1}(n,x)$$

$$\Rightarrow P_{n-1}(n,x) = \frac{1}{n\left[N' + \frac{n+1}{2}D''\right]}\frac{dP_n(n,x)}{dx}$$

substituting in (109), we get

$$P_{n+1}(n+1,x) = [N+(n+1)D']P_n(n,x) + \frac{N'+(n+1)D''}{N' + \frac{(n+1)}{2}D''}\cdot D\cdot \frac{dP_n(n,x)}{dx} \tag{110}$$

Now in looking over the relations existing for the Laguerre polynomials, we show that the relation

$$P_n'(n,x) - nP_{n-1}'(n,x) + nP_{n-1}(n-1,x) = 0$$

for the Laguerre polynomials $P_n(n,x)$ is a special case of another form of formula (110).

To show this, we proceed as follows:

Differentiating (110) gives

$$\frac{dP_{n+1}(n+1,x)}{dx} = [N'+(n+1)D'']P_n(n,x) + [N+(n+1)D']\frac{dP_n(n,x)}{dx}$$

$$+ \frac{N'+(n+1)D''}{N' + \frac{(n+1)}{2}D''}\cdot D'\cdot \frac{dP_n(n,x)}{dx} + \frac{N'+(n+1)D'}{N' + \frac{(n+1)}{2}D''}\cdot D\cdot \frac{d^2P_n(n,x)}{dx^2}$$

Substituting from (107) for $\frac{d^2P_n(n,x)}{dx^2}$ above, we get

$$\frac{dP_{n+1}(n+1,x)}{dx} = [N'+(n+1)D'']P_n(n,x) + [N+(n+1)D']\frac{dP_n(n,x)}{dx}$$

$$+ \frac{N'+(n+1)D''}{N' + \frac{(n+1)}{2}D''}\cdot D'\cdot \frac{dP_n(n,x)}{dx} + \frac{N'+(n+1)D'}{N' + \frac{(n+1)}{2}D''}$$

$$\left[-(N+D')\frac{dP_n(n,x)}{dx} + n\left(\frac{n+1}{2}D''+N'\right)P_n(n,x)\right]$$

$$\Rightarrow \quad \frac{dP_{n+1}(n+1,x)}{dx} = (n+1)\,[N'+(n+1)D'']\,P_n(n,x)$$

$$+ \left[\{N+(n+1)D'\} - \frac{N'+(n+1)D''}{N' + \frac{n+1}{2}D''} \cdot N \right] \frac{dP_n(n,x)}{dx} \qquad (111)$$

Now, bearing in mind that for the Laguerre polynomials the differential equation is of the form

$$\frac{dy}{dx} = \frac{a-x}{x}\,y,$$

we find that substitution of x for D and $(a-x)$ for N reduces (111) to

$$P'_{n+1}(n+1,x) = -(n+1)P_n(n,x)+(n+1)P'_n(n,x)$$

or, $\quad P'_{n+1}(n+1,x)-(n+1)P'_n(n,x)+(n+1)P_n(n,x) = 0$

which is similar to (83).

(v) SOME FURTHER OBSERVATIONS AND CONCLUSIONS:

The first four of the following results were obtained by Frank S. Beale [2] as an extension of Hildebrandt's results:

1. $P_{n+1}(k,x) = [N+(k-n)D']P_n(k,x)+DP'_n(k,x)$

2. $P'_{n+1}(k,x) = (n+1)\left[N' + \frac{2k-n}{2}D''\right]P_n(k,x)$

3. $P_{n+1}(k,x) = [N+(k-n)D']P_n(k,x)+n\left[N' + \frac{2k-n+1}{2}D''\right]DP_{n-1}(k,x)$

4. $P^{q}_{n+1} = \prod_{i=0}^{q-1} (n+1-i)\left[N' + \frac{2k-n+i}{2} D''\right] P_{n-q+1}(k,x)$, $q = 1,2,\ldots(n+1)$

5. $DP''_n(k,x) + [N+(k-n+1)D']P'_n(k,x) - n\left[N' - \frac{2k-n+1}{2} D''\right] P_n(k,x) = 0$

6. $$P'_{n+1}(n+1,x) = (n+1)[N'+(n+1)D'']P_n(n,x) + \left\{[N+(n+1)D'] - \frac{N'+(n+1)D''}{N' + \frac{(n+1)}{2} D''} \cdot N\right\} P'_n(n,x)$$

Comparing $P_n(k,x)$ with the Rodrigues' formula representation of the classical orthogonal polynomials for $n = k+1$, we may determine expressions for N and D in equation (77) for all the classical orthogonal polynomials. This enables us to associate a first order differential equation of the form (77) with the classical orthogonal polynomials. Then the result 5 above leads us to derive the well-known and time-honoured second order differential equations of the same type. The recurrence relations, pure and involving first derivatives, may be obtained by means of results 1,2,3 and 6 above. The results 1-4 were used by Frank S. Beale to derive some interesting theorems concerning common zeros of $P_n(k,x)$ and D. In particular, result 4 above gives the points of inflexion for each polynomial concerned, and result 6 leads us to derive the relation (83) for the Laguerre polynomials as already shown in the preceding section.

To conclude this chapter, it can be safely said that the well-known, widely studied and the time-honoured classical orthogonal polynomials may be studied in a simpler and more systematic way by means of the study as outlined in this chapter. Since the classical polynomials are employed extensively in statistical theory, the significance of the study in this chapter is obvious.

CHAPTER V

EXTENSIONS OF THE PEARSON SYSTEM OF FREQUENCY CURVES

29. INTRODUCTION AND SUMMARY

The differential equation

$$\frac{d \log f(x)}{dx} = \frac{\gamma_0+\gamma_1 x+\gamma_2 x}{\delta_0 x+\delta_1 x^2+\delta_2 x^3} \tag{112}$$

where $\gamma_0,\gamma_1,\gamma_2,\delta_0,\delta_1$ and δ_2 are real numbers, and its solution in the form

$$f(x) = cx^{r_1}(a_1+a_2x)^{r_2}(b_1+b_2x)^{r_3} \tag{113}$$

where $a,a_1,a_2,b_1,b_2,r_1,r_2$ and r_3 are real parameters, are fundamental in the discussion of this chapter.

First we show that the following distributions (i) Uniform, (ii) Normal, (iii) Exponential, (iv) Gamma, (v) Cauchy, (vi) Student's t (vii) χ^2, (viii) χ, (ix) Rayleigh, (x) Maxwell, (xi) F, (xii) Beta, (xiii) Inverse Gaussian and (xiv) Pareto, are special cases of $f(x)$ in (113) with suitable values of the parameters of $f(x)$.

Then we show that the differential equation (112) with the solution (113), or, a function as given by (113) leading to a differential equation of the form (112) gives rise to a Pearson curve under certain specified conditions. We shall specify these conditions and use the differential equation (112) to derive five curves whose parameters depend on the first seven population moments. The Pearson curves are shown to be solutions of a special case of the differential equation (112). Another extension of the Pearson's Differential Equation (1a) in the form

$$\frac{d \log f(x)}{dx} = \frac{x-a}{b_0+b_1x+b_2x^2+b_3x^3} \tag{112a}$$

and the curves associated with it are also discussed.

30. SOLUTION OF THE DIFFERENTIAL EQUATION (112) IN THE FORM (113)

To begin with our present topic of discussion, we first derive the solution of the differential equation (112) in the form (113).

To do this we proceed as follows:

$$\text{R.H.S. of (112)} = \frac{1}{x}\left(\frac{\gamma_0+\gamma_1x+\gamma_2x}{\delta_0+\delta_1x+\delta_2x^2}\right)$$

$$= \frac{1}{x}\left[1 + \frac{Ax+B}{(\alpha_0+\alpha_1x)(\beta_0+\beta_1x)}\right], [\delta_0+\delta_1x+\delta_2x^2 = (\alpha_0+\alpha_1x)(\beta_0+\beta_1x)]$$

$$= \frac{1}{x}\left[1 + \frac{A_1}{\alpha_0+\alpha_1x} + \frac{B_1}{\beta_0+\beta_1x}\right]$$

$$= \frac{1}{x} + \frac{A_1}{x(\alpha_0+\alpha_1x)} + \frac{B_1}{x(\beta_0+\beta_1x)}$$

$$= \frac{1}{x} + \frac{A_2}{x} + \frac{A_3}{\alpha_0+\alpha_1x} + \frac{B_2}{x} + \frac{B_3}{\beta_0+\beta_1x}$$

$$= \frac{C_1}{x} + \frac{C\alpha_1}{\alpha_0+\alpha_1x} + \frac{D\beta_1}{\beta_0+\beta_1x},$$

choosing the constants suitably.

Integration of (112) gives:

$$\log f(x) = C_1 \log x + C \log(\alpha_0+\alpha_1 x)+D \log(\beta_0+\beta_1 x)+\log E$$
$$\Rightarrow f(x) = Ex^{C_1}(\alpha_0+\alpha_1 x)^C(\beta_0+\beta_1 x)^D$$

Thus we see that the differential equation (112) has its solution of the form (113). For our present disucssion, we would use the form of $f(x)$ as given by (113).

31. IMPORTANT CONTINUOUS PROBABILITY LAWS

We shall see that by giving particular values to the parameters of $f(x)$ in (113) we are able to tobain some important continuous frequency functions that are evidently special cases of $f(x)$ in (113). But before going to do this, a brief account of these continuous frequency functions will be helpful towards our present purpose.

(i) THE UNIFORM PROBABILITY LAW:

The uniform probability law over the interval a to b, where a and b are any finite real numbers such that $a < b$, is specified by the probability density function

$$f(x) = \frac{1}{b-a} \quad \text{for } a < x < b$$
$$= 0 \quad \text{otherwise}$$

(ii) THE NORMAL PROBABILITY LAW:

The normal probability law with parameters m and σ where $-\infty < m < \infty$ and $\sigma > 0$, is specified by the probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-m}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

(iii) THE EXPONENTIAL PROBABILITY LAW:

The exponential probability law with paramter λ, in which $\lambda > 0$, is specified by the probability density function

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x > 0$$

$$= 0 \quad \text{otherwise}$$

(iv) THE GAMMA PROBABILITY LAW:

The gamma probability law with parameters r and λ, in which $r = 1,2,\ldots$ and $\lambda > 0$, is specified by the probability density function

$$f(x) = \frac{\lambda}{(r-1)!} (\lambda x)^{r-1} e^{-\lambda x} \quad \text{for} \quad x \geq 0$$

$$= 0 \quad \text{otherwise}$$

Note: (iii) is a special case of (iv) with $r = 1$.

(v) THE CAUCHY PROBABILITY LAW:

The Cauchy probability law with parameters α and β, in which $-\infty < \alpha < \infty$ and $\beta > 0$, is specified by the probability density function

$$f(x) = \frac{1}{\pi\beta\left\{1 + \left(\frac{x-\alpha}{\beta}\right)^2\right\}}, \quad -\infty < x < \infty.$$

(vi) STUDENT'S DISTRIBUTION:

Student's distribution with parameter $n = 1,2,\ldots$ (also called Student's t-distribution with n degrees of freedom) is specified by the probability density function

$$f(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$$

[Note: Student's distribution with parameter $n = 1$ coincides with the Cauchy probability law with $\alpha = 0$ and $\beta = 1$.]

(vii) THE χ^2-DISTRIBUTION:

The χ^2-distribution with parameters $n = 1,2,\ldots$ and $\delta > 0$ is specified by the probability density function

$$f(x) = \frac{1}{2^{n/2}\sigma^n\Gamma(n/2)} x^{(n/2)-1} e^{-(x/2\sigma^2)} \quad \text{for } x > 0$$

$$= 0 \quad \text{for } x < 0$$

[Note: The χ^2-distribution with parameters n and $\sigma = 1$ is called the χ^2-distribution with n degrees of freedom. The χ^2-distribution with parameters n and σ coincides with the gamma distribution with parameters $r = n/2$ and $\lambda = 1/2\sigma^2$. To define the gamma probability law for non-integer r, we replace $(r-1)!$ by $\Gamma(r)$.]

(viii) THE χ-DISTRIBUTION:

The χ-distribution with parameters $n = 1,2,\ldots$ and $\sigma > 0$ is specified by the probability density function

$$f(x) = \frac{2(n/2)^{n/2}}{\sigma^n \Gamma(n/2)} x^{n-1} e^{-(n/2\sigma^2)x^2} \quad \text{for } x > 0$$

$$= 0 \quad \text{for } x < 0$$

[Note: The χ-distribution with parameters n and $\sigma = 1$ is often called the χ-distribution with n degrees of freedom. The relation between the χ^2 and χ distributions is given by the following: If X has a χ^2 distribution with parameters n and σ, then $Y = \sqrt{X/n}$ has a χ-distribution with parameters n and σ.]

(ix) THE RAYLEIGH DISTRIBUTION:

The Raleigh distribution with parameter $\alpha > 0$ is specified by the probability density function

$$f(x) = \frac{1}{\alpha^2} x e^{-\frac{1}{2}(x/\alpha)^2} \qquad \text{for} \quad x > 0$$

$$= 0 \qquad \text{for} \quad x < 0$$

[Note: The Rayleigh distribution coincides with the χ-distribution with parameters $n = 2$ and $\sigma = \alpha\sqrt{2}$.]

(x) THE MAXWELL DISTRIBUTION:

The Maxwell distribution with parameter $\alpha > 0$ is specified by the probability density function

$$f(x) = \frac{4}{\sqrt{\pi}} \frac{1}{\alpha^3} x^2 e^{-x^2/\alpha^2} \qquad \text{for} \quad x > 0$$

$$= 0 \qquad \text{for} \quad x < 0$$

[Note: The Maxwell distribution with parameter α coincides with the χ-distribution with parameters $n = 3$ and $\sigma = \alpha\sqrt{3/2}$.]

(xi) THE F-DISTRIBUTION:

The F-distribution with parameters $m = 1,2,\ldots$ and $n = 1,2,\ldots$ is specified by the p.d.f.

$$f(x) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)} (m/n)^{m/2} \frac{x^{(m/2)-1}}{[1+(m/n)x]^{(m+n)/2}} \qquad \text{for} \quad x > 0$$

$$= 0 \qquad \text{for} \quad x < 0$$

(xii) THE BETA PROBABILITY LAW:

The beta probability law with parameters a and b, in which a and b are posotive real numbers, is specified by the p.d.f.

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \qquad 0 < x < 1$$

$$= 0 \qquad \text{otherwise}$$

where $B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}dy$

(xiii) INVERSE GAUSSIAN DISTRIBUTION:

The family of probability density functions

$$f(x) = [\lambda/2\pi x^3]^{1/2}\exp[-\lambda(x-\mu)^2/2\mu^2 x], \quad \mu, \ \lambda > 0$$

for a variate x and parameters μ and λ, with x,μ,λ each confined to $(0,\infty)$ is called the Inverse Gaussian distribution. The expectation of x is μ, while λ is a measure of relative precision.

(xiv) PARETO'S DISTRIBUTION:

In certain kinds of economic statistics, we often meet truncated distributions. Thus for example in income statistics the data supplied are usually concerned with the distribution of the incomes of persons whose income exceeds a certain limit x_0

fixed by taxation rules. This distribution, and certain analogous distributions of property values, sometimes agree approximately with the Pareto distribution defined by the relation

$$P(X > x) = \left(\frac{x_0}{x}\right)^{\alpha}, \quad (x > x_0,\ \alpha > 0)$$

The frequency function of this distribution is

$$f(x) = \frac{\alpha}{x_0}\left(\frac{x_0}{x}\right)^{\alpha+1} \qquad \text{for } x > x_0$$

$$= 0 \qquad \text{for } x \leq x_0$$

The mean is finite for $\alpha > 1$, and is then equal to $\frac{\alpha}{\alpha-1}x_0$. The median of the distribution is $2^{\frac{1}{\alpha}}\ x_0$.

32. DEDUCTIONS OF THE PROBABILITY LAWS OF SECTION 31 AS SPECIAL CASES OF f(x) AS GIVEN BY (113)

(i) To deduce the Uniform Distribution with parameters $a < x < b$, we take

$$C = \frac{1}{b-a}, \quad r_1 = r_2 = r_3 = 0,$$

and the result is obvious.

(ii) To get the Normal Distribution with parameters $-\infty < m < \infty$ and $\sigma > 0$, we first take

$$C = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-m^2/2\sigma^2}, \quad r_1 = 0,\ r_2 = r_3 = r$$

Then we get from (113),

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-m^2/2\sigma^2} \{(a_1+a_2x)(b_1+b_2x)\}^r$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-m^2/2\sigma^2} \{a_1b_1+(a_1b_2+a_2b_1)x+a_2b_2x^2\}^r$$

Next we take

$$a_1b_1 = 1, \ a_1b_2+a_2b_1 = \frac{m}{r\sigma^2}, \quad a_2b_2 = -\frac{1}{r\cdot 2\sigma^2}$$

Thus $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-m^2/2\sigma^2}\left(1 + \frac{mx}{r\sigma^2} - \frac{1}{2r\sigma^2}x^2\right)^r$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-m^2/2\sigma^2}\left[1 + \frac{1}{r}\left(\frac{mx}{\sigma^2} - \frac{x^2}{2\sigma^2}\right)\right]^r$$

Now letting $r \to \infty$, we get ultimately

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-m^2/2\sigma^2} e^{\frac{mx}{\sigma^2} - \frac{x^2}{2\sigma^2}} \quad \left[\text{Since } \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x\right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

which is the result as desired for $-\infty < x < \infty$.

(iii) For the Exponential Probability Law with parameter $\lambda > 0$, we take

$$C = \lambda, \ r_1 = 0 = r_2, \ b_1 = 1, \ b_2 = -\frac{\lambda}{r_3}$$

We then have

$$f(x) = \lambda\left(1 - \frac{\lambda x}{r_3}\right)^{r_3}$$

Now letting $r_3 \to \infty$, we get

$$f(x) = \lambda e^{-\lambda x}$$

which is the required result for $x > 0$.

(iv) For the Gamma Probability Law with parameters $r = 1,2,\ldots$ and $\lambda > 0$, we take

$$C = \frac{\lambda^r}{(r-1)!}, \quad r_1 = r-1,\ r_2 = 0,\ b_1 = 1,\ b_2 = -\frac{\lambda}{r_3}$$

Then

$$f(x) = \frac{\lambda}{(r-1)!}(\lambda x)^{r-1}\left(1 - \frac{\lambda x}{r_3}\right)^{r_3}$$

Letting $r_3 \to \infty$, we get the required result as

$$f(x) = \frac{\lambda}{(r-1)!}(\lambda x)^{r-1}e^{-\lambda x} \qquad \text{for } x > 0.$$

(v) To deduce the Cauchy Probability Law with parameters $-\infty < \alpha < \infty$ and $\beta > 0$, we take first

$$C = \frac{1}{\pi\beta}, \quad r_1 = 0,\ r_2 = r_3 = -1$$

whence we get

$$f(x) = \frac{1}{\pi\beta} \cdot \frac{1}{(a_1+a_2x)(b_1+b_2x)}$$

$$= \frac{1}{\pi\beta} \cdot \frac{1}{[a_1b_1+(a_1b_2+a_2b_1)x+a_2b_2x^2]}$$

Next we take

$$a_1b_1 = 1 + \frac{\alpha^2}{\beta^2}, \quad a_1b_2+a_2b_1 = -2\alpha/\beta^2, \quad a_2b_2 = 1/\beta^2$$

Then we have

$$f(x) = \frac{1}{\pi\beta} \cdot \frac{1}{\left\{1 + \left(\frac{x-\alpha}{\beta}\right)\right\}^2}$$

which is the result for $-\infty < x < \infty$.

(vi) To get Student's Distribution with n degrees of freedom, we take first

$$C = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}, \quad r_1 = 0,\ r_2 = r_3 = -(n+1)/2$$

We get

$$f(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \quad [a_1b_1+(a_1b_2+a_2b_1)x+a_2b_2x^2]^{-(n+1)/2}$$

Next we take

$$a_1b_1 = 1,\ a_1b_2+a_2b_1 = 0,\ a_2b_2 = \frac{1}{x}$$

whence

$$f(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$$

which is the desired result for $-\infty < x < \infty$

(vii) For the deduction of the χ^2-distribution with parameters $n = 1,2,\ldots$ and $\sigma > 0$, we take

$$C = \frac{1}{2^{n/2}\sigma^n\Gamma(n/2)}, \quad r_1 = \left(\frac{n}{2}\right) - 1, \ r_2 = 0, \ b_1 = 1,$$

$$b_2 = -\frac{1}{2\sigma^2 r_3}$$

so that we have in this case

$$f(x) = \frac{1}{2^{n/2}\sigma^n\Gamma(n/2)} x^{\left(\frac{n}{2}\right) - 1} \left(1 - \frac{1}{r_3 \cdot 2\sigma^2} x\right)^{r_3}$$

Letting $r_3 \to \infty$ we get the required result as

$$f(x) = \frac{1}{2^{n/2}\sigma^n\Gamma(n/2)} x^{\left(\frac{x}{2}\right) - 1} e^{-(x/2\sigma^2)} \qquad \text{for} \quad x > 0$$

(viii) To get the χ-distribution with parameters $n = 1,2,\ldots$ and $\sigma < 0$, the following substitutions are obvious at the first instance:

$$C = \frac{2\left(\frac{n}{2}\right)^{n/2}}{\sigma^n\Gamma(n/2)}, \quad r_1 = n-1, \ r_2 = r_3 = r$$

so that we have

$$f(x) = \frac{2(n/2)^{n/2}}{\sigma^n \Gamma(n/2)} x^{n-1} [a_1b_1+(a_1b_2+a_2b_1)x+a_2b_2x^2]^r$$

Then we put

$$a_1b_1 = 1, \ a_1b_2+a_2b_1 = 0, \ a_2b_2 = -\frac{n}{2\sigma^2 r}$$

and accordingly we get

$$f(x) = \frac{2(n/x)^{n/2}}{\sigma^n \Gamma(n/2)} x^{n-1} \left(1 - \frac{nx^2}{2\sigma^2 r}\right)^r$$

Now letting $r \to \infty$, we get finally

$$f(x) = \frac{2(n/2)^{n/2}}{\sigma^n \Gamma(n/2)} x^{n-1} e^{-(n/2\sigma^2)x^2}$$

which is the required result for $x > 0$.

(ix) For the Rayleigh Distribution with parameter $\alpha > 0$, the obvious substitutions are

$$C = \frac{1}{\alpha^2}, \ r_1 = 1, \ r_2 = r_3 = r$$

whence we get

$$f(x) = \frac{1}{\alpha^2} x \ [a_1b_1+(a_1b_2+a_2b_1)x+a_2b_2x^2]^r$$

Next we take

$$a_1b_1 = 1, \ a_1b_2+a_2b_1 = 0, \ a_2b_2 = -\frac{1}{2\alpha^2 r},$$ and accordingly

we get

$$f(x) = \frac{1}{\alpha^2} x \left(1 - \frac{x^2}{2\alpha^2 r}\right)^r$$

Not letting $r \to \infty$ we have

$$f(x) = \frac{x}{\alpha^2} e^{-\frac{1}{2}(x/\alpha)^2}$$

which is the required result for $x > 0$.

(x) To get the Maxwell Distribution with parameter $\alpha > 0$, we have

$$C = \frac{4}{\sqrt{\pi}} \cdot \frac{1}{\alpha^3}, \ r_1 = 2, \ r_2 = 0, \ b_1 = 1, \ b_2 = -\frac{1}{\alpha^2 r_3}$$

whence

$$f(x) = \frac{4}{\sqrt{\pi\alpha^3}} x^2 \left(1 - \frac{1}{\alpha^2 r_3} x^2\right)^{r_3}$$

Letting $r_3 \to \infty$, we get

$$f(x) = \frac{4}{\sqrt{\pi\alpha^3}} x^2 e^{-x^2/\alpha^2}$$

which is the required distribution for $x > 0$.

(xi) To deduce the F-distribution with parameters $m = 1,2,...$ and $n = 1,2,...,$ we take

$$C = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} (m/n)^{m/2}, \quad r_1 = \left(\frac{m}{2}\right) - 1, \ r_2 = 0, \ b_1 = 1,$$

$$b_2 = m/n, \ r_3 = -\frac{m+n}{2}$$

and we get

$$f(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} (m/n)^{m/2} \frac{x^{\left(\frac{m}{2}\right)-1}}{\left(1 + \frac{m}{n} x^2\right)^{(m+n)/2}}$$

which is the required result for $x > 0$.

(xii) For the Beta Distribution with parameters $a,b > 0$, we take

$C = \frac{1}{B(a,b)}$ where $B(a,b)$ is already defined in section 31,

$r_1 = a-1, \ r_2 = 0, \ b_1 = 1 = -b_2, \ r_3 = b-1$

and we get

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$$

which is the result as desired for $0 < x < 1$.

(xiii) To deduce the Inverse Gaussian Distribution with parameters $\lambda > 0$, we take

$$C = \sqrt{\frac{\lambda}{2\pi}} e^{\lambda/\mu}, \ r_1+r_2 = -3/2$$

Thus we get

$$f(x) = \sqrt{\frac{\lambda}{2\pi}} e^{\lambda/\mu} x^{r_1+r_2}\left(\frac{a_1}{x} + a_2\right)^{r_2} (b_1+b_2x)^{r_3}$$

Then we put

$$r_2 = r_3 = r,$$

whence we get

$$f(x) = \sqrt{\frac{\lambda}{2\lambda}}\, e^{\lambda/\mu} x^{-3/2} \left[(a_1 b_2 + a_2 b_1) + a_2 b_2 x + \frac{a_1 b_1}{x}\right]^r$$

Finally, we put $a_1 b_2 + a_2 b_1 = 1$, $a_2 b_2 = -\frac{\lambda}{2\mu^2 r}$, $a_1 b_1 = -\frac{\lambda}{2r}$,

and accordingly we get

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{\lambda/\mu} \left[1 - \frac{1}{r}\left(\frac{\lambda x}{2\mu^2} + \frac{\lambda}{2x}\right)\right]^r$$

Letting $r \to \infty$, we have ultimately

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{\lambda/\mu} . e^{-\left(\frac{\lambda x}{2\mu^2} + \frac{\lambda}{2x}\right)}$$

$$= \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{-\frac{\lambda (x-\mu)^2}{2\mu^2 x}}$$

which is the required result for $x > 0$.

(xiv) Lastly, to deduce the Pareto Distribution with parameter $\alpha > 0$, we take

$$C = \alpha x_0^{\alpha},\ r_1 = -(\alpha+1),\ r_2 = r_3 = 0$$

and we get

$$f(x) = \frac{\alpha}{x_0}\left(\frac{x_0}{x}\right)^{\alpha+1}$$

which is the required result for $x > x_0$.

33. OBSERVATIONS ON THE PREVIOUS RESULTS

For the Inverse Gaussian Distribution, it is interesting to take note of the fact that all the real parameters C, a_1, a_2, b_1, b_2, r_1, r_2 and r_3 of $f(x)$ in (113) have non-zero values. The same is the case for the χ, Rayleigh and Maxwell Distributions. In all other distributions as discussed, at least one of the parameters of $f(x)$ in (113) is zero.

In this context, it would be quite relevant to indicate that the relationship between the Inverse Gaussian Distribution and the Pearson Type VI Distribution was first studied by L.K. Roy in his M.Sc. thesis in Queen's University [26].

34. DIRECT DEDUCTIONS OF ALL THE PEARSON CURVES FROM f(x) IN (113)

(i) Type I:

$$y = y_0\left(1 + \frac{x}{a_1}\right)^{m_1}\left(1 - \frac{x}{a_2}\right)^{m_2} \quad \text{where} \quad \frac{m_1}{a_1} = \frac{m_2}{a_2}$$

Substitutions are: $C = y_0$, $r_1 = 0$, $a = 1$, $a_2 = \frac{1}{a_1}$, $r_2 = m_1$, $b_1 = 1$, $b_2 = -\frac{1}{a_2}$, $r_3 = m_2$.

(ii) Type II:

$$y = y_0\left(1 - \frac{x^2}{a^2}\right)^m$$

Obvious substitutions are $C = y_0$, $r_1 = 0$, $r_2 = r_3 = m$, $a_1 = b_1 = 1$,

$a_2 = \frac{1}{a} = -b_2.$

(iii) Type III:

$$y = y_0 e^{-\gamma x}\left(1 + \frac{x}{a}\right)^{\gamma a}$$

The substitutions at the first instance are

$$c = y_0,\ r_1 = 0,\ a_1 = 1,\ a_2 = -\frac{\gamma}{r_2},\ b_1 = 1,\ b_2 = \frac{1}{a},\ r_3 = \gamma a,$$

whence

$$f(x) = y_0\left(1 - \frac{\gamma x}{r_2}\right)^{r_2}\left(1 + \frac{x}{a}\right)^{\gamma a}$$

Now letting $r_2 \to \infty$, we get the result.

(iv) Type IV:

$$y = y_0\left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-\nu \tan^{-1}\frac{x}{a}}$$

First we take $c = y_0'$, $r_1 = 0$, $a_1 = -k_1$, $a_2 = 1$, $b_2 = -k_2$, $b_2 = 1$

Then $f(x) = y_0'(x-k_1)^{r_2}(x-k_2)^{r_3}$

Next we take $k_1 = -p+iq$, $k_2 = -p-iq$ $(p,q$ real)

$$r_2 = \frac{\nu i}{2} - r,\ r_3 = -\frac{\nu i}{2} - r \quad (\nu, r \text{ real})$$

$$\therefore f(x) = y_0'[\{(x+p)-iq\}^{\frac{\nu i}{2} - r}\{(x+p)+iq\}^{-\frac{\nu i}{2} - r}$$

$$= y_0'\left(\frac{x+p-iq}{x+p+iq}\right)^{\frac{\nu i}{2}} \times [(x+p)^2+q^2]^{-r}$$

Then we put

$$x+p = R\cos\theta$$
$$q = R\sin\theta$$

so that

$$\left(\frac{x+p-iq}{x+p+iq}\right)^{\frac{\nu i}{2}} = \left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right)^{\frac{\nu i}{2}} = \left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^{\frac{\nu i}{2}}$$
$$= e^{\nu\theta}$$

and

$$\cot\theta = \frac{x+p}{q} \Rightarrow \tan\left(\frac{\pi}{2} - \theta\right) = \frac{x+p}{q} \Rightarrow \frac{\pi}{2} - \theta = \tan^{-1}\frac{x+p}{q}$$

$$\Rightarrow \frac{\nu\pi}{2} - \nu\theta = \nu\tan^{-1}\frac{x+p}{q} \Rightarrow \nu\theta = \frac{\nu\pi}{2} - \nu\tan^{-1}\frac{x+p}{q}$$

Thus from above, we have

$$f(x) = y_0' e^{\frac{\nu\pi}{2}} [(x+p)^2+q^2]^{-r} e^{-\tan^{-1}\frac{x+p}{q}}$$
$$= y_0' e^{\frac{\nu\pi}{2}} q^{-2r}\left[1 + \left(\frac{x+p}{q}\right)^2\right]^{-r} e^{-\tan^{-1}\frac{x+p}{q}}$$

Ultimately, putting

$$y_0 = y_0' e^{\frac{\nu\pi}{2}} q^{-2r},\ p = 0,\ q = a,\ r = m$$

we get the result.

[Note: For type IV curve, the above written complex substitutions are justified with reference to our discussion in Chapter II.]

(v) Type V:

$$y = y_0 x^{-p} e^{-\gamma/x}$$

We write,

$$f(x) = Cx^{r_1}.x^{r_2}\left(\frac{a_1}{x} + a_2\right)^{r_2}(b_1+b_2x)^{r_3}$$

Then we take $C = y_0$, $r_1+r_2 = -p$, $r_2 = r_3 = r$ so that

$$f(x) = y_0 x^{-p}\left[(a_1b_2+a_2b_1) + \frac{a_1b_1}{x} + a_2b_2x\right]^r$$

Next taking $a_1b_2+a_2b_1 = 1$, $a_1b_1 = -\frac{\gamma}{r}$, $a_2b_2 = 0$ and letting $r \to \infty$, we get the result.

(vi) Type VI :

$$y = y_0(x-a)^{q_2}x^{-q_1}$$

The obvious substitutions are $C = y_0$, $r_1 = -q_1$, $r_2 = 0$, $b_1 = -a$, $b_2 = 1$, $r_3 = q_2$.

(vii) Type VII :

$$y = y_0 e^{-x^2/2\sigma^2}$$

The substitutions at the first instance are

$$C = y_0, \; r_1 = 0, \; r_2 = r_3 = r$$

$$\therefore \; f(x) = y_0[a_1b_1+(a_1b_2+a_2b_1)x+a_2b_2x^2]^r$$

Then we take $a_1b_1 = 1$, $a_1b_2+a_2b_1 = 0$, $a_2b_2 = -\frac{1}{2\sigma^2 r}$. Now letting $r \to \infty$, we get the result.

(viii) Type VIII:

$$y = y_0\left(1 + \frac{x}{a}\right)^{-m}$$

The abvious substitutions are: $C = y_0$, $r_1 = r_2 = 0$, $b_1 = 1$, $b_2 = \frac{1}{a}$, $r_3 = -m$.

(ix) Type IX:

$$y = y_0\left(1 + \frac{x}{a}\right)^{m}$$

The substitutions are immediate.

(x) Type X:

$$y = \frac{n}{\sigma}\, e^{\pm \frac{x}{\sigma}}$$

The substitutions are $C = \frac{n}{\sigma}$, $r_1 = r_2 = 0$, $b_1 = 1$, $b_2 = \pm \frac{1}{\sigma r_3}$. Next, letting $r_3 \to \infty$, we get the result.

(xi) Type XI:

$$y = y_0 x^{-m}$$

The substitutions are obvious.

(xii) Type XII:

$$y = y_0\left(\frac{a_1+x}{a_2-x}\right)^{p}$$

The obvious substitutions are

$$C = y_0,\ a_1 = \alpha_1,\ a_2 = 1 = -b_2,\ b_1 = \alpha_2,\ r_2 = p = -r_3$$

35. AN EXTENSION OF PEARSON'S SYSTEM

We have in (113)

$$f(x) = Cx^{r_1}(a_1+a_2x)^{r_2}(b_1+b_2x)^{r_3}$$

whence $\log f(x) = \log C + r_1 \log x + r_2 \log(a_1+a_2x)+r_3 \log(b_1+b_2x)$

$$\frac{d \log f(x)}{dx} = \frac{r_1}{x} + \frac{r_2a_2}{a_1+a_2x} + \frac{r_3b_2}{b_1+b_2x}$$

$$= \frac{a_1b_1r_1+(a_1b_2r_1+a_2b_1r_1+a_2b_1r_2+a_1b_2r_3)x+(a_2b_2r_1+a_2b_2r_2+a_2b_2r_3)x^2}{a_1b_1x+(a_1b_2+a_2b_1)x^2+a_2b_2x^3}$$

Thus we can easily see that (113) can be expressed in the form

$$\frac{1}{f}\frac{df}{dx} = \frac{\gamma_0+\gamma_1x+\gamma_2x^2}{\delta_0x+\delta_1x^2+\delta_2x^3}$$

which coincides with (112).

Now we have, from above,

$$(\delta_0x+\delta_1x^2+\delta_2x^3)df = (\gamma_0+\gamma_1x+\gamma_2x^2)fdf$$

$$\Rightarrow \int x^n(\delta_0x+\delta_1x^2+\delta_2x^3)df = \int x^n(\gamma_0+\gamma_1x+\gamma_2x^2)fdf$$

$$\Rightarrow x^n(\delta_0x+\delta_1x^2+\delta_2x^3)f - \int f[\delta_0(n+1)x^n+\delta_1(n+2)x^{n+1}$$

$$+\delta_2(n+3)x^{n+2}]dx = \int x^n(\gamma_0+\gamma_1x+\gamma_2x^2)dx \qquad (114)$$

which is obtained just integrating the L.H.S. by parts.

Now set $\mu_n' = \int x^n f\,dx$ so that from (114), we get

$$x^n(\delta_0 x+\delta_1 x^2+\delta_2 x^3)f-(n+1)\delta_0\mu_n'-(n+2)\delta_1\mu_{n+1}'-(n+3)\delta_2\mu_{n+2}'$$

$$= \gamma_0\mu_n'+\gamma_1\mu_{n+1}'+\gamma_2\mu_{n+2}' \qquad (115)$$

Now if the first term in (115) vanishes when limits are taken, then (115) $\Rightarrow$

$$\mu_n'[\gamma_0+(n+1)\delta_0]+\mu_{n+1}'[\gamma_1+(n+2)\delta_1]+\mu_{n+2}'[\gamma_2+(n+3)\delta_2] = 0 \qquad (116)$$

Now, $\mu_0 = \int f\,dx = 1.$

Then setting $n = 0,1,2,3,4,$ in (116) we have

$$\gamma_0+\delta_0+\mu_1'(\gamma_1+2\delta_1)+\mu_2'(\gamma_2+3\delta_2) = 0$$

$$\mu_1'(\gamma_0+2\delta_0)+\mu_2'(\gamma_1+3\delta_1)+\mu_3'(\gamma_2+4\delta_2) = 0$$

$$\mu_2'(\gamma_0+3\delta_0)+\mu_3'(\gamma_1+4\delta_1)+\mu_4'(\gamma_2+5\delta_2) = 0 \qquad (117)$$

$$\mu_3'(\gamma_0+4\delta_0)+\mu_4'(\gamma_1+5\delta_1)+\mu_5'(\gamma_2+6\delta_2) = 0$$

$$\mu_4'(\gamma_0+5\delta_0)+\mu_5'(\gamma_1+6\delta_1)+\mu_6'(\gamma_2+7\delta_2) = 0$$

Thus in (117) we have got six equations involving the six unknowns $\gamma_0,\gamma_1,\gamma_2,\delta_0,\delta_1$ and δ_2 that come in the differential equation (112), and these equations can be solved to determine the unknowns in terms of the first seven moments $\mu_1',\mu_2',\mu_3',$ $\mu_4',\mu_5',\mu_6',\mu_7'$.

These 5 equations can be used to obtain $\gamma_1, \gamma_2, \delta_0, \delta_1, \delta_2$ in terms of γ_0 and the first six moments. Choice of the value of γ_0 is arbitrary (but $\gamma_0 \neq 0$) since we could have divided the numerator and denominator of the right side of (112) by γ_0.

(a) CONDITIONS FOR A PEARSON CURVE:

The following conditions will give rise to a Pearson curve:

1. Any or all of a_1, a_2, b_1, b_2 are zero.

Proof: (i) $a_1 = 0,\ a_2, b_1, b_2 \neq 0$

$$f(x) = Cx^{r_1} a_2^{r_2} x^{r_2} (b_1 + b_2 x)^{r_3}$$

$$= Ax^{p} (b_1 + b_2 x)^{r_3}$$

which can be transformed into Pearson Type VI curve by assigning suitable values to the constants as appeared above.

(ii) $a_1 \neq 0,\ a_2 = 0,\ b_1, b_2 \neq 0$

$$f(x) = Cx^{r_1} a_1^{r_2} (b_1 + b_2 x)^{r_3}$$

$$= Ax^{r_1} (b_1 + b_2 x)^{r_3}$$

Now similar arguments as in (i)

(iii) $a_1 = a_2 = 0,\ b_1, b_2 \neq 0$

$$f(x) = Cx^{r_1} (b_1 + b_2 x)^{r_3}$$

Proceed as before.

(vi) $a_1 = 0,\ b_1 = 0 \Rightarrow f(x)$ is Type XI curve

(v) $a_1 = b_2 = 0$ or $a_2 = b_1 = 0 \Rightarrow f(x)$ is a Type XI curve.

(vi) $a_1 = a_2 = b_1 = 0$ or $a_1 = b_1 = b_2 = 0$ or $a_1 = a_2 = b_1 = b_2 = 0$.

Proceed as before.

The last condition includes $f(x) = 0$ under the Pearson system as in Chapter I

2. Both $a_1 = b_1$ and $a_2 = b_2$.

Proof: Let $a_1 = b_1 = p$, $a_2 = b_2 = q$. Then $f(x) = Cx^{r_1}(p+qx)^{r_2+r_3}$

$= f(x)$ is Type VI

3. At least one, but not all of r_1, r_2, r_3 are zero ($r_1 = r_2 = r_3 = 0$ eventually leads to the Uniform Distribution).

Proof: (i) $r_1 = 0$, $r_2, r_3 \neq 0$

$$f(x) = C(a_1+a_2x)^{r_2}(b_1+b_2x)^{r_3}$$

which can evidently be transformed into Type I or II or XII by assigning suitable values to the constants involved.

(ii) $r_1 = r_2 = 0$ or $r_1 = r_3 = 0$ or $r_2 = r_3 = 0$

$\Rightarrow f(x)$ is Type IX or XI.

(b) SOME FURTHER OBSERVATIONS:

(i) $a_1 = 0$ or $b_1 = 0 \Leftrightarrow \delta_0 = 0$

and $a_1 = 0$ or $b_1 = 0 \Rightarrow \gamma_0 = 0$

Simply note that $\delta_0 = a_1 b_1$ and $\gamma_0 = a_1 b_1 r_1$. Also note that $\delta_0 = 0 \Rightarrow \gamma_0 = 0$.

(ii) $a_2 = 0$ or $b_2 = 0 \Leftrightarrow \delta_2 = 0$

Note that $\delta_2 = a_2 b_2$.

(iii) $\delta_2 = 0 \Rightarrow \gamma_2 = 0$

Note that $\gamma_2 = \delta_2(r_1+r_2+r_3)$.

Thus we get Pearson curves from (113) if and only if either

(i) $\gamma_0 = \delta_0 = 0$

or (ii) $\gamma_2 = \delta_2 = 0$

We further observe that γ_0 may be zero while δ_0 is not because of condition 3 of section 35(a) above. Nevertheless, this condition gives rise to Pearson curves. For non-zero γ_0, δ_0 and δ_2 we get non-Pearson curves, because in this case the differential equation (112) cannot be reduced to the form (20) in Chapter II (which gives Pearson curves as its solutions).

(c) SOME SPECIAL CASES OF (112) AND ITS SOLUTIONS IN THESE CASES:

Case 1: $\gamma_1 = \delta_1 = 0$

The differential equation (1) under these conditions become

$$\frac{d}{dx}[\log f(x)] = \frac{\gamma_0+\gamma_1 x^2}{x(\delta_0+\delta_2 x^2)} = \frac{\gamma_0+\gamma_1 x^2}{x(x+d)(x-d)}$$

where

$$d = \sqrt{-\frac{\delta_0}{\delta_2}}, \quad -\frac{\delta_0}{\delta_2} > 0 \ (\neq 0 \ \text{ or } \ \infty, \quad \text{since } \delta_0 \neq 0, \ \delta_2 \neq 0)$$

From above, we have

$$\frac{d}{dx}[\log f(x)] = \frac{A}{x} + \frac{B}{x+d} + \frac{C}{x-d} \tag{118}$$

where

$$\gamma_0+\gamma_1 x^2 = A(x^2-d^2)+B(x^2-dx)+C(x^2+dx)$$

$$\Rightarrow A = \frac{\gamma_0\delta_2}{\delta_0}, \quad B = C = \frac{1}{2}\left(\gamma_2 - \frac{\gamma_0\delta_2}{\delta_0} \right)$$

Integrating (118),

$$\log f(x) = \log K + A \log x + B \log(x+d) + C \log(x-d)$$

$$\Rightarrow f(x) = kx^A(x^2-d^2)^B$$

$$\text{or,} \quad f(x) = kx^{m_1}(x^2-d^2)^{m_2} \tag{119}$$

where $m_1 = A$, $m_2 = B$ are evaluated above

Case 2: $\delta_1 = 0$, $-\frac{\delta_0}{\delta_2} > 0$

The differential equation (112) becomes:

$$\frac{d}{dx}[\log f(x)] = \frac{\gamma_0+\gamma_1 x+\gamma_2 x^2}{x(\delta_0+\delta_2 x^2)}$$

$$= \frac{\gamma_0+\gamma_1 x+\gamma_2 x}{x(x+d)(x-d)} \quad \text{where} \quad d = \sqrt{-\frac{\delta_0}{\delta_2}}$$

$$= \frac{A}{x} + \frac{B}{x+d} + \frac{C}{x-d} \tag{120}$$

where $\gamma_0+\gamma_1 x+\gamma_2 x^2 = A(x^2-d^2)+B(x^2-dx)+C(x^2+dx)$

$$\Rightarrow \; A = \frac{\gamma_0\delta_2}{\delta_0}, \; B = \frac{1}{2}\left(\gamma_2 - \frac{\gamma_0\delta_2}{\delta_0} - \sqrt{-\frac{\delta_2}{\delta_0}}\right),$$

$$C = \frac{1}{2}\left(\gamma_2 - \frac{\gamma_0\delta_2}{\delta_2} + \sqrt{-\frac{\delta_2}{\delta_0}}\right)$$

Integrating (120), we get

$$f(x) = kx^A(x+d)^B(x-d)^C$$

$$\Rightarrow f(x) = kx^{m_1}(x+d)^{m_2}(x-d)^{m_3} \tag{121}$$

where $m_1 = A$, $m_2 = B$, $m_3 = C$ are already evaluated above.

Case 3: Either $-\frac{\delta_0}{\delta_2} > 0$ or $\delta_1{}^2 > 4\delta_0\delta_2$.

From (112), we have

$$\frac{d}{dx}[\log f(x)] = \frac{\gamma_0+\gamma_1 x+\gamma_2 x^2}{x(x-d_1)(x-d_2)} = \frac{A}{x} + \frac{B}{x-d_1} + \frac{C}{x-d_2} \tag{122}$$

where

$$d_1, d_2 = \frac{-\delta_1 \pm \sqrt{\delta_1{}^2-4\delta_0\delta_2}}{2\delta_2} \quad \text{are real.}$$

Integrating (122), we get

$$f(x) = kx^A(x-d_1)^B(x-d_2)^C$$

$$= kx^{m_1}(x-d_1)^{m_2}(x-d_2)^{m_3} \tag{123}$$

where $m_1 = A$, $m_2 = B$, $m_3 = C$ are determined from (122), by the usual method of partial fractions, as:

$$A = m_1 = \frac{\gamma_0}{d_1 d_2}$$

$$B = m_2 = \frac{\gamma_0+\gamma_1 d_1+\gamma_2 d_1^2}{d_1(d_1-d_2)}$$

$$C = m_3 = \frac{\gamma_0+\gamma_1 d_2+\gamma_2 d_2^2}{d_2(d_1-d_2)}$$

Case 4: $\delta_1 = 0$, $\frac{\delta_0}{\delta_2} > 0$ i.e. δ_0 and δ_2 have the same sign

i.e. either $\delta_0 < 0$, $\delta_2 < 0$ or $\delta_0 > 0$, $\delta_2 > 0$.

The differential equation (112) becomes:

$$\frac{d}{dx} \log f(x) = \frac{\gamma_0+\gamma_1 x+\gamma_2 x^2}{\delta_2 x\left(x^2 + \frac{\delta_0}{\delta_2}\right)}$$

$$= \frac{\gamma_0}{\delta_2 x\left(x^2 + \frac{\delta_0}{\delta_2}\right)} + \frac{\gamma_1}{\delta_2\left(x^2 + \frac{\delta_0}{\delta_2}\right)} + \frac{\gamma_2}{2\delta_2} \cdot \frac{2x}{x^2 + \frac{\delta_0}{\delta_2}} \tag{124}$$

Now let $\frac{1}{x\left(x^2 + \frac{\delta_0}{\delta_2}\right)} = \frac{A}{x} + \frac{Bx+C}{x^2 + \frac{\delta_0}{\delta_2}}$

$$\Rightarrow A = \frac{\delta_2}{\delta_0} = -B, \ C = 0$$

$\therefore$ From (124) we get

$$\frac{d}{dx} \log f(x) = \frac{\gamma_0}{\delta_0} \cdot \frac{1}{x} + \frac{\gamma_1\delta_0-\gamma_0\delta_2}{\delta_0\delta_2} \cdot \frac{1}{x^2 + \frac{\delta_0}{\delta_2}} + \frac{\gamma_2}{2\delta_2} \frac{2x}{x^2 + \frac{\delta_0}{\delta_2}}$$

Integrating,

$$\log f(x) = \frac{\gamma_0}{\delta_0}\log x + \frac{\gamma_1\delta_0-\gamma_0\delta_2}{\delta_0\delta_2}\sqrt{\frac{\delta_2}{\delta_0}}\tan^{-1}\frac{\delta_2 x}{\delta_0} + \frac{\gamma_2}{2\delta_2}\log\left(x^2+\frac{\delta_0}{\delta_2}\right)$$

$$+ \log k$$

$$\Rightarrow f(x) = kx^{\gamma_0/\delta_0}\left(x^2+\frac{\delta_0}{\delta_2}\right)^{\gamma_2/2\delta_2}\exp\left(\frac{\gamma_1\delta_0-\gamma\delta_2}{\delta_0\delta_2}\right)\sqrt{\frac{\delta_2}{\delta_0}}\tan^{-1}\frac{\delta_2 x}{\delta_0}$$

or, $f(x) = Cx^{m_1}(x^2-d^2)^{m_2}\exp[C_1\tan^{-1}(C_2x)]$, (125)

d having the same value as in Case 1, and choosing m_1, m_2, C_1 and C_2 in an obvious way.

Case 5: $\delta_1^2 < 4\delta_0\delta_2$, $\frac{\delta_0}{\delta_2} > 0$

$$(112) \Rightarrow \frac{d}{dx}\log f(x) = \frac{\gamma_0+\gamma_1x+\gamma_2x^2}{\delta_2x\left(x^2+\frac{\delta_1}{\delta_2}x+\frac{\delta_0}{\delta_2}\right)}$$

$$= \frac{\gamma_0+\gamma_1x+\gamma_2x^2}{\delta_2x\left[\left(x+\frac{\delta_1}{2\delta_2}\right)^2+\frac{1}{4\delta_2^2}(4\delta_0\delta_2-\delta_1^2)\right]} \qquad (126)$$

But $x+\frac{\delta_1}{2\delta_2} = y \Rightarrow x = y-\frac{\delta_1}{2\delta_2} = y-b$, $b = \frac{\delta_1}{2\delta_2}$

$dx = dy$

$\frac{1}{4\delta_2^2}(4\delta_0\delta_2-\delta_1^2) = a^2 > 0$

$\therefore$ (126) $= \frac{\gamma_0+\gamma_1(y-b)+\gamma_2(y-b)^2}{(y-b)(y^2+a^2)}$

$$= \frac{(\gamma_0 - b\gamma_1 + b^2\gamma_2) + (\gamma_1 - 2b\gamma_2)y + \gamma_2 y^2}{\delta_2(y-b)(y^2+a^2)}$$

$$= \frac{\alpha_0 + \alpha_1 y + \alpha_2 y^2}{\delta_2(y-b)(y^2+a^2)}$$

the expressions for α_0, α_1 and α_2 being obvious from above.

Now let

$$\frac{\alpha_0 + \alpha_1 y + \alpha_2 \gamma^2}{\delta_2(y-b)(y^2+a^2)} = \frac{A}{y-b} + \frac{By+C}{y^2-a} \tag{127}$$

$$\Rightarrow A = \frac{1}{a^2+b^2}(\alpha_0 + \alpha_1 b + \alpha_2 b^2)$$

$$= \frac{\gamma_0\delta_2}{\delta_0},$$

$$B = \frac{1}{a^2+b^2}(a^2\alpha_2 - \alpha_0 - \alpha_1 b)$$

$$= \gamma_2 - \frac{\gamma_0\delta_2}{\delta_0}$$

$$C = \alpha_1 + bB = \gamma_1 - \frac{\gamma_0\delta_1}{2\delta_0} - \frac{\gamma_2\delta_1}{2\delta_2}$$

Hence integrating (126) by means of (127), we get

$$\log f(y-b) = A\log(y-b) + \frac{1}{2}B\log(y^2+a^2) + \frac{C}{a}\tan^{-1}\frac{\gamma}{a} + \log k'$$

(Int. const.)

$$f(y-b) = k'(y-b)^A(y^2+a^2)^{\frac{B}{2}} \exp\frac{C}{a}\tan^{-1}\frac{y}{a}$$

$$f(x) = k'x^A\left[\frac{1}{\delta_2}(\delta_0 + \delta_1 x + \delta_2 x^2)\right]^{\frac{B}{2}} \exp\frac{C}{a}\tan^{-1}\frac{x+b}{a}$$

$$= kx^{m_1}(\delta_0+\delta_1 x+\delta_2 x\ )^{m_2}\exp\left[C_1\tan^{-1}\frac{x+C_2}{C_3}\right] \tag{128}$$

Choosing the constants k, m_1 m_2, C_1, C_2 and C_3 in an obvious way, where the constants A,B and C are already determined.

Note: The cases in which $\gamma_2 = 0$ follow from the above five cases.

(d) A SYNOPSIS OF THE EXTENDED CASES:

Case 1: $\gamma_1 = \delta_1 = 0$, $d = \sqrt{-\frac{\delta_0}{\delta_2}}$ real

$$f(x) = Cx^{m_1}(x^2-d^2)^{m_2}$$

Case 2: $\delta_1 = 0$, $\frac{\delta_0}{\delta_2} < 0$

$$f(x) = Cx^{m_1}(x-d)^{m_2}(x+d)^{m_3}$$

Case 3: Either $\frac{\delta_0}{\delta_2} < 0$ or $\delta_1{}^2 > 4\delta_0\delta_2$

$$d_1,d_2 = \frac{-\delta_1 \pm\sqrt{\delta_1{}^2-4\delta_0\delta_2}}{2\delta_2}$$

$$f(x) = Cx^{m_1}(x-d_1)^{m_2}(x-d_2)^{m_2}$$

Case 4: $\delta_1 = 0$, $\frac{\delta_0}{\delta_2} > 0$

$$f(x) = C_1x^{m_1}(x^2-d^2)^{m_2}\exp[C_2\ \tan^{-1}(C_3x)]$$

Case 5: $\delta_1{}^2 < 4\delta_0\delta_2$, $\frac{\delta_0}{\delta_2} > 0$

$$f(x) = C_1x^{m_1}(\delta_0+\delta_1x+\delta_2x^2)^{m_2}\exp\left[C_2\ \tan^{-1}\left(\frac{x+C_3}{C_4}\right)\right]$$

(e) TRANSITION BETWEEN THE EXTENDED AND THE PEARSON SYSTEMS:

(i) Pearson Type I and Case 3 of extension:

$$\text{Case 3} \Rightarrow f(x) = Cx^{m_1}(x-d_1)^{m_2}(x-d_2)^{m_3}$$

$$= C\,\frac{(-1)^{m_2+m_3}}{d_1^{m_1}d_2^{m_2}}\,x^{m_1}\left(1 - \frac{x}{d_1}\right)^{m_2}\left(1 - \frac{x}{d_2}\right)^{m_3}$$

$$= y_0\left(1 + \frac{x}{a_1}\right)^{m_1}\left(1 - \frac{x}{a_2}\right)^{m_2}$$

$$\Rightarrow \text{Type I}$$

where

$$y_0 = C\,\frac{(-1)^{m_2+m_3}}{d_1^{m_1}d_2^{m_2}} \quad (\text{real}), \quad d_1 = -a_1,\ d_2 = a_2,\ m_1 = 0,$$

$m_2 = m_1$, $m_2 = m_2$.

$$\text{Now} \quad m_1 = 0 \Rightarrow \frac{\gamma_0}{d_1 d_2} = 0 \Rightarrow \gamma_0 = 0$$

$$\text{and} \quad \frac{m_1}{a_1} = \frac{m_2}{a_2} \Rightarrow \frac{\gamma_0+\gamma_1 d_1+\gamma_2 d_1^2}{d_1(d_1-d_2)(-d_1)} = \frac{\gamma_0+\gamma_1 d_2+\gamma_2 d_2^2}{d_2(d_2-d_1)d_2}$$

$$\Rightarrow \gamma_1(d_1-d_2) = 0 \quad (\gamma_0 = 0)$$

$$\Rightarrow \gamma_1 = 0 \quad (\because d_1 \neq d_2)$$

The differential equation (112) becomes

$$\frac{d \log f(x)}{dx} = \frac{\gamma_2 x}{\delta_0+\delta_1 x+\delta_2 x^2}$$

(ii) Pearson Type II and Case 4 of extension:

$$\text{Case 4} \Rightarrow f(x) = Cx^{m_1}(x^2-d^2)^{m_2}$$

$$= y_0\left(1 - \frac{x^2}{a^2}\right)^m$$

$$\Rightarrow \text{Type II}$$

Where $y_0 = \frac{C(-1)^m}{d^{2m}}$ (real), $m_1 = 0$, $d = a$, $m_2 = m$

$$\gamma_0 = \gamma_1 = \delta_1 = 0 \Rightarrow (112),$$

$$\frac{d \log f(x)}{dx} = \frac{\gamma_2 x}{\delta_0+\delta_2 x^2}$$

(iii) Pearson Type III and Case 3 of extension:

$$\text{Case 3} \Rightarrow f(x) = y_0 x^{m_1}\left(1 - \frac{x}{d_1}\right)^{m_2}\left(1 - \frac{x}{d_2}\right)^{m_3} \quad [y_0 \text{ as in (i)}]$$

Now taking $m_1 = 0$, $d_1 = \frac{\gamma}{m_2}$, $d_2 = -a$, $m_3 = \gamma a$,

$$f(x) = y_0\left(1 - \frac{\gamma x}{m_2}\right)^{m_2}\left(1 + \frac{x}{a}\right)^{\gamma a}$$

Letting $m_2 \to \infty$,

$$f(x) = y_0 e^{-\gamma x}\left(1 + \frac{x}{a}\right)^{\gamma a}$$

$$\Rightarrow \text{Type III}$$

Now $m_1 = 0 \Rightarrow \gamma_0 = 0$

$m_2 \to \infty \Rightarrow d_1 = d_2$

$$\Rightarrow \frac{-\delta_1+\sqrt{\delta_1{}^2-4\delta_0\delta_2}}{2\delta_2} = \frac{-\delta_1-\sqrt{\delta_1{}^2-4\delta_0\delta_2}}{2\delta_2}$$

$$\Rightarrow \delta_1{}^2 = 4\delta_0\delta_2$$

(iv) Type IV and Case 5 of extension:

$C_1 = y_0$, $m_1 = 0$, $\delta_0 = 1$, $\delta_1 = 0$, $\delta_2 = \frac{1}{a^2}$, $m_2 = -m$, $C_2 = -\nu$, $C_3 = 0$, $C_4 = a$ give

$$f(x) = y_0\left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-\nu\tan^{-1}\frac{x}{a}}$$

$\Rightarrow$ Type IV

$$m_1 = 0 \Rightarrow \gamma_0 = 0.$$

The differential equation (112) becomes

$$\frac{d \log f(x)}{dx} = \frac{a^2(\gamma_1+\gamma_2 x)}{x^2+a^2}$$

(v) Type V and Case 3:

We write

$$f(x) = Cx^{m_1+m_2}\left(1 - \frac{d_1}{x}\right)^{m_2}(x-d_2)^{m_3}$$

Now $C = y_0$, $m_1+m_2 = -p$, $d_1 = -\frac{\gamma}{m_2}$, $m_3 = 0$

$$\Rightarrow f(x) = y_0 x^{-p} \left(1 - \frac{\gamma}{x m_2}\right)^{m_2}$$

Letting $m_2 \to \infty \Rightarrow f(x) = y_0 x^{-p} e^{-\gamma/x} \Rightarrow$ Type V

$$m_3 = 0 \Rightarrow \gamma_0 + \gamma_1 d_2 + \gamma_2 d_2^2 = 0$$

$$m_2 \to \infty \Rightarrow d_1 \to 0$$

$$\Rightarrow \delta_1 \to \sqrt{\delta_1^2 - 4\delta_0\delta_2}$$

$$\Rightarrow \delta_1^2 = \delta_1^2 - 4\delta_0\delta_2$$

$$\Rightarrow \delta_0\delta_2 = 0$$

$$\Rightarrow \delta_0 = 0 \quad (\because \delta_2 \neq 0)$$

(vi) Type VI and Case 3:

Type VI follows directly from Case 3.

(vii) Type VII can be obtained from all the five cases of extension.

(viii) Type VIII and Case 2:

$$C = y_0,\ m_1 = m_2 = 0,\ d = \frac{1}{a},\ m_3 = -m$$

Now $m_1 = 0 \Rightarrow \gamma_0 = 0$

$$m_2 = 0 \Rightarrow \gamma_2 + \gamma_1 \sqrt{-\frac{\delta_2}{\delta_0}} = 0 \quad (\because \gamma_0 = 0)$$

$$\Rightarrow \frac{\gamma_2^2}{\gamma_1^2} + \frac{\delta_2}{\delta_0} = 0$$

(ix) Type IX can be treated exactly in a similar way.

(x) Type X and case 5:

$$C_1 = \frac{n}{\sigma},\ \delta_0 = 1,\ \delta_1 = \pm\frac{1}{\sigma m_2},\ \delta_2 = 0,\ m_1 = 0,\ C_2 = 0$$

$$\Rightarrow\ y = \frac{n}{\sigma}\left(1 \pm \frac{x}{\sigma m_2}\right)^{m_2}$$

Letting $m_2 \to \infty$,

$$y = f(x) = \frac{n}{\sigma}\, e^{\pm\frac{x}{\sigma}}$$

$$m_1 = 0 \Rightarrow \gamma_0 = 0$$

$$m_2 \to \infty \Rightarrow \gamma_2 \to \infty$$

(xi) Type XI can be obtained from all the five cases of extension.

(xii) Type XII follows from Case 3.

(f) DEDUCTION OF THE INVERSE GAUSSIAN DISTRIBUTION FROM CASE 5 OF EXTENSION:

In section 33 we have indicated that the Inverse Gausisan Distribution is Type VI Pearson, as proved by L.K. Roy. We conclude this chapter indicating how the Inverse Gaussian Distribution can be derived from Case 5 of extension.

In Case 5, we first put $C_2 = 0$ which implies

$$f(x) = C_1x^{m_1}(\delta_0+\delta_1x+\delta_2x^2)^{m_2}$$

$$= C_1x^{m_1+m_2}\left(\delta_1 + \frac{\delta_0}{x} + \delta_2x\right)^{m_2}$$

$$= \left(\frac{C_1}{\delta_1{}^{m}}\right) x^{m_1+m_2}\left[1 + \left(\frac{\delta_0}{\delta_1}\right)\frac{1}{x} + \left(\frac{\delta_2}{\delta_1}\right)x\right]^{m_2}$$

$$= Cx^{m_1+m_2}\left[1 + \frac{p}{x} + qx\right]^{m_2}$$

by means of obvious substitutions.

Next take $C = \sqrt{\frac{\lambda}{2\pi}}\, e^{\lambda/\mu}$, $m_1+m_2 = -3/2$, $p = -\frac{\lambda}{2m_2}$, $q = -\frac{\lambda}{2\mu^2 m_2}$

$$\Rightarrow f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{\lambda/\mu}\left[1 - \frac{1}{m_2}\left(\frac{\lambda}{2x} + \frac{\lambda x}{2\mu^2}\right)\right]^{m_2}$$

Letting $m_2 \to \infty$, we get ultimately

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{\lambda/\mu} .\ e^{-\left(\frac{\lambda}{2x} + \frac{\lambda x}{2\mu^2}\right)}$$

$$= \sqrt{\frac{\lambda}{2\pi x^3}}\, e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}$$

which is the required Inverse Gaussian Distribution as discussed in section 31 (xiii).

36. ANOTHER EXTENSION OF PEARSON'S SYSTEM BY EDWIN D. MOUZON

In a paper in the Annals of Mathematical Statistics, Edwin D. Mouzon, Jr. [18] extended Pearson's Differential Equation in the form

$$\frac{dy}{dx} = \frac{(x-a)y}{b_0+b_1x+b_2x^2+b_3x^3} \qquad (129)$$

The recurrence relation (8) of section 6 in Chapter I, in this case, is transformed into

$$nb_0\mu'_{n-1}+(n+1)b_1\mu'_n+(n+2)b_2\mu'_{n+1}+(n+3)b_3\mu'_{n+2} = a\mu'_n-\mu'_{n+1} \qquad (130)$$

and the set of equations (10) becomes

$$\begin{aligned} 0b_0+b_1+0b_2+3\mu_2b_3 &= a \\ b_0+0b_1+3\mu_2b_2+4\mu_3b_3 &= -\mu_2 \\ 0b_0+3\mu_2b_1+4\mu_3b_2+5\mu_4b_3 &= a\mu_2-\mu_3 \\ 3\mu_2b_0+4\mu_3b_1+5\mu_4b_2+6\mu_5b_3 &= a\mu_3-\mu_4 \end{aligned} \qquad (131)$$

This set of equations enables us to determine $b_0, b_1, b_2,$ and b_3 in terms of $\mu_2, \mu_3, \mu_4, \mu_5$ and a, as follows:

$$b_0 = \begin{vmatrix} a & 1 & 0 & 3\mu_2 \\ -\mu_2 & 0 & 3\mu_2 & 4\mu_3 \\ a\mu_2-\mu_3 & 3\mu_2 & 4\mu_3 & 5\mu_4 \\ a\mu_3-\mu_4 & 4\mu_3 & 5\mu_4 & 6\mu_5 \end{vmatrix} \Bigg/ \begin{vmatrix} 0 & 1 & 0 & 3\mu_2 \\ 1 & 0 & 3\mu_2 & 4\mu_3 \\ 0 & 3\mu_2 & 4\mu_3 & 5\mu_4 \\ 3\mu_2 & 4\mu_3 & 5\mu_4 & 6\mu_5 \end{vmatrix}$$

$= A/\Delta$ (say), $\Delta \neq 0$

$$\begin{aligned} \text{similarly,} \quad b_1 &= B/\Delta, \\ b_2 &= C/\Delta, \\ b_3 &= D/\Delta \text{ (say)}, \end{aligned} \qquad (132)$$

where B, C, D are obtained for b_1, b_2, b_3 in the numerator just as A above for b_0.

The differential equation (130) then becomes

$$\frac{dy}{y} = \frac{(x-a)dx}{\frac{A}{\Delta} + \frac{Bx}{\Delta} + \frac{Cx^2}{\Delta} + \frac{Dx^3}{\Delta}} = \frac{\Delta(x-a)dx}{A+Bx+Cx^2+Dx^3} \tag{133}$$

The solution of the differential equation (134) depends on the nature of the zeros of the denominator of the right hand side, that is on the discriminant of the general cubic

$$b_0+b_1x+b_2x^2+b_3x^3 = 0$$

which is

$$18b_0b_1b_2b_3-4b_0b_2{}^3+b_1{}^2b_2{}^2-4b_1{}^3b_2-27b_0{}^2b_3{}^2 \tag{134}$$

The cubic has

(i) three distinct real zeros,

(ii) one real, two imaginary zeros, (135)

(iii) at least two real and equal zeros

according as

$$(134) \gtreqless 0$$

We will expect, therefore, three general types of curves when integration is effected. The following classification was made by Edwin Mouzon:

Class A [curves obtained under condition (i), 135]

Class B [curves obtained under condition (ii), 135]

Class C [curves obtained under condition (iii), 135]

Under Class A, he got six curves, namely:

Type A-1: (all the zeros are positive)

$$y = y_0 \frac{\left(1 - \frac{x}{a_1}\right)^{m_1 a_1} \left(1 - \frac{x}{a_3}\right)^{m_3 a_3}}{\left(1 - \frac{x}{a_2}\right)^{m_2 a_2}}$$

Type A-2: (two positive and one negative zeros)

$$y = \frac{y_0 \left(1 - \frac{x}{a_1}\right)^{m_1 a_1}}{\left(1 - \frac{x}{a_2}\right)^{m_2 a_2} \left(1 - \frac{x}{a_3}\right)^{m_3 a_3}}$$

Type A-3: (Two negative and one positive zero)

$$y = y_0 \frac{\left(1 - \frac{x}{a_1}\right)^{m_1 a_1} \left(1 + \frac{x}{a_2}\right)^{m_2 a_2}}{\left(1 + \frac{x}{a_3}\right)^{m_3 a_3}}$$

Type A-4: (All the three zeros are negative):

$$y = y_0 \frac{\left(1 + \frac{x}{a_2}\right)^{m_2 a_2}}{\left(1 + \frac{x}{a_1}\right)^{m_1 a_1} \left(1 + \frac{x}{a_3}\right)^{m_3 a_3}}$$

Type A-5: (Special Type A-3 when one negative zero = the positive zero):

$$y = y_0 \frac{\left(1 - \frac{x}{a_1}\right)^{m_1} \left(1 + \frac{x}{a_2}\right)^{m_2}}{\left(1 + \frac{x}{a_3}\right)^{m_3}}$$

Type A-6: (Special case of Type A-3 when one of the zeros is zero):

$$y = y_0 \frac{x^{m_1}\left(1 + \frac{x}{a_2}\right)^{m_2}}{\left(1 + \frac{x}{a_3}\right)^{m_3}}$$

where the parameters in the equations of the curves are suitably adjusted.

Under Class B, two curves were obtained, viz:

Type B-1: (Two of the zeros are complex):

$$y = y_0 \frac{(x+C)^{2m} e^{\gamma \tan^{-1} \frac{x}{M}}}{(m^2+M^2)^m}$$

Type B-2: (When two of the zeros are pure imaginaries):

$$y = y_0 \frac{(x-a_1)^{2m} e^{\gamma \tan^{-1} \frac{x}{a_3}}}{(x^2+a_2{}^2)^m}$$

The parameters of the curves have suitably assigned values.

Under Type C, the following are the curves:

Type C-1: (Two zeros are equal):

$$y = y_0 \frac{e^{-\frac{m_1}{x}} (1+x/k)^{m_2}}{x^{m_2}}$$

Type C-2: (When all the zeros are equal):

$$y = y_0 \, e^{\frac{p-x}{q(x-r)^2}}$$

The parameters are suitably adjusted.

The set of frequency curves as indicated above gives a better fit to the modal neighborhood of the data to which it is applied than is often found in the existing methods (as one discussed in Chapter I), one example being given by E.D. Mouzon, taking $b_3 = 0$ in the extended Type A curves. Mouzon also observed that the bell-shaped (or 'cocked hat') Pearsonian curves often yield a very poor fit to the data for a good number of distributions derived from the financial ratios of public utility companies, and furthermore, in some cases, on the left extremity of the distribution, the rise of the curve to the mode is too steep for a good fit, while the fit about the mode is of primary importance in much economic data.

In Chapter I, the constants a, b_0, b_1 and b_2 were determined by equating the moments of the raw data to the moments of the theoretical distribution. The new assumption made by Mouzon is that the value of the constant, a, the mode, is determined first from the observed data, and equated to the value of the mode in the theoretical distribution. This method of procedure is particularly adapted to economic data, as it assures a good fit about the mode.

If $b = 0$, equations (131) reduce to

$$b_1 = a$$

$$b_0+3b_2\mu_2 = -\mu_2$$

$$3b_1\mu_2+4b_2\mu_3 = -\mu_3+a\mu_2$$

the solutions of which give

$$b = \frac{-\mu_2\mu_3+6a\mu_2{}^2}{4\ \mu_3}$$

$$b_1 = a \tag{136}$$

$$b_2 = \frac{-\mu_3-2a\mu_2}{4\ \mu_3}$$

Thus, we see that the constants are determined in terms of the mode, and the second, and third moments of the raw data. Evidently the expressions for b_0, b_1 and b_2 in (136) are much simpler than those in (12) of Chapter I.

37. CONCLUSIONS (DUE TO KARL PEARSON):

The theory of curves given by

$$\frac{1}{y}\frac{dy}{dx} = \frac{x-a}{b_0+b_1x+b_2x^2+b_3x^3+\ldots}$$

is worked out by Dr. David Heron, but Karl Pearson [25] did not publish that in the Biometrika because (a) good fits in the case of homogeneous material are as a rule found from using the first four moments, and because (b) of Pearson's profound distrust - based largely on experience of the different results for the freqeuncy curves obtained from different samples of the same material, when high moment coefficients are involved - of the use of high moments at all.

In Karl Pearson's own statement, "I have a very firm conviction that the mathematician who uses high moments may make interesting contributions to mathematics, but he removes his work from any contact with practical statistics."[2]

2 "Skew Correlation and Non-Linear Regression", Drapers' Co. Res. Mem. 1905, p. 9.

A SELECTED BIBLIOGRAPHY

[1] Bateman, Harry, Higher Transcendental Functions, Bateman Manuscript Project, California Institute of Technology, Volume 1, 1953.

[2] Beale, Frank S., "On the polynomials related to the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0+a_1x}{b_0+b_1x+b_2x^2} \equiv \frac{N}{D}\text{"}$$

Annals of Mathematical Statistics, Volume 8 (1937), pp. 206-210.

[3] Butkov, Eugene, Mathematical Physics, Addision-Wesley Publishing Company, 1968.

[4] Carlson, John L., "A study of the distribution of means estimated from small samples by the method of maximum likelihood for Pearson type II curve", Annals of Mathematical Statistics, Volume 3 (1932), pp. 86-95.

[5] Chester, Clive R., Techniques in Partial Differential Equations, International Series in Pure and Applied Mathematics, McGraw-Hill Book Company.

[6] Craig, Cecil C., "A new exposition and chart for the Pearson system of frequency curves", Annals of Mathematical Statistics, Volume 7 (1936), pp. 16-28.

[7] Cramer, Harold, Mathematical Methods of Statistics, Princeton University Press, 1961.

[8] Elderton, William Palin, and Norman Lloyd Johnson, Systems of Frequency Curves, Cambridge University Press, 1969.

[9] Fisher, R.A., "On the Mathematical Foundations of Theoretical Statistics", Phil. Trans. Series A, Vol. 222 (1922), pp. 327-328.

[10] Henderson, Robert, "Frequency curves and moments", Transactions of the Acturial Society of America, Volume 8 (1904), pp. 30-41.

[11] Hildebrandt, Emanuel Henry, "Systems of polynomials connected with the Charlier expansions and the Pearson differential and difference equations", Annals of Mathematical Statistics, Volume 2 (1931), pp. 393-411.

[12] Hochstadt, Harry, The Functions of Mathematical Physics, Wiley-Interscience, Volume XXIII.

[13] Kendall, M.G., and Stuart A., The Advanced Theory of Statistics, Volume 1, Charles Griffen and Co. Ltd., 1958.

[14] Kreyszig, Erwin, Advanced Engineering Mathematics, Ohio State University.

[15] Lebedev, N.N., Special Functions and their Applications, Prentice-Hall, Inc., N.J., 1965.

[16] Magnus, W., Oberhettinger, F., Soni, R.P., Formulas and Theorems for the Special Functions of Mathematical Physics, Third Edition, Band 52 Springer-Verlag Berlin Heidelberg New York 1966.

[17] McBride, Elna B., Obtaining Generating Functions, Springer Tracts in Natural Philosophy, Volume 21, 1971.

[18] Mouzon, Edwin D., "Equimodal frequency distributions", Annals of Mathematical Statistics, Volume 1 (1930), pp. 137-158.

[19] Parzen, Emanuel, Modern Probability Theory and its Applications, John Wiley and Sons, Inc., Publishers, 1967.

[20] Pearson, E.S., and Hartley, H.O., Biometrika Tables for Statisticians Volume 1, Cambridge University Press, 1966.

[21] Pearson, E.S., The Selected Papers of E.S. Pearson, University of California Press, Berkeley and Los Angeles, 1966.

[22] Pearson, Karl, Tables of the Incomplete Beta-Function, Second Edition, Cambridge University Press, 1968.

[23] Rainville, Earl D., Special Functions, The Macmillan Company, New York, 1965.

[24] Rietz, H.L., Mathematical Statistics, The Carus Mathematical Monographs, Number Three, The Mathematical Association of America, The Open Court Publsihing Company, La Salle, Illinois.

[25] Romanovsky, "Generalization of some types of the frequency curves of Professor Pearson", Biometrika, Vol. 16 (1924), pp. 106-117.

[26] Roy, Lawrence Keith, Some Properties and Applications of the Inverse Gaussian Distribution, Queen's University, M.Sc. Thesis, 1967.

[27] Roy, Lawrence Keith, "An extension of the Pearson system of frequency curves", Mathematical Report, #69-4, August, 1969, Department of Mathematics, Lakehead University.

[28] Salvosa, Luis R., "Tablesof Pearson's type III function", Annals of Mathematical Statistics, Volume 1 (1930), pp. 191-198.

[29] Thompson, William R., "On a criterion for the rejection of observations and the distribution of the ratio of deviation to sample standard deivation", Annals of Mathematical Statistica, Volume 6 (1935), pp. 214-219.

[30] Tweedie, M.C.K., "Statistical properties of inverse Gaussian distributions", Annals of Mathematical Statistics, Volume 28 (1957).

[31] Wasan, M.T., First Passage Time Distribution of Brownian Motion with Positive Drift, Queen's Paper in Pure and Applied Mathematics, No. 19.

[32] Zoch, Richmond T., "Some interesting features of frequency curves", Annals of Mathematical Statistics, Volume 6 (1935), pp. 1-10.