DIFFERENTIABLE AND ROUGH NORMS

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ABSTRACT

The principal question discussed in this dissertation is the problem of characterizing the existence of admissible Fréchet differentiable norms on Banach spaces.

In the first chapter the basic concepts of normed linear spaces are introduced and a summary of differential calculus on Banach spaces is given.

The following three chapters of the paper are concerned with the existence of an admissible Fréchet differentiable norm on a separable Banach space. A construction of such a norm is given for a separable space which has a separable dual. Also, it is shown that if such a norm exists on a Banach space, then the density character of the space equals the density character of its dual.

In Chapter V, it is shown that if the density characters of a space and its dual are not equal, then the space admits a rough norm. As a consequence of this, there is no Fréchet differentiable function on this space with bounded non-empty support. This implies that the space does not admit a Fréchet differentiable norm.

Finally, the notion of C^p - smoothness is introduced. It is shown that if a Banach space admits a C^p - norm, then it is C^p - smooth. This fact is one of the reasons why it is of interest to determine the class of Banach spaces that admit C^p - norms. Also, those spaces that are C^p - smooth are characterized as those for which

the Γ^p - topology is the norm topology. The Γ^p - topology is the weakest topology for which the functions of class C^p on the space are continuous. Some further properties of these topologies are also studied.

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CHAPTER I

INTRODUCTORY CONCEPTS

1.1 Introduction

In this chapter we will introduce some of the basic definitions and fundamental propositions of linear analysis on which the results of this paper are based. However, we will not include all of the terminology and ideas used in this paper, particularly those concepts employed which pertain to topology and classical analysis. The reader is referred to Kelley [10] and Robertson [18] for clarification of the topological notions used. Royden [19] is a good reference for concepts of classical analysis. The definitions of certain concepts have been, more appropiately, deferred until later chapters.

We will first consider the underlying structure of Banach spaces, paying special attention to those aspects of convexity and functionals which are naturally involved in any study of differential calculus. In particular, we shall introduce the Hahn Banach theorem, which is a useful tool in any study of functional analysis. Wilansky [21] is a good reference source for the many concepts of linear spaces mentioned in this paper.

Following the development of the basic ideas of linear spaces, we will go on to define norms and the resulting topologies, which give rise to the notion of continuity of functionals. In the light of these topologies, we will consider some of the implications of the

Hahn Banach theorem, and we also consider those definitions and propositions which characterize a normed linear space by its unit sphere. The book by Dunford and Schwarz [7] complements this development very well.

We will conclude this chapter with a study of infinite dimensional differential calculus. The concepts of Gateaux and Fréchet differentiability are developed and several propositions are given which illustrate that strong differentials of the norm of a Banach space can be represented by continuous linear functionals. A complete treatment of differential calculus on Banach spaces is given in Dieudonné's book [6] and Cudia [5] does a comprehensive survey of the differentiation of norms.

Many of the elementary and well known propositions of functional analysis, which follow, have not been proved here; however, standard proofs of these theorems can be found in the resource material indicated.

1.2 Linear Spaces

1.2.1 Definition

A linear space X is a set on which is defined addition, $+$, so that $(X, +)$ is a commutative group; and multiplication by scalars satisfying the distributive laws

$$t(x + y) = tx + ty \quad \text{and} \quad (s + t)x = sx + tx,$$

where s, t are scalars, $x, y \in X$, and satisfying $(st)x = s(tx)$, and $1 \cdot x = x$.

The elements of X will be called vectors and for the purposes of this dissertation, the field of scalars will always the reals, denoted R.

1.2.2 Definition

A subset of linear space is convex if and only if it includes the line segment joining any two of its points. More precisely, a set A is convex provided $tx + (1 - t)y$ is in A, for all scalars t satisfying $0 \leq t \leq 1$, and $x, y \in A$.

1.2.3 Definition

A set A is called balanced if $t A \subset A$ for all scalars t satisfying $|t| \leq 1$, where $tA = \{tx \mid x \in A\}$.

1.2.4 Definition

A nonempty subset A of a linear space is called affine if $tA + (1 - t) A \subset A$ for all scalars t. Thus, a set A is affine if it is a translate of a linear subspace.

1.2.5 Definition

The convex hull of a set A is the smallest convex set which includes A, and so, every element of the convex hull of A is a finite linear combination of elements of A with non-negative real coefficients adding up to 1.

1.2.6 Definition

If M is a subset of the linear space X, then a point $x \in M$ is called an internal point of M if, for each $y \in X$, there exists an $\mu > 0$ such that $x + ty \in M$ for $|t| \leq \mu$.

A point $x \in X$ is called a bounding point of M if x is not an internal point of M or $X - M$. A bounding point of M is called an extremal of M if it is not interior to any line segment contained in M.

1.2.7 Definition

Let X and Y be two linear spaces. The mapping f of X into Y is called linear if

$$f(x + y) = f(x) + f(y), \; f(tx) = tf(x)$$

for all $x, y \in X$ and $t \in R$.

If the image space of a linear mapping f is the scalar field of X, namely R, then f is called a linear function or functional. Associated with every functional is the set $\{x \mid f(x) = 0$ and $x \in X\}$, denoted $f^{\perp}$ and called the kernel of f.

1.2.8 Definition

A proper subspace of a linear space X is called maximal provided its only superspace is the whole space X. A hyperplane is a translation of a maximal subspace.

In the following propositions we will characterize maximal subspaces and hyperplanes.

1.2.9 Theorem

Let f be a non-identically-zero linear functional, then $f^{\perp}$ is a maximal subspace and conversely, if S is a maximal subspace there exists a linear functional f such that $S = f^{\perp}$.

The following lemma shows that a linear functional is almost determined by its set of zeros. It is in fact, determined up to a multiplicative constant.

1.2.10 Lemma

Let f, g be linear functionals such that $f^{\perp}$ is contained in $g^{\perp}$. Then there exists a constant t such that $g = tf$. If g is not identically 0, then $t \neq 0$, and $f^{\perp} = g^{\perp}$.

Proof:

Let a be a vector such that $f(a) \neq 0$. Given any vector x, let $y = x - \frac{f(x)}{f(a)} a$. Then $f(y) = 0$, hence $g(y) = 0$, from which we get $g(x) = \frac{g(a)}{f(a)} f(x)$. Set $t = \frac{g(a)}{f(a)}$ to get the above result.

1.2.11 Theorem

If f is a linear functional not identically zero and t is a scalar, $\{x \mid f(x) = t \text{ and } x \varepsilon X\}$ is a hyperplane. Conversely, every hyperplane has this form.

1.2.12 Corollary

If f, g are linear functionals which are equal on a hyperplane, which does not contain the origin, then $f = g$.

1.2.13 Definition

If x is a bounding point of a set A, a non-zero linear functional f is said to support A at x if there exists a real constant t such that $f(A) \leq t$, $f(x) = t$, where $f(A) = \{f(y) \mid y \varepsilon A\} \subset R$. If such a functional exists it is called a support functional of A at x. The hyperplane determined by f and t is called the support

<u>hyperplane</u> of A at x.

We note that if f is a support functional of A at x, then so is every positive multiple of f. Conversely, if every support functional of A at x is a positive multiple of f, we say that A has a unique tangent at x.

<u>1.2.14 Definition</u>

Two sets, A, B are said to be <u>separated</u> by a linear functional f if $f \neq 0$ and there exists a real number t such that $f(x) \leq t \leq f(y)$ for all $x \in A, y \in B$.

Next we state the Hahn Banach extension theorem which is of basic importance. We will also list some corollaries of this theorem which are applicable to linear spaces; later in the chapter we will indicate some results of the theorem pertaining to normed linear spaces.

<u>1.2.15 Theorem (Hahn Banach)</u>

Let the real function p on the linear space X satisfy

$$0 \leq p(x + y) \leq p(x) + p(y),$$

$$p(tx) = tp(x); \quad t \geq 0, \quad x, y \in X.$$

Let f be a linear functional on a subspace Y of X with

$$f(x) \leq p(x), \quad x \in Y .$$

Then there is a linear functional F on X for which

$$F(x) = f(x), \quad x \in Y; \quad F(x) \le p(x), \quad x \in X.$$

For a proof of this theorem see [21, p. 65].

1.2.16 Theorem (Basic separation theorem)

Let A and B be disjoint convex subsets of a linear space X, and let A have an internal point. Then there exists a linear functional f which separates A and B.

1.2.17 Theorem (Basic support theorem)

If a convex set A in a linear space X has an internal point, then A has a non-zero support functional at each of its bounding points.

1.3 Normed Linear Spaces

For the remainder of this chapter we will be concerned with normed linear spaces.

1.3.1 Definition

A prenorm is a real valued function, β, defined on a linear space and satisfying, for all vectors x, y and scalars t,

(1) $\beta(x) \ge 0$,

(2) $\beta(x + y) \le \beta(x) + \beta(y)$,

(the triangle inequality),

(3) $\beta(tx) = |t| \, \beta(x)$, (homogeneity).

A norm is a prenorm satisfying the following condition;

(4) $\beta(x) > 0$ if $x \ne 0$.

A linear space X, together with a norm β is called a normed linear space and is denoted (X, β) or X, the norm being understood.

1.3.2 Definition

Let (X, β) be a normed linear space. Then the function d defined by,

$$d(x, y) = \beta(x - y)$$

where $x, y \in X$, is a metric, called the induced metric on X. This metric gives rise to a topology on X called the norm or strong topology. Robertson [18] proves that the norm topology is the weakest topology on X compatible with its algebraic structure, and in which the norm β is continuous.

Let (X, β) be a normed linear space and ρ another norm defined on X. Then we say that ρ is equivalent to β or that ρ is admissible on X, if the norm topology of ρ is equivalent to the norm topology of β. A useful characterization of admissibility results from the following proposition:

1.3.3 Lemma

Two norms ρ and β defined on a linear space X are equivalent if and only if there exists two real constants, $a > 0$, $b > 0$, such that,

$$a \cdot \rho(x) \leq \beta(x) \leq b \cdot \rho(x)$$

for any $x \in X$.

Clearly, a norm ρ is continuous on (X, β) if its induced topology is courser than the norm topology of β. As above, wecan characterize the continuity of ρ, by the necessary and sufficient existence of a constant $a > 0$, such that $\rho(x) \leq a \cdot \beta(x)$ for every $x \in X$.

1.3.4 Definition

A normed linear space which is complete in its norm topology is called a Banach space. In particular, the set of reals, together with the absolute value function, is a Banach space, whose norm topology is in fact, the usual topology of the reals.

1.3.5 Definition

The closed unit ball of a normed linear space (X, β) is denoted B_β and is defined by $B_\beta = \{x \mid x \in X$ and $\beta(x) \leq 1\}$.

The unit sphere, denoted S_β, is defined by $S_\beta = \{x \mid x \in X$ and $\beta(x) = 1\}$.

1.3.6 Definition

If (X, β) and (Y, ρ) are normed linear spaces, the symbol $L(X, Y)$ the linear space of all linear maps from X to Y which are continuous with respect to the norm topologies. The norms β and ρ generate a natural norm α on $L(X, Y)$, called the supremum or sup norm, defined by $\alpha(f) = \sup \{\rho(f(x)) \mid x \in B_\beta\}$, where $f \in L(X, Y)$. It is well known, for example see Royden [19], that if (Y, ρ) is a Banach space then $L(X, Y)$ together with the sup norm is also a Banach space.

Associated with a normed linear space, X, is the space of continuous linear functionals on X. This Banach space, denoted X^*, is called the conjugate or dual space of X. The sup norm on X^* is called the dual norm and is denoted β^* to indicate its relationship with the norm β on X.

Similarly, we denote the dual space of X^*, by X^{**}, and its sup norm by β^{**}.

1.3.7 Definition

Let X be a normed linear space. The mapping $\phi: X \to X^{**}$, defined by $(\phi(x)) f = f(x)$ for f in X^*, is called the natural embedding of X into X^{**}. ϕ maps X isometrically into X^{**} and, when ϕ is onto X^{**}, then X is said to be reflexive.

1.3.8 Definition

Let (X, ρ) be a normed linear space. The weakest topology of X under which all the linear functionals of X^* are continuous will be called the weak topology of X. The terms w-open, w-continuous, etc. will refer to the weak topology of X.

If X is the dual space of some normed linear space Y, then the weakest topology of X under which all the linear functionals of $\phi(Y) \subset X^*$ are continuous, will be called the weak * - topology of X. (The mapping ϕ is the natural embedding of Y into X^*). When referring to this topology we will prefix terms with w^*. It should be noted that the w^*-topology is only defined on dual spaces.

1.3.9 Theorem (Alaoglu)

Let (X, β) be a Banach space. The dual unit ball, B_{β^*}, is compact in the w^*-topology.

For a proof of this theorem see [19; p. 173].

We will now consider ideas and results obtained by interrelating the topological and algebraic properties of normed linear spaces. First, we will consider implications of the Hahn Banach extension theorem on normed linear spaces. If the prenorm p, used in the Hahn Banach theorem (p. 6) is continuous in the norm topology then f and its extension F are both continuous linear functionals and so, it is evident that in normed linear spaces, the basic separation and support theorems determine the existence of continuous linear functionals with the appropriate properties.

The following corollaries are immediate results of the Hahn Banach theorem applied to a normed linear space (X, β).

1.3.10 Corollary

Let Y be a subspace of the normed linear space X. Then β restricted to Y is a norm on Y and we have that for every element f in Y^* corresponds a g in X^* with

$$\beta^*(g) = \beta^*(f); \quad f(x) = g(x), \quad x \in Y.$$

1.3.11 Corollary

Let Y be a subspace of the normed linear space X and let $x \in X$ be such that

$$\inf \{\beta(y - x) \mid y \in Y\} = t > 0.$$

Then there is a functional $f \in X^*$ with

$$f(x) = 1; \quad \beta^*(f) = \frac{1}{t}; \quad f(y) = 0, \quad y \in Y.$$

1.3.12 Corollary

Let x be a vector not in the closed subspace Y of X. Then there is a continuous linear functional $f \in X^*$ with

$$f(x) = 1; \quad f(y) = 0, \quad y \in Y.$$

1.3.13 Corollary

For every $x \neq 0$ in the normed linear space X, there is an $f \in X^*$ with

$$\beta^*(f) = 1 \quad \text{and} \quad f(x) = \beta(x).$$

This proposition shows that there are enough functionals in the dual space X^* to distinguish the points of X. Any set of functions with this property is called total. In view of the following definition and theorem, this proposition implies that the weak topology on X is locally convex. Similar proofs show that the norm and weak*-topologies are also locally convex.

1.3.14 Definition

A topology on a linear space is said to be locally convex if it possesses a base consisting of convex sets.

1.3.15 Theorem

If Y is a total linear space of linear functionals on X, then the weakest topology on X for which the elements of Y are continuous is locally convex.

1.3.16 Theorem

For every x in a normed linear space (X, β),

$$\beta(x) = \sup \{ | f(x) | \mid f \in B_{\beta^*} \}$$

1.3.17 Definition

A normed linear space is called separable if it contains a countable dense subset.

1.3.18 Theorem

If the dual space X^* is separable, so is X.

We will now turn our attention to the closed unit ball and the unit sphere. It is obvious that every element in a normed linear space is the product of a non-negative scalar and some element of the unit sphere. In this sense, the unit sphere completely characterizes the norm.

1.3.19 Definition

Let (X^*, β^*) be the dual space of (X, β). Then $f \in X^*$ is said to be a normalized functional if $\beta^*(f) = 1$, that is, $f \in S_{\beta^*}$.

Since the unit ball in convex and the unit sphere is the set of its bounding points, Corollary 1.2.17 and the comments following definition 1.2.13 assure the existence of normalized support functionals to the unit ball at every point of the unit sphere.

1.3.20 Definition

A norm β is said to be smooth if there is a unique normalized support functional at each point of S_β.

1.3.21 Definition

A norm β is said to be rotund if S_β contains no line segments, or equivalently, every point of S_β is an extremal point.

A characterization of rotundity is given in the following lemma.

1.3.22 Lemma

Let (X, β) be a normed linear space. Then β is rotund if and only if for every pair of non-zero elements $x, y \in X$ that satisfy $\beta(x + y) = \beta(x) + \beta(y)$ we have that $x = ty$ for some scalar $t > 0$.

1.3.23 Theorem

Let (X, β) be a normed linear space, and let ρ be a continuous norm on X. Then, if ρ^* is rotund, ρ is smooth.

Proof:

Assume ρ^* is rotund and ρ is not smooth. Then, for some $X_0 \in S_\rho$ there exists two different normalized support functionals f_1 and f_2 at X_0. Since f_1 and f_2 are different it follows that f_1 is not a scalar multiple of f_2. However, it can be shown that $\rho^*\left(\frac{f_1 + f_2}{2}\right) = 1$. Therefore,

$$\rho^*(f_1 + f_2) = \rho^*(f_1) + \rho^*(f_2)$$

so, by Lemma 1.3.22, $f_1 = tf_2$ for some $t > 0$, which is a contradiction.

1.3.24 Theorem

If ρ^* is smooth, then ρ is rotund.

Proof:

Suppose ρ^* is smooth and ρ is not rotund. Then, for some $x, y \in S_\rho$, $\frac{x+y}{2}$ is in S_ρ. Therefore, there is a normalized support functional f at $\frac{x+y}{2}$ and it follows that f is also a normalized support functional at x and y. But then $\phi(x), \phi(y) \in X^{**}$ would be different normalized support functionals at f, which contradicts the smoothness property of ρ^*.

1.3.25 Theorem

A norm defined as the sum of two norms, one of which is rotund, is rotund.

Proof:

Assume ρ and ω are norms, and ρ is rotund. Define the norm α by $\alpha(x) = \rho(x) + \omega(x)$. Let $x, y \in X$ such that $x \neq 0$, $y \neq 0$ and $\alpha(x+y) = \alpha(x) + \alpha(y)$. Then

$$\begin{aligned}\rho(x) + \omega(x) + \rho(y) + \omega(y) &= \rho(x+y) + \omega(x+y)\\ &\leq \rho(x+y) + \omega(x) + \omega(y)\end{aligned}$$

Cancelling we get:

$$\rho(x) + \rho(y) \leq \rho(x+y).$$

Hence $\rho(x+y) = \rho(x) + \rho(y)$ which implies that $x = ty$ for some real $t > 0$, and the theorem is proved.

The following definitions and propositions offer a very useful characterization of unit balls.

1.3.26 Definition

Let (X, β) be a normed linear space. For $A \subset X$, define A^0, the polar of A, by

$$A^0 = \{f \mid f \in X^*, \mid f(x) \mid \leq 1 \text{ for all } x \in A\}.$$

Similarly, we define the polar of $B \subset X^*$, by

$$B^0 = \{x \mid x \in X, \mid f(x) \mid \leq 1 \text{ for all } f \in B\}.$$

The following list of facts concerning polars follows almost immediately from the definition. A and B can be subsets of X or X^* unless otherwise specified.

1.3.27 Lemma

i) If $A \subset B$ then $A^0 \supset B^0$.

ii) $A \subset (A^0)^0$.

iii) $(A^0)^0$ includes the convex balanced hull of A.

iv) For scalar $t \neq 0$, $(tA)^0 = \frac{1}{t} A^0$.

v) If $A \subset X^*$, then A^0 is closed.

1.3.28 Theorem

The polar of the unit ball in (X, ρ) is the unit ball of X^*. The converse is also true.

Proof:

Only the converse need be proved, since from the definition of the dual norm we get immediately that $(B_\rho)^0 = B_{\rho^*}$.

From this, and the fact that $(A^0)^0 \supset A$ for any subset A, we see that $(B_\rho*)^0 = (B_\rho^0)^0 \supset B\rho$.

To prove containment in the other direction, let $x \in (B_\rho*)^0$, and suppose $x \notin B_\rho$, then $\rho(x) > 1$. By using Corollary 1.3.13 of the Hahn Banach theorem, there exists a continuous linear functional f such that $\rho*(f) = 1$ and $f(x) = \rho(x)$, hence $| f(x) | > 1$. But this is a contradiction, and thus we have proven that $(B_\rho*)^0 = B_\rho$.

1.4 Differential Calculus

We shall now give a summary of the concepts of differential calculus on Banach spaces used in this paper.

1.4.1 Definition

A real or vector valued function f on a Banach space X is said to have an upper Gateaux differential at $x \in X$, denoted $f'x$, provided

$$(f'x)u = \lim_{t \to 0^+} \frac{f(x + tu) - f(x)}{t}$$

exists for all $u \in X$.

We say that f is upper Gateaux differentiable at x if f has an upper Gateaux differential at x. If f is upper Gateaux differentiable at every non-zero element of X, then f is upper Gateaux differentiable.

1.4.2 Lemma

If a continuous norm ρ on (X, β), a Banach space, is upper

Gateaux differentiable at each point $x \in S_\rho$, then ρ is upper Gateaux differentiable and $(\rho'(tx))u = (\rho'x)u$ for each $t > 0$ and every $u \in X$.

Proof:

Using the linear property of ρ, we have immediately, since $t > 0$, that

$$(\rho'(tx))u = \lim_{s\to 0^+} \frac{\rho(tx + su) - \rho(tx)}{s}$$

$$= \lim_{s\to 0^+} \frac{\rho(x + \frac{s}{t}\cdot u) - \rho(x)}{\frac{s}{t}}$$

$$= (\rho'x)u .$$

This fact, and the result that every non-zero element of X is the product of a positive scalar with some element of S_ρ imply the desired result.

It is obvious from this proposition that the existence of a differentiable norm on X is equivalent to the existence of a norm which is differentiable at every point of S_ρ.

The following results are due to James [8] and Mazur [14]. We first show that the differential of a continuous norm is a continuous prenorm. Then, using this prenorm in the Hahn Banach theorem, we characterize the support functionals at $x \in X$ with respect to the upper Gateaux differential of the norm at x.

1.4.3 Lemma

Let (X, β) be any arbitrary Banach space and ρ a continuous norm on X. Let $x \in X$, then ρ is upper Gateaux differentiable at x, and

1) $(\rho' x)(y + z) \leq (\rho' x)y + (\rho' x)z$,

2) $(\rho' x)(ty) = t(\rho' x)y$ for $t > 0$,

3) $(\rho' x)y \leq \rho(y)$, and

4) $-(\rho' x)(-y) \leq (\rho' x)y$,

for every $y, z \in X$.

Proof:

If $t_1 > t_2 > 0$, then for $y \in X$

$$\frac{\rho(x + t_1 y) - \rho(x)}{t_1} - \frac{\rho(x + t_2 y) - \rho(x)}{t_2}$$

$$= \frac{t_2 \cdot \rho(x + t_1 y) - t_1 \cdot \rho(x + t_2 y) + (t_1 - t_2) \cdot \rho(x)}{t_1 \cdot t_2}$$

≥ 0, since it follows from the subadditive property of the norm that,

$$t_2 \cdot \rho(x + t_1 y) - t_1 \cdot \rho(x + t_2 y)$$

$$= \rho(t_2 x + t_1 t_2 y) - \rho(t_1 x + t_1 t_2 y)$$

$$\geq -(t_1 - t_2) \cdot \rho(x).$$

It also is a consequence of the same property that,

$$\frac{\rho(x + ty) - \rho(x)}{t} \geq -\rho(y).$$

Thus we have shown that the ratio $\frac{\rho(x + ty) - \rho(x)}{t}$ is a monotonically increasing function of t, which is also bounded below. It follows that $(\rho'x)y$ exists for every y.

1) Let $s > 0$, then for $y, z \in X$ we have

$$2\rho(x + \frac{s}{2}(x + z))$$

$$= \rho(x + sy + x + \ z)$$

$$\leq \rho(x + sy) + \rho(x + sz)$$

and so

$$2 \cdot \frac{\rho(x + \frac{s}{2} \cdot (y + z)) - \rho(x)}{s}$$

$$\leq \frac{\rho(x + sy) - \rho(x)}{s} + \frac{\rho(x + sz) - \rho(x)}{s}$$

which implies that

$$2 \cdot (\rho'x)\left(\frac{y + z}{2}\right) \leq (\rho'x)y + (\rho'x)z.$$

By using property 2) of this theorem we immediately get the desired inequality.

2) Suppose $t > 0$, then for $y \in X$, we have

$$\frac{\rho(x + tsy) - \rho(x)}{s}$$

$$= t \cdot \frac{\rho(\frac{x}{t} + sy) - \rho(\frac{x}{t})}{s}$$

This implies that $(\rho'x)(ty) = t \cdot (\rho' \frac{x}{t})y$; hence, by lemma 1.4.2, we have

$$(\rho' x)(ty) = t \cdot (\rho' x) y.$$

3), 4) These inequalities follow immediately from the subadditive property of the norm.

We note, in passing, that property 2) implies that a norm is differentiable at x if the norm is differentiable at each $x \in S_\rho$.

1.4.4 Lemma

Let (X, β) be any arbitrary Banach space and ρ a continuous norm on X. If $x_0, y_0 \in X$ and a is a real such that $-(\rho' x_0)(-y_0) \leq a \leq (\rho' x_0)\, y_0$, then there exists a continuous linear functional f on X, such that $\rho^*(f) = 1$, $f(x_0) = \rho(x_0)$ and $f(y_0) = a$.

Proof:

Consider the set of elements

$$C = \{x \mid x = tx_0 + sy_0\}, \quad t, s \in R$$

and define g on C by $g(x) = t\rho(x_0) + sa$. We note that C is, in fact, a linear subspace and that g is a well defined continuous linear functional on C.

Let y be an element of C, then $y = t_0 x_0 + s_0 y_0$ for some $t_0, s_0 \in R$, and for $r > 0$, we have

$$\frac{\rho(x_0 + ry) - \rho(x_0)}{r}$$

$$= t_0 \cdot \rho(x_0) + \frac{1 + t_0 r}{r} \left[\rho(x_0 + \frac{rs_0}{1 + rt_0} \cdot y_0) - \rho(x_0) \right]$$

from which we get

$$(\rho' x_0)y = t_0 \cdot \rho(x_0) + (\rho' x_0)\ (s_0 y_0).$$

But, since $s_0 a \leq s_0 \cdot (\rho' x_0)y_0 = (\rho' x_0)\ (s_0 y_0)$

and

$$g(y) = t_0 \cdot \rho(x_0) + s_0 a$$

we have $g(y) \leq (\rho' x_0)y$.

Now by lemma 1.4.3, $\rho' x_0$ and g satisfy the hypothesis of the Hahn Banach extension theorem, so there exists an $f \in X^*$, such that $f(x) \leq (\rho' x_0)x$ for all $x \in X$ and $f(x) = g(x)$ on C. It follows that $f(x_0) = \rho(x_0)$ and $f(y_0) = a$. Further, by the previous lemma, $f(x) \leq \rho(x)$ for all $x \in X$ which implies immediately that $\rho^*(f) = 1$.

It follows directly that if $x_0 \in S\rho$ then f is a normalized support functional of x_0.

<u>1.4.5 Definition</u>

A function f on a Banach space X to a Banach space Y is said to have a <u>Gateaux differential</u> at x X, provided

$$(f' x)u = \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t}$$

exists for all $u \in X$. It follows from Lemma 1.4.2 that a norm ρ has a Gateaux differential at $x \in X$ if and only if $(\rho' (\frac{x}{\rho(x)})u$ exists for every $u \in S_\rho$).

1.4.6 Theorem

If (X, β) is a Banach space, and ρ is a continuous norm on X, then, if ρ is Gateaux differentiable at $x_0 \varepsilon X$, $\rho' x_0$ is a continuous linear functional, with $\rho^*(\rho' x_0) = 1$ and $(\rho' x_0)x_0 = \rho(x_0)$.

Proof:

Consider any $y_0 \varepsilon X$; using lemma 1.4.4, we can find a continuous linear functional f such that,

$$\rho^*(f) = 1, \quad f(x_0) = \rho(x_0) \quad \text{and} \quad f(y_0) = (\rho' x_0)y_0.$$

From this we get

$$\pm\, t\, f(y) \leq \rho(x_0 \pm ty) - \rho(x_0), \quad \text{for} \quad t > 0.$$

Hence

$$\frac{\rho(x_0 - ty) - \rho(x_0)}{-t} \leq f(y) \leq \frac{\rho(x_0 + ty) - \rho(x_0)}{t}$$

which implies

$$-(\rho' x_0)(-y) \leq f(y) \leq (\rho'(x_0))y$$

for all y.

Since ρ is Gateaux differentiable at x_0 we immediately get $f = \rho' x_0$ and the theorem is proved.

In the following proposition, we demonstrate the intuitive classical notion of the equivalence of differentials and unique tangents.

1.4.7 Lemma

Assume ρ is a continuous smooth norm on (X, β). Then ρ is

Gateaux differentiable on S_ρ.

Proof:

Suppose ρ is not Gateaux differentiable at $x_0 \in S_\rho$. Then there are reals, a and b, and $y_0 \in X$ such that,

$$(\rho' x_0)\ y_0 \geq a > b \geq -(\rho' x_0)\ (-y_0).$$

Then, by lemma 1.4.4, there are normalized support functionals f, g at x_0 such that $f(y_0) = a \neq b = g(y_0)$. But this contradicts the smoothness property of ρ.

<u>1.4.8 Definition</u>

Let (X, β) be any arbitrary Banach space and assume ρ is a continuous smooth norm on X. The mapping $\gamma: S_\rho \to S_{\rho^*}$, called the <u>support mapping</u>, assigns to each $x \in S_\rho$ the unique normalized support functional at x.

<u>1.4.9 Corollary</u>

Assume ρ is a continuous smooth norm on (X, β). Then $(\gamma(x_0))y = (\rho' x_0)y$ for all $y \in X$ and $x_0 \in S_\rho$.

Proof:

By lemma 1.4.7, we see that $(\rho' x_0)y$ exists for every $y \in X$, and, by corollary 1.4.6, $\rho' x_0$ is a normalized support functional at x_0. Smoothness immediately implies the conclusion of the theorem.

It is also true that a continuous Gateaux differentiable norm is smooth.

1.4.10 Definition

Let (X, β) and (Y, ρ) be any two Banach spaces. The function $f\colon X \to Y$ has a Fréchet differential at $x_0 \in X$, provided there is $f' x_0 \in L(X, Y)$ such that for every $\mu > 0$ there is a $\delta > 0$ such that $\beta(x - x_0) < \delta$ implies

$$\rho[f(x) - f(x_0) - (f'x_0)(x - x_0)] \leq \mu \cdot \beta(x - x_0).$$

We will say that f is differentiable at x_0 if f has a differential at x_0.

The Fréchet differential is the Gateaux differential for which the limit is uniform with respect to direction, and so, if a function f is Fréchet differentiable at $x \in X$, then f is Gateaux differentiable at x and the differentials are equal.

If f is Fréchet differentiable at all points of X, then we get a map $f'\colon X \to L(X, Y)$, given by $x \to f'x$, which is called the Fréchet derivative of f. Note that the differentiability of f is independent of the choice of equivalent norms on X and Y.

Since $L(X, Y)$ is a Banach space it is possible to consider continuity and differentiability of f' and hence higher order derivatives of f.

Let $L_n(X, Y)$ denote the Banach space of continuous n-multilinear maps. Define $L_0(X, Y)$ to be Y. Now, since $L_{n+1}(X, Y)$ is naturally isometric to $L(X, L_n(X, Y))$, with its sup norm, we may consider the n-th derivative of f (if it exists) as a map $f^{(n)}\colon X \to L_n(X, Y)$.

1.4.11 Definition

The function $f\colon X \to Y$ is said to be of class C^n, $0 < n < \infty$, on X, if f is n-times Fréchet differentiable on X and $f^{(n)}\colon X \to L_n(X, Y)$ is continuous. If f is continuous on X, then f is of class C^0. If f is of class C^n for all n, $0 \leq n < \infty$, then f is of class C^∞.

The following propositions illustrate the strong similarity between infinite dimensional calculus and calculus of the real line.

1.4.12 Theorem

Let X, Y, Z be Banach spaces, and let $f\colon X \to Y$ be differentiable at $x \in X$, and let $g\colon Y \to Z$ be differentiable at $f(x) \in Y$. Then $g \circ f\colon X \to Z$ is differentiable at $x \in X$, and $(g \circ f)'x = g'(f(x)) \circ f'(x)$.

1.4.13 Theorem

Let (X, β) and (Y, ρ) be Banach spaces. Then,

1) if $f\colon X \to Y$ is a constant, $f'(x) = 0$ for each $x \in X$;

2) if $f\colon X \to Y$ is a continuous linear map, f is differentiable and $f'(x) = f$ for each $x \in X$.

1.4.14 Definition

By a Hilbert space we mean a Banach space (X, β) on which there is defined a function (x, y) from $X \times X$ to R with the following properties

(i) $(tx_1 + sx_2, y) = t(x_1, y) + s(x_2, y)$

(ii) $(x, y) = (y, x)$

(iii) $(x, x) = [\beta(x)]^2$.

We call (x, y) the inner product of x and y. Since $\beta(0) \geq 0$ with equality only for $x = 0$, we have $0 \leq [\beta(x - ty)]^2$

$$= (x - ty, x - ty)$$

$$= (x, x) - 2t\,(x, y) + t^2(y, y).$$

If $t > 0$, we have $2(x, y) \leq t^{-1}\,[\beta(x)]^2 + t\,[\beta(y)]^2$. Setting $t = \frac{\beta(x)}{\beta(y)}$, we obtain $(x, y) \leq \beta(x) \cdot \beta(y)$, and we see that equality can only occur when $y = 0$ or $x = ty$ for some scalar $t \geq 0$. This inequality is variously known as the Schwarz or Cauchy - Schwarz inequality.

As an example of a Hilbert space consider the Banach space ℓ^2, the set of all real sequences x such that $\sum_{k=1}^{\infty} |x_k|^2 < \infty$, with the norm ρ defined by $\rho(x) = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2}$. The natural inner product making ℓ^2 a Hilbert space is defined by $(x, y) = \sum_{k=1}^{\infty} x_k y_k$.

CHAPTER II

A SMOOTH NORM

2.1 Introduction

V. Klee in [11] proves the following: if X is a Banach space whose dual space X^* is separable, then there is an admissible norm ρ on X such that the dual norm ρ^* on X^* is rotund and such that whenever $\{f_n\}$ is a sequence in X^*, $f \in X^*$ and $f_n \to f$ in the w^*-topology of X^*, and $\rho^*(f_n) \to \rho^*(f)$, then $\rho^*(f_n - f) \to 0$, that is $f_n \to f$ in the norm topology of X^*. Klee's proof is indirect in the sense that he uses a construction, due to Kadec [9], of such a norm for $C[0, 1]$ and the fact that every separable Banach space is isometrically isomorphic to a subspace of $C[0, 1]$.

The principle result in this chapter is a direct and constructive proof of Klee's theorem, which uses a modification of Whitfield's [20] approach, suggested by Rainwater [16]. To this end, we define two norms, α^* and ω^* on X^*, the separable dual of a Banach space (X, β), which generate a third norm ρ^* that has the properties of Klee's result. The norm α^* on X^*, derived using the separability property of X is rotund and continuous with respect to the given norm β^* on X^*. The separability of the dual space X^* is used to generate the equivalent norm ω^*. The norm ρ^* is defined as the algebraic sum of the above norms, and we will show that it is the dual of an equivalent norm ρ on X and that its inherited properties satisfy the hypothesis of Klee's theorem.

Many of the following propositions are slight generalizations of ideas used in the construction and may be useful in their own right.

The proofs that certain functions defined in this chapter are, in fact, norms are elementary and will be omitted. On the other hand, we will attempt to establish some of the more obscure properties of these functions, such as semi-continuity, which we now introduce.

2.1.1 Definition

A real valued mapping f of a Banach space X into the reals is called $(\omega-, \omega^*-)$ lower semicontinuous if $\{x | x \in X$ and $f(x) \leq t\}$ is closed in the norm $(\omega-, \omega^*-)$ topology on X for each real t. The following result is useful characterization of semicontinuity: f is lower semicontinuous if and only if for each sequence $\{x_n\} \subset X$, with $x_n \to x$ we have $\lim\inf f(x_n) \geq f(x)$.

If $-f$ is lower semicontinuous, we call f upper semicontinuous.

2.2 The Norm ω^*

2.2.1 Lemma

Let (X, β) be a Banach space and A a bounded subset of X. Define the real valued function ω^* on X^*, the dual of X, by $\omega^*(f) = \sup\{|f(x)| \mid x \in A\}$. Then ω^* is w^*-lower semicontinuous.

Proof:

Since A is bounded, we have that $\omega^*(f)$ exists for every $f \in X^*$.

Let $\{g_n\}$ be any arbitrary w^*-convergent sequence in X^*, that is $g_n \to g$ (weak*), and suppose $\mu > 0$. There exists $x_0 \in A$ such that $|g(x_0)| > \omega^*(g) - \frac{\mu}{2}$. Also, by w^*-convergence, there exists a positive integer N such that if $n \geq N$ then $|g_n(x_0)| > |g(x_0)| - \frac{\mu}{2}$, but $\omega^*(g_n) \geq |g_n(x_0)|$ for every n, so for $n \geq N$ we have $\omega^*(g_n) \geq \omega^*(g) - \mu$, hence $\liminf \omega^*(g_n) \geq \omega^*(g)$ and the theorem is proved.

2.2.2 Lemma

The geometric sum of positive real valued uniformly bounded w^*-lower semicontinuous functions on X^* is w^*-lower semicontinuous.

Proof:

Assume ω_n^* are positive real valued uniformly bounded functions on X^*, which are w^*-lower semicontinuous. Define ω^* on X^* by $\omega^*(f) = \sum_{n=1}^{\infty} 2^{-n} \omega_n^*(f)$ and let $\{f_m\}$ be any sequence in X^* such that $f_m \to f$ (weak*).

Suppose we are given $\mu > 0$. Then, since the ω_n^* are uniformly bounded, there exists a positive integer M such that

$$\omega^*(f) \leq \sum_{n=1}^{M} 2^{-n} \omega_n^*(f) + \frac{\mu}{2}$$

Further, since the ω_n^* are w^*-lower semicontinuous, we can choose, for every n, a $N_{(n)}$ such that if $m \geq N_{(n)}$ then $\omega_n^*(f_m) \geq \omega_n^*(f) - \frac{\mu}{2}$. Choose $N = \max \{N_{(n)}, |1 \leq n \leq M\}$. Then, if $m \geq N$, we have

$$\omega^*(f_m) \geq \sum_{n=1}^{M} 2^{-n} \omega^*_n(f_m)$$

$$\geq \sum_{n=1}^{M} 2^{-n} [\omega^*_n(f) - \frac{\mu}{2}]$$

$$\geq \sum_{n=1}^{M} 2^{-n} \omega^*_n(f) - \frac{\mu}{2}$$

$$\geq \omega^*(f) - \frac{\mu}{2} - \frac{\mu}{2} = \omega^*(f) - \mu,$$

hence, $\lim\inf \omega^*(f_m) \geq \omega^*(f)$, which proves that ω^* is w^*-semi-continuous.

2.2.3 Lemma

Let $\{h_n\}$ be a countable set of continuous linear functionals in the unit sphere of X^*, the dual space of a Banach space (X, β). Let $B_n = \{x \mid x \in B_\beta$ and $|h_n(x)| \leq \frac{1}{n}\}$ and define the functions ω_n^* on X^*, by $\omega^*_n(f) = sup\{|f(x)| \mid x \in B_n\}$. The functions ω^*_n are equivalent, w^*-lower semicontinuous norms on X^*.

Proof:

Since $\frac{1}{n} B_\beta \subset B_n \subset B_\beta$, we have $\frac{1}{n} \cdot \beta^*(f) \leq \omega^*_n(f) \leq \beta^*(f)$ for every $f \in X^*$, hence ω^*_n is an equivalent norm on X^* for every n. Each ω^*_n satisfies the hypothesis of lemma 2.2.1 and is, thus w^*-lower semicontinuous.

2.2.4 Lemma

The geometric sum of equivalent norms, which are uniformly bounded above, is an equivalent norm.

Proof:

Assume ω_n are admissible norms defined on a Banach space (X, β)

and that they are uniformly bounded. Then there are constants $a_n > 0$, $b > 0$ such that

$$a_n \cdot \beta(x) \le \omega_n(x) \le b \cdot \beta(x)$$

for every $x \in X$ and every n.

Consider the function ω defined on X by $\omega(x) = \sum_{n=1}^{\infty} 2^{-n} \omega_n(x)$. It is certainly a norm and, since $\omega(x) \le \sum_{n=1}^{\infty} 2^{-n} b \cdot \beta(x) \le b \cdot \beta(x)$ and $\omega(x) \ge \frac{1}{2} \omega_1(x) \ge \frac{a_1}{2} \cdot \beta(x)$ for every $x \in X$, it is admissible on X.

If a Banach space (X, β) has a separable dual space, then the unit sphere of the dual space contains a countable dense subset. Denote such a set by $\{h_n\}$, and define the norm ω^* on X^* by,

$$\omega^*(f) = \sum_{n=1}^{\infty} 2^{-n} \omega^*_n(f),$$

where the norms ω^*_n are defined as in lemma 2.2.3 using the dense set $\{h_n\}$ above. Since $\omega^*_n(f) \le \beta^*(f)$ for each n, ω^* is an equivalent w^*-lower semicontinuous norm on X^*.

In the following lemma, we consider an additional property of the norm ω^*.

<u>2.2.5 Lemma</u>

Consider the equivalent norm ω^* defined above. Suppose $\{f_m\}$ is a sequence in X^* such that $f_m \to f$ (weak*) and $\omega^*(f_m) \to \omega^*(f)$; then there exists a subsequence $\{f_\zeta\} \subset \{f_m\}$ such that

$\lim_{j\to\infty} \omega^*_n(f_\varsigma)$ exists for each n, and $\omega^*_n(f) = \lim_{j\to\infty} \omega^*_n(f_\varsigma)$.

Proof:

Consider $\{\omega_1{}^*(f_m)\}$; it is a bounded set of reals and hence has a cluster point or is finite. Therefore, we can choose a subsequence of $\{f_m\}$: $f^1{}_{m_1}, f^1{}_{m_2}, \ldots, f^1{}_{m_s}, \ldots$ such that $\lim_{s\to\infty} \omega_1{}^*(f^1{}_{m_s})$ exists. Similarly, we can choose a subsequence $\{f^2{}_{m_s}\}$ of $\{f^1{}_{m_s}\}$ such that $\lim_{s\to\infty} \omega_2{}^*(f^2{}_{m_s})$ exists, and so on.

Denote the sequence $f^1{}_{m_1}, f^2{}_{m_2}, f^2{}_{m_3}, \ldots$ by $\{f_\varsigma\}$. Clearly, $\lim_{j\to\infty} \omega_n{}^*(f_\varsigma)$ exists for every n.

Furthermore, because of the w^*-lower semicontinuity of the $\omega_n{}^*$, we have that $\omega_n{}^*(f) \le \lim\inf \omega_n{}^*(f_\varsigma) = \lim \omega_n{}^*(f_\varsigma)$ for every n.

Also

$$\omega(f) = \sum_{n=1}^{\infty} 2^{-n}\, \omega_n{}^*(f),$$

and

$$\omega^*(f) = \lim \omega^*(f_s) = \sum_{n=1}^{\infty} 2^{-n} \lim \omega_n{}^*(f_s),$$

hence $\omega_n{}^*(f) = \lim \omega_n{}^*(f_\varsigma)$ for every n, which concludes the proof.

2.3 The Norm α^*

We now develop those concepts involved in the development of the norm α^* mentioned in the introduction.

2.3.1 Lemma

Let (X, β) be a Banach space and x_0 a fixed element of X.

Define the real valued function α_0^* on X^* by $\alpha_0^*(f) = [f(x_0)]^2$. Then α_0^* is w^*- lower semicontinuous.

Proof:

Let $\{f_n\}$ be a w^*-convergent sequence in X^* such that $f_n \to f$ (weak*). Then there exists a constant t such that $|f_n(x_0)| \leq t$ for every n.

Suppose $\mu > 0$, and assume, without loss of generality, that $\mu \leq 8t$. Then, by w^*-convergence there exists a positive integer N such that, if $n \geq N$, then $| f_n(x_0) | + \frac{\mu}{4t} > | f(x_0) | \geq 0$.

Thus, if $n \geq N$, we have

$$[f_n(x_0)]^2 + \frac{| f_n(x_0) | \cdot \mu}{2t} + \frac{\mu^2}{16t} \geq [f(x_0)]^2$$

so

$$[f_n(x_0)]^2 \geq [f(x_0)]^2 - \frac{| f_n(x_0) | \cdot \mu}{2t} - \frac{\mu^2}{16t}$$

$$\geq [f(x_0)]^2 - \mu;$$

hence, $\liminf \alpha_0^*(f_n) \geq \alpha_0^*(f)$ and the theorem is proved.

By a similar proof, it can be shown that the square root of a w^*-lower semicontinuous function is w^*-lower semicontinuous. Using this fact and lemma 2.2.2, it is evident that the norm α^* on X^* defined in the following lemma is w^*-lower semicontinuous.

2.3.2 Lemma

Assume (X, β) is a separable Banach space. Then its unit sphere contains a countable dense subset, say $\{y_n\}$. Define α^* of X^* by,

$$\alpha^*(f) = \left(\sum_{n=1}^{\infty} 4^{-n} f(y_n)^2 \right)^{\frac{1}{2}}$$

Then α^* is a continuous norm on X^*.

Proof:

Obviously α^* is a prenorm, since $\alpha^*(tf) = |t| \alpha^*(f)$, $\alpha^*(f) \geq 0$, and $\alpha^*(f + g) \leq \alpha^*(f) + \alpha^*(g)$ for $f, g \in X^*$ and $t \in R$.

Suppose $\alpha^*(f) = 0$ for some $f \in X^*$. Then $f(y_n) = 0$ for every n. Let x_0 be any non-zero element in X; then $\frac{x_0}{\beta(x_0)} \in S_\beta$. Further, suppose $\mu > 0$. Then, since f is continuous, there exists a real number $\delta > 0$ such that if $\beta(\frac{x_0}{\beta(x_0)} - x) < \delta$, $x \in X$, then $\left| f\left(\frac{x_0}{\beta(x_0)}\right) - f(x) \right| < \frac{\mu}{\beta(x_0)}$. But $\{y_n\}$ is dense in S_β, hence there exists n_0 such that $\beta(\frac{x_0}{\beta(x_0)} - y_{n_0}) < \delta$. Combining these results, we have that $\left| f\left(\frac{x_0}{\beta(x_0)}\right) - f(y_{n_0}) \right| < \frac{\mu}{\beta(x_0)}$, which implies that $| f(x_0) | < \mu$. Now, since μ and x_0 are arbitrary, we see that f is identically zero on X, which proves that α^* is, in fact, a norm.

Also, since $| f(x) | \leq \beta^*(f) \cdot \beta(x)$,

$$\alpha^*(f) = \left(\sum_{n=1}^{\infty} 4^{-n} f(y_n)^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{n=1}^{\infty} 4^{-n} \beta^*(f)^2 \right)^{\frac{1}{2}}$$

$$\leq \beta^*(f)$$

for every $f \in X^*$, which proves that α^* is continuous with respect to β^*.

2.3.3 Lemma

The norm α^* on X^*, defined above, is w^*-continuous on bounded subsets of X^*.

Proof:

The norm α^* is w^*-lower semicontinuous everywhere on X^*, hence we have only to show that it is w^*-upper semicontinuous on bounded sets, since a function which is w^*-upper and w^*-lower semicontinuous is w^*-continuous.

Let $\{f_\zeta\}$ be a bounded w^*-convergent sequence such that $f_\zeta \to f$ (weak*) and suppose $\mu > 0$. Then, since $\{f_\zeta\}$ is bounded, there exists a positive integer M such that

$$\sum_{n=1}^{\infty} 4^{-n} [f_\zeta (y_n)]^2$$

$$\leq \sum_{n=1}^{M} 4^{-n} [f_\zeta (y_n)]^2 + \frac{\mu}{2},$$

for all ζ. Further, by a proof similar to lemma 2.3.1, we know that $[f(x_0)]^2$, where x_0 is a fixed element of X, is a w^*-upper semicontinuous function on X^*. So, we can choose, for every n, a natural number $N_{(n)}$ such that if $\zeta \geq N_{(n)}$, then

$$[f_\zeta (x_n)]^2 \leq [f(x_n)]^2 + \frac{3}{2} \cdot \frac{\mu}{2}.$$

Choose $N = max \{N_{(n)} \mid 1 \leq n \leq M\}$. Then, if $\zeta \geq N$, we have

$$[\alpha^*(f_\zeta)]^2 \leq \sum_{n=1}^{M} 4^{-n} [(f_\zeta (x_n)]^2 + \frac{\mu}{2}$$

$$\leq \sum_{n=1}^{M} 4^{-n} \quad [f(x_n)]^2 + \frac{\mu}{2} \cdot \mu \quad + \frac{\mu}{2}$$

$$\leq [\alpha^*(f)]^2 + \mu .$$

So, given $\delta > 0$, and since α^* is a non-negative function, we can define

$$\mu = 2 \cdot \alpha^*(f) \cdot \delta + \delta^2 > 0$$

and using the above argument, get

$$\alpha^*(f_\zeta) \leq \alpha^*(f) + \delta \quad \text{for} \quad \zeta \geq N,$$

hence $\alpha^*(f) \geq \limsup \alpha^*(f_\zeta)$, and so α^* is w^*-upper semicontinuous on $\{f_\zeta\}$ which is arbitrary hence α^* has this property on bounded sets.

2.3.4 Lemma

The norm α^* on X^* is rotund.

Proof:

Let f, g be non-zero elements of X^* such that $\alpha^*(f + g) = \alpha^*(f) + \alpha^*(g)$. Then we have,

$$\sum_{n=1}^{\infty} 4^{-n} \mid f(y_n) \mid \cdot \mid g(y_n) \mid$$

$$= \left(\sum_{n=1}^{\infty} 4^{-n} [f(y_n)]^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{n=1}^{\infty} 4^{-n} [g(y_n)]^2 \right)^{\frac{1}{2}}$$

We observe that $\{\mid f(y_n) \mid /2^n\}$ and $\{\mid g(y_n) \mid /2^n\}$ are, in fact, elements of the Hilbert space ℓ^2. The factors on the right of the equality are the respective norms of these elements and the terms on the left is the inner product of these elements. The Cauchy - Schwarz inequality for inner products immediately implies that there exists a real constant $t \geq 0$, such that $tf = g$. It follows from lemma 1.3.22 that α^* is rotund.

2.4 ρ and ρ^*

Suppose (X, β) is a Banach space, whose dual space, X^*, is separable. Then X is separable, and as we have done above, we can fix countable dense subsets of the respective unit spheres, and define the associated norms ω^* and α^* on X^*.

On X^* define another norm ρ^* by

$$\rho^*(f) = \omega^*(f) + \alpha^*(f).$$

Clearly, ρ^* is w^*-lower semicontinuous, and by theorem 1.3.25, rotund. Also, since $\rho^*(f) = \alpha^*(f) + \omega^*(f) \leq 2\beta^*(f)$ and $\rho^*(f) \geq \omega^*(f) \geq \frac{1}{2}\beta^*(f)$ for every $f \in X^*$, ρ^* is an equivalent norm on X^*.

In the following lemmas, we will show that this norm, ρ^*, is, in fact, the dual norm of an equivalent norm ρ on X.

The following lemma, which is actually a generalization of theorem 1.3.28, has, to some extent, motivated our use of the concept of w^*-semicontinuity.

2.4.1 Lemma

Let (X, β) be a Banach space and suppose $A \subset X^*$ is a convex, balanced and w^*-closed set, which has the origin as an internal point. Then $(A^0)^0 = A$.

Proof:

By lemma 1.3.27, we immediately see that $(A^0)^0 \supset A$.

Let $f \in X^* - A$. Then, since A is w^*-closed, f and A can be strictly separated (theorem 1.2.16). That is, there exists $x_0 \in X$ and a real number $t > 0$ such that $f(x_0) > t$ and $g(x_0) < t$ for all g in A. It follows that $|\ g(x_0)\ | \leq t$ for all g in A, since A is balanced, and it follows that $\frac{x_0}{t}$ is in A^0.

But $|\ f(\frac{x_0}{t})\ | > 1$, hence f is not in $(A^0)^0$, and since f is arbitrary we have $(A^0)^0 \subset A$.

Since the dual unit ball is convex, balanced and w^*-closed, by Alaoglu's theorem (theorem 1.3.9), this lemma immediately implies that $(S^{*}_{\beta}{}^{0})^0 = S^{*}_{\beta}$.

<u>2.4.2 Lemma</u>

Assume (X, β) is a Banach space and that B is a convex, balanced subset of X, which has the origin as an internal point. Define the real valued function ρ on X by

$$\rho(x) = inf \{t \mid t > 0,\ t \in R \text{ and } t^{-1} x \in B\}.$$

Then ρ is a prenorm on X.

Proof:

We first show that ρ is subadditive. Let x, y be elements of B and suppose $r > \rho(x) + \rho(y)$, then $r = s + t$ where $s > \rho(x)$ and $t > \rho(y)$. $s^{-1}x$ and $t^{-1}y$ are in B. Then, since B is convex, we have

$$\frac{(x + y)}{r} = \frac{(x + y)}{(s + t)}$$

$$= \frac{s(s^{-1}x) + t(t^{-1}y)}{(s + t)}$$

is in B, and it follows that $\rho(x + y) \leq r$, hence $\rho(x + y) \leq \rho(x) + \rho(y)$.

From the definition we see that $\rho(tx) = t\rho(x)$ for $t \geq 0$. Now, since B is balanced we have that $-tx \in B$ if and only if $tx \in B$ and so $\rho(tx) = \rho(-tx) = -t\ \rho(x) = |t|\ \rho(x)$ for $t < 0$. This establishes the homogeneity property of ρ.

Obviously, since the infimum is taken over positive scalars, $\rho(x) \geq 0$ for all x in X, and, thus, the proposition is proved.

This prenorm is often called the support function of the set B.

2.4.3 Lemma

Suppose (X, β) is a Banach space, and assume that ρ^* is an admissible norm on X^*, the dual of X. Then the support function ρ of $(\beta_{\rho^*})^0$ is an equivalent norm on X.

Proof:

The fact that $(B_{\rho^*})^0$ is convex, balanced, and has the origin as an interior point, is an immediate result of the elementary properties of polars; therefore, the above lemma shows that ρ is a prenorm.

Since ρ^* is an equivalent dual norm on X^*, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\ \beta^*(x) \leq \rho^*(x) \leq c_2\ \beta^*(x) \quad \text{for all } x \in X^*,$$

hence

$$c_1 \cdot B_{\beta^*} \subset B_{\rho^*} \subset c_2 \cdot E_{\beta^*}$$ and, using

the properties of polars proved in lemma 1.3.27 we have

$$(c_1 \cdot B_{\beta^*})^0 \supset (B_{\rho^*})^0 \supset (c_2 \cdot B_{\beta^*})^0$$

so

$$\frac{1}{c_1} \cdot (B_{\beta^*})^0 \supset (B_{\rho^*})^0 \supset \frac{1}{c_2} \cdot (B_{\beta^*})^0.$$

By theorem 1.9, we then have

$$\frac{1}{c_1} \cdot B_\beta \supset (B_{\rho^*})^0 \supset \frac{1}{c_2} B_\beta$$

and it follows that ρ is a norm, for if $\rho(x) = 0$ for some $x \in X$, then $\beta(x) = 0$, which implies that $x = 0$.

Clearly, ρ is a admissible norm on X, since $\frac{1}{c_1} \cdot \beta(x) \geq \rho(x) \geq \frac{1}{c_2} \cdot \beta(x)$ for every $x \in X$.

Recall the norm ρ^* defined on the separable dual space X^*, at the beginning of 2.4. If we define ρ, as above, to be the support function of the polar of the unit ball of ρ^*, then, by the above lemma, ρ is an equivalent norm on X. Further, ρ^* is w^*-lower semicontinuous. This implies that its unit ball is w^*-closed; therefore, theorem 1.3.28 and lemma 2.4.1, together, imply that ρ^* is indeed the dual norm of ρ.

2.5 Properties of ρ and ρ^*

To prove Klee's theorem we have only to show that ρ^* has the desired convergence property.

We digress somewhat and prove an elementary theorem due to Phelps

[15], which asserts the intuitively obvious fact that if the kernel of two functionals, with equal norms, are sufficiently close, then either the value of their sum, or the value of their difference is small.

2.5.1 Theorem

Suppose $\mu > 0$ and that (X, β) is a Banach space. If f and g are in the dual, X^*, and if $\beta^*(f) = \beta^*(g) = 1$ and $f(x) = 0$, $x \in B_\beta$ implies that $|g(x)| \leq \frac{\mu}{2}$, then either $\beta^*(f - g) \leq \mu$ or $\beta^*(f + g) \leq \mu$.

Proof:

By the Hahn Banach theorem, we can choose h in X^* such that $h = g$ on $f^\perp$ and $\beta^*(h) = sup\ \{|g(B_\beta \cap f^\perp)|\}$. Then $\beta^*(h) \leq \frac{\mu}{2}$.

Furthermore, since $g - h = 0$ on $f^\perp$ there exists $t \in R$ such that $g - h = t \cdot f$. (lemma 1.2.10). Hence $\beta^*(g - tf) = \beta^*(h) \leq \frac{\mu}{2}$.

Suppose $t \geq 0$. If $t \geq 1$, then $t^{-1} \leq 1$ and $\beta^*(g - f)$

$$= \beta^*[(1 - t^{-1})\, g + t^{-1}\, (g - tf)]$$

$$\leq 1 - t^{-1} + t^{-1} \cdot \beta^*(g - tf).$$

Also $$t = \beta^*(tf) \leq \beta^*(g) + \beta^*(g - tf)$$

so $$1 - t^{-1}$$

$$\leq [1 + \beta^*(g - tf)]^{-1} \cdot \beta^*(g - tf)$$

$$\leq \beta^*(g - tf).$$

Hence $\beta^*(g - f) \leq 2 \cdot \beta^*(g - tf) \leq \mu$.

If $0 \leq t < 1$, then $\beta^*(g - f)$

$\leq \beta^*(g - tf) + \beta^*((1 - t)f)$

$\leq \beta^*(g - tf) + 1 - t$

$\leq \beta^*(g - tf) + \beta^*(g) - \beta^*(t \cdot f)$

$\leq 2 \cdot \beta^*(g - tf) \leq \mu$

If $t \leq 0$, the same proof, using $(-t) \cdot f$ shows that $\beta(f + g) \leq \varepsilon$.

We are now ready to prove that the norm ρ^* on the separable dual space X^*, has the convergence property of Klee's norm

2.5.2 Theorem

If (X, β) is a Banach space, such that its dual (X^*, β^*) is separable, then X^* admits a dual norm ρ^*, which is rotund and which satisfies the following convergence property; if $\{f_m\}$ is a sequence such that $\rho^*(f_m) \to \rho^*(f)$ and $f_m \to f$ (weak*), then $\rho^*(f_m - f) \to 0$.

Proof:

Consider the norm ρ^*, previously defined; it is rotund and is the dual of an equivalent norm on X. It only remains to show that it has the convergence property.

Let $\{f_m\}$ be a sequence such that $\rho^*(f_m) \to \rho^*(f)$ and $f_m \to f$ (weak*). We can assume without loss of generality that $\beta^*(f_m) = 1$ for every m, because $\rho^*(f_m - f) \to 0$ if and only if $\rho^*\left(\frac{f_m}{\beta^*(f_m)} - \frac{f}{\beta^*(f)}\right) \to 0$ for $f \neq 0$. If $f = 0$, then the theorem is trivially true.

Since $\rho^*(f_m) \to \rho^*(f)$, $\{f_m\}$ is bounded, therefore $\alpha^*(f_m) \to \alpha^*(f)$ since α^* is w^*-continuous on such sets. It follows that $\omega^*(f_m) \to \omega^*(f)$.

Suppose $\rho^*(f_m - f)$ does not converge to zero, then neither does $\beta^*(f_m - f)$, hence given $0 < \delta < 1$, there is a subsequence, still denoted $\{f_m\}$, such that $\beta^*(f_m - f) \geq \delta$. Also, using lemma 2.2.5, we can choose another subsequence, still denote $\{f_m\}$, such that $\omega^*_n(f) = \lim \omega^*_n(f_m)$ for each n.

Consider the set $\{h_n\}$ associated with the norm ω^*_n (lemma 2.2.3). It is dense in S_{β^*}, hence there are infinitely many n such that $\beta^*(f - h_n) < \frac{\delta}{8}$; choose one so that $\frac{1}{n_0} \leq \frac{\delta}{8}$. Since $\omega^*_{n_0}(f - h_{n_0}) \leq \beta^*(f - h_{n_0})$, we have

$$\begin{aligned} \omega^*_{n_0}(f) &\leq \beta^*(f - h_{n_0}) + \omega^*_{n_0}(h_{n_0}) \\ &\leq \frac{\delta}{8} + \frac{1}{n_0} \\ &< \frac{\delta}{4}. \end{aligned}$$

Now, since $\omega^*_{n_0}(f_m) \to \omega^*_{n_0}(f)$, we can choose a positive integer M such that $\omega^*_{n_0}(f_m) < \frac{\delta}{4}$, if $m \geq M$.

Furthermore, for such M, $\frac{\delta}{4} > \omega^*_{n_0}(f_m) \geq | f_m(x) |$, for $x \in B_{n_0}$, (lemma 2.2.3), and since $f_m^{\perp} \cap B_\beta \subset B_{n_0}$, we have satisfied the hypothesis of theorem 2.5.2, therefore, either $\beta^*(f_m - h_{n_0}) \leq \frac{\delta}{2}$ or $\beta^*(f_m + h_{n_0}) \leq \frac{\delta}{2}$.

But $\delta < \beta^*(f_m - f) \leq \beta^*(f_m - h_{n_0}) + \beta^*(h_{n_0} - f) \leq \frac{\delta}{2} + \frac{\delta}{8}$ if the first is true. Since this is impossible we have that $\beta^*(f_m + h_{n_0}) \leq \frac{\delta}{2}$

for $m \geq M$.

Since $\beta^*(h_{n_0}) = 1$, we can choose $x_0 \in B_\beta$ such that $h_{n_0}(x_0) > 1 - \frac{\delta}{2}$, then $\frac{\delta}{2} \geq \beta^*(f_m + h_{n_0}) \geq f_m(x_0) + h_{n_0}(x_0) \geq f_m(x_0) + 1 - \frac{\delta}{2}$, so $f_m(x_0) \leq -1 + \delta$, for $m \geq M$.

It follows from the w^*-convergence of $\{f_m\}$, that $f(x_0) \leq -1 + \delta$, and we have

$$\frac{1}{2} < 2 - \frac{3}{2} \cdot \delta = (1 - \frac{\delta}{2}) + 1 - \delta)$$

$$\leq h_{n_0}(x_0) - f(x_0) \leq \beta^*(h_{n_0} - f) < \frac{\delta}{8}$$

which is certainly a contradiction, and the theorem is proven.

CHAPTER III

EXISTENCE OF A DIFFERENTIABLE NORM

3.1 Introduction

Using Klee's norm Restrepo [17] showed the existence of an admissible, Fréchet differentiable norm on a Banach space which has a separable dual.

In this chapter, we reach the same conclusion by using a proof, due to Cudia [5] which avoids some of the more difficult concepts of Restrepo's theorem. To this end, we develop two theorems, both important in their own right. The first theorem shows that the support mapping, γ (p. 25) of the unit sphere of a smooth norm on a Banach space X is continuous (if the w^*-topology is considered in X^*). The second theorem, a modification of an ingeneous proof by Cudia, shows that, if the support mapping of the unit ball of a smooth norm is continuous, the norm is Fréchet differentiable.

We conclude the chapter with a third theorem in which we show that Restrepo's result follows as a consequence of the above theorems and the norm defined in Chapter II.

3.2 Theorem

Let (X, β) be a Banach space and assume that ρ is a continuous smooth norm on X. Then the support mapping $\gamma: S_\rho \to S_{\rho^*}$, which assigns to each $x \in S_\rho$ the unique normalized support functional at x, is continuous if the norm topology of ρ is considered in S_ρ and the w^*-topology is considered in S_{ρ^*}.

Proof:

Let x be any element in the unit sphere, S_ρ, and let $\{x_n\} \subset S_\rho$ be any sequence converging to x in the norm topology.

Assume the $\gamma(x_n)$ does not converge to $\gamma(x)$ in the w^*-topology, then there is a w^*-neighbourhood, U, of $\gamma(x)$, such that for each n, there is some $m \geq n$ and $\gamma(x_m)$ is not in U. The subsequence $\{\gamma(x_m)\}$ has a subsequence, still denoted $\{\gamma(x_m)\}$, which converges to some $g \in \bar{B}_{\rho^*}$ (the closed unit ball of ρ^*), because $\bar{B}_{\rho^*}$ is w^*-compact (theorem 1.3.9) and hence sequentially compact.

But,

$$\begin{aligned} & | (\gamma(x_m)) (x_m - x) | \\ &= | \gamma(x_m) (x_m) - \gamma(x_m) (x) | \\ &= | 1 - \gamma(x_m) (x) | \\ &< \rho^*(\gamma(x_m) \cdot \rho(x_m - x) \\ &< \rho(x_m - x) \to 0, \end{aligned}$$

therefore $\lim \gamma(x_m) (x) = g(x) = 1$, and g is a normalized support functional at x different from $\gamma(x)$. This, however, contracdicts the smoothness property of ρ; hence, $\gamma(x_n) \to \gamma(x)$ (weak*), which implies that γ is continuous as specified above.

3.3 Theorem

Assume ρ is a continuous, smooth norm defined on a Banach space (X, β). If the support mapping, γ: $S_\rho \to S_{\rho^*}$ is norm to norm continuous, then ρ is Fréchet differentiable on S_ρ relative to S_ρ.

Proof:

First we note that, since ρ is smooth, ρ is Gateaux differentiable on S_ρ relative to S_ρ. That is, for $x, y \in S_\rho$

$$\lim_{t\to 0} \frac{\rho(x + ty) - \rho(x)}{t} = (\rho' x)y$$

exists.

Also, for any $x, y \in X$ and any real t, the real - valued function of the real variable s, $0 \le s \le 1$,

$$F(s) = \rho(x + tsy)$$

is absolutely continuous. Hence, by standard theorems of real analysis (Royden [19], p. 90), $F(s)$ has a finite derivative almost everywhere on $[0, 1]$, $F'(s)$ is Lebesque integrable on $[0, 1]$, and $\rho(x + ty) - \rho(x) = \int_0^1 F'(s)ds$, for any $x, y \in X$ and any real t.

Therefore

$$F'(s) = \lim_{r\to 0} \frac{\rho(x + tsy + try) - \rho(x + tsy)}{r}$$

$$= \rho'(x + tsy)\ (ty)$$

exists for s in the complement of some set (depending on x, y, and t) of Lebesque measure zero. Thus for each $x, y \in S_\rho$ and real t,

$$(1) \quad \left| \frac{\rho(x + ty) - \rho(x)}{t} - (\rho' x)y \right|$$

$$\leq \int_0^1 |(\rho'(x + tsy))(y) - (\rho' x)y| \, ds.$$

Now, if x and y are on S_ρ and $0 \leq s \leq 1$, then if $|t| < 1$,

$$\rho\left(x - \frac{x + tsy}{\rho(x + tsy)}\right)$$

$$\leq \rho(x - x + tsy) + \rho\left(x + tsy - \frac{x + tsy}{\rho(x + tsy)}\right)$$

$$\leq |ts| + \frac{\rho((\rho(x + tsy) - 1)x)}{\rho(x + tsy)} + \frac{\rho((\rho(x + tsy) - 1)tsy)}{\rho(x + tsy)}$$

$$\leq |ts| + \frac{|\rho(x + tsy) - \rho(x)|}{\rho(x + tsy)} + \frac{|ts| \cdot |\rho(x + tsy) - \rho(x)|}{\rho\,(x + tsy)}$$

$$(2) \quad \leq |ts| + \frac{|ts| + |ts|^2}{\rho(x + tsy)}$$

$$\leq |ts| + \frac{|ts| + |ts|^2}{|1 - |ts||}$$

$$\leq |t| + \frac{|t| + |t|^2}{|1 - |t||}.$$

Let $x_0 \in S_\rho$ and choose $\mu > 0$. Since γ is continuous in the norm topologies, there is a $\delta(x_0, \mu) > 0$ such that with $z \in S_\rho$ and $\rho(x_0 - z) < \delta(x_0, \mu)$ then $\rho^*(\gamma(x_0) - \gamma(z)) < \mu$. According to inequality (2) above, there is a $\eta(x_0, \mu) > 0$ such that if $|t| < \eta(x_0, \mu) < 1$ then,

$$\rho\left(\frac{x_0 + tsy}{\rho(x + tsy)} - x_0\right) < \delta(x_0, \mu)$$

for all $y \in S_\rho$. Now let t_0 be a fixed real number such that $|t_0| < \eta(x_0, \mu)$ and let y_0 be any arbirary element of S_ρ. Then except for a set of Lebesque measure zero

$$\rho'\left(\frac{(x_0 + t_0 s y_0)}{\rho(x_0 + t_0 s y_0)}\right) y_0$$

exists, and so, by corollary 1.4.9,

$$\gamma \frac{(x_0 + t_0 s y_0)}{\rho(x_0 + t_0 s y_0)} y_0$$

$$= \rho'\left(\frac{(x_0 + t_0 s y_0)}{\rho(x_0 + t_0 s y_0)}\right) y_0$$

hence

$$(\rho'(x_0 + t_0 s y_0))\, y_0 - (\rho'(x_0))\, y_0$$

$$= \rho'\left(\frac{(x_0 + t_0 s y_0)}{\rho(x_0 + t_0 s y_0)}\right) y_0 - (\rho'(x_0))\, y_0$$

$$= \gamma \frac{(x_0 + t_0 s y_0)}{\rho(x_0 + t_0 s y_0)} y_0 - (\gamma(x_0)) y_0$$

for almost all s, since for any positive real r, $(\rho'(x))y = (\rho'(rx))y$.

Using the hypothesis of continuity, we see that the integrand in (1) is less than μ for almost all s. Hence by (1),

$$\left| \frac{\rho(x_0 + t_0 y_0) - \rho(x_0)}{t_0} - (\rho'(x_0))y_0 \right| < \mu$$

and so, for all t such that $|t| < \eta(x_0, \mu)$

$$\left| \frac{\rho(x_0 + ty) - \rho(x_0)}{t} - (\rho'(x_0)y_0 \right| < \mu$$

Now since $\eta(x_0, \mu)$ is independent of y_0 which is arbitrary, the theorem is proved.

It is an easy consequence of this theorem that ρ is Fréchet differentiable everywhere and, indeed, that ρ is of class C^1.

3.4 Theorem

Assume that (X, β) is a Banach space, whose dual is separable, Then X admits an equivalent norm ρ of class C^1.

Proof:

Consider the norm ρ^* on X^* defined in the second chapter. It is the dual of the norm ρ which is admissible and smooth; therefore, the support mapping γ: $S_\rho \to S_{\rho^*}$ is continuous if the w^*-topology is considered in S_{ρ^*} (theorem 3.2).

Let x be any element in S_ρ, and let $\{x_n\} \subset S_\rho$ be a sequence such that $x_n \to x$. Using the properties of ρ and ρ^*, developed in theorem 2.5.2, it follows that

$$\gamma(x_n) \to \gamma(x), \quad \text{(weak*)}$$

and

$$\rho^*(\gamma(x_n)) \to \rho^*(\gamma(x));$$

and consequently that

$$\rho^*(\gamma(x_n) - \gamma(x)) \to 0.$$

Thus the support mapping, γ, is continuous in the norm topologies, and the above theorem immediately implies that ρ is of class C^1.

CHAPTER IV

DIFFERENTIABLE NORMS AND DENSITY CHARACTER

4.1 Introduction

The conclusions of this chapter are essentially the converse of those of the previous chapter; that is, we will show that the existence of an admissible Fréchet differentiable norm on a separable Banach space X, implies that the dual space, X^*, is separable. Indeed, the first theorem we prove here is the converse of theorem 3.3. In it, we prove that if ρ is a continuous, Fréchet differentiable norm, then the support mapping of its unit sphere is continuous in the norm topologies.

Following the above theorem, we cite a modification of a significant result, due to Bishop and Phelps [1; 2], in which they prove that every Banach space is subreflexive.

We subsequently give a proof of the principal theorem of this chapter and conclude, by indicating some of the important results which follow as a consequence of the ideas developed here and in the previous two chapters.

4.2 Theorem

Let (X, β) be a Banach space and assume ρ is a continuous norm on X and that ρ is Fréchet differentiable on the unit sphere of ρ, then the support mapping $\gamma: S_\rho \to S_{\rho*}$ is continuous in the norm topologies.

Proof:

This proof is based on the work of Phelps [15].

Since ρ is Fréchet differentiable on the unit sphere of ρ, it is Gateaux differentiable there. It follows that ρ is smooth and the support mapping γ is well defined.

If γ is not continuous at some point $x \in S_\rho$, then there exists a sequence $\{x_n\} \subset S_\rho$ such that $x_n \to x$ but $\rho^*(\gamma(x_n) - \gamma(x))$ does not converge to zero.

Also, for some $\mu > 0$, there is a subsequence, still denoted by $\{x_n\}$, such that $\rho^*(\gamma(x_n) - \gamma(x)) > 2 \cdot \mu$, for each n, and $x_n \to x$. Thus, for each n, there is some element $y_n \in S_\rho$ such that $(\gamma(x_n) - \gamma(x))\ y_n \geq 2 \cdot \mu$.

Define a new sequence, $\{z_n\}$, by $z_n = \dfrac{(1 - \gamma(x_n)\ x)}{\mu} \cdot y_n$. Obviously, $\rho(z_n) \to 0$, since

$$|1 - \gamma(x_n)\ x\ | \leq \rho^*(\gamma(x_n)) \cdot \rho(x_n - x)$$

$$\leq \rho(x_n - x) \to 0.$$

Also,

$$\rho(x + z_n) - \rho(x) - \gamma(x)\ z_n$$

$$\geq (\gamma(x_n))\ (x + z_n) - 1 - \gamma(x\ )z_n$$

$$\geq (\gamma(x_n) - \gamma(x))\ z_n + \gamma(x_n)\ x - 1$$

$$\geq 2 \cdot \mu \cdot \frac{1 - \gamma(x_n)\ x}{\mu} + \gamma(x_n)\ x - 1$$

$$\geq 1 - \gamma(x_n)x$$

hence

$$\left| \frac{\rho(x + z_n) - \rho(x) - \gamma(x)z_n}{\rho(z_n)} \right|$$

$$\geq \frac{|\gamma(x_n)x - 1|}{\rho(z_n)}$$

$$\geq \mu \cdot \frac{|\gamma(x_n)x - 1|}{|1 - \gamma(x_n)\, x| \cdot \rho(y_n)}$$

$$\geq \mu, \text{ for all } n.$$

But $\nu(x)z_n = (\rho' x)z_n$ and we have a contradiction to the fact that ρ is Fréchet differentiable at x, which proves the theorem.

4.3 Theorem

Let ρ be a continuous norm on a Banach space (X, β). The set of normalized support functionals is dense in the dual unit sphere ρ.

Proof:

Suppose f is an element in the unit sphere of ρ^* and that we have an arbitrary constant $\mu > 0$.

Define the set A by

$$A = \{x \mid x \in X,\ f(x) = 0, \text{ and } \rho(x) \leq \frac{2}{\mu}\}$$

and let C be the convex hull of the union of the sets A and the unit ball B_ρ, and suppose there exists an element x_0 in the ball B_ρ, which is also in the boundary of C. Since C has a non-empty interior, it follows, by the support theorem 1.2.17, that there exists a functional g in S_{ρ^*} such that $\sup\{g(x) \mid x \in C\} = g(x_0)$.

It follows that $g(x_0) = 1 = \rho(x_0)$, hence on $\frac{\mu}{2} \cdot A$, we have $|g(x)| \leq \frac{\mu}{2}$ and so by theorem 2.5.1, either $\rho^*(f - g) \leq \mu$ or $\rho^*(f + g) \leq \mu$. Since g and $(-g)$ are both normalized support functionals, the theorem is proved if we show that $B_\rho \cap \alpha C$ is non-empty.

Choose z in B_ρ such that $f(z) > 0$ and let $a = \dfrac{1 + \frac{2}{\mu}}{f(z)}$.

Define a partial ordering on the set $Z = \{x \mid x \in B_\rho \text{ and } f(x) \geq f(z)\}$ as follows: $x > y$ if

i) $f(x) > f(y)$

ii) $\rho(x - y) \leq a(f(x) - f(y))$.

Assume that W is a totally ordered subset of Z. By (i), the net of real numbers, $\{f(x), x \in W\}$, is monotone (and bounded), and hence converges to its supremum. From this fact and (ii) it follows that W is a Cauchy net. But X is complete and so W, in fact, converges to a point y in B_ρ. By continuity of f and the norm, we have for every x in W that $f(y) > f(x)$ and $\rho(y - x) \leq a(f(y) - f(x))$, thus y is an upper bound for W. It follows, by Zorn's lemma, that there exists a maximal element x_0 of Z.

Since $x_0 \in B_\rho \subset C$, we need only show that x_0 is in the boundary of C.

If not, then x_0 is in the interior of C, and there exists $t > 0$ such that $x_0 + tz$ is an element of C. Also, from the definition of C, there exists elements y in B_ρ and x in A, and a constant s, $0 \le s \le 1$ such that $x_0 + tz = sy + (1 - s)x$. Thus

$$f(z) \le f(x_0) < f(x_0 + tz) = sf(y) \le f(y),$$

and so $y \in Z$.

Also

$$y - x_0 = (1 - s)\ (y - x) + tz,$$

thus

$$\begin{aligned} \rho(y - x\ ) &\le (1 - s)\ \rho(y - x) + t \\ &\le (1 - s)\ (\rho(y) + \rho(x)) + t \\ &\le (1 - s)\ (1 + \frac{2}{\mu}) + t \\ &< (1 - s + t)\ (1 + \frac{2}{\mu}). \end{aligned}$$

However,

$$\begin{aligned} f(y - x_0) &= (1 - s)\ f(y) + t \cdot f(z) \\ &\ge (1 - s + t)\ f(z) \end{aligned}$$

so

$$\rho(y - x_0) \le a(f(y) - f(x_0))$$

hence $y > x_0$, which contradicts the maximality of x_0, thus x_0 is in the boundary of C.

4.4 Density Character

4.4.1 Definition

The density character of a Banach space X, denoted dens X, is the minimum cardinal number of a dense subset of X, or equivalently, it is the maximum cardinal number of a discrete subset of X.

The following theorem implies the converse of theorem 3.4.

4.4.2 Theorem

If a Banach space (X, β) admits an equivalent norm ρ of class C', then dens $X =$ dens X^*.

Proof:

Suppose a norm ρ on X fulfills the above hypothesis, then, by theorem 4.2 the support mapping $\gamma: S_\rho \to S_{\rho^*}$ is well defined and continuous. Clearly, the image of γ is the set of all normalized support functionals.

Let D be a dense subset of S_ρ with card $D =$ dens X, and let f be any arbitrary element in S_{ρ^*}. Also, choose U to be any neighbourhood of f.

By the previous theorem, the set of normalized support functionals is dense in S_{ρ^*}; hence, since $\gamma^{-1}(u)$ is open, by continuity, and D is dense in S_ρ, it follows that there exists $x \in D \bigcap \gamma^{-1}(u)$, with $\gamma(x) \in U$. Since f and U are arbitrary, this implies that $\gamma(D)$ is dense in S_{ρ^*}. Thus, dens $X^* =$ dens $S_{\rho^*} =$ card $\gamma(D) =$ card $D =$ dens X. Since dens $X \leq$ dens X^* is always true, the theorem is proved.

This theorem completes our first main objective of presenting a simplified but comprehensive proof of Restrepo's main result.

4.5 Theorem (Restrepo)

A separable Banach space (X, β) admits an equivalent norm ρ of class C' if and only if (X^*, β^*) is also separable.

4.5.1 Corollary

Let (X, β) be a separable Banach space. Then, if both β and β^* are of class C', (X, β) is reflexive.

A more direct application of this theorem answers the question of existence or non-existence of admissible C'-norms on well known separable Banach spaces.

4.5.2 Corollary

The topology of c_0 can be defined by a norm of class C'. (c_0 is the Banach space of all sequences $\{x_n\}$ of real numbers such that $x_n \to 0$, with the norm β defined by $\beta(x) = sup \{x_n\}$).

4.5.3 Corollary

The topology of $C[0, 1]$ cannot be defined by any norm of class C^1 ($C[0, 1]$ is the Banach space of continuous functions on $[0, 1]$ with norm $\beta(x) = sup \{x(t) \mid 0 \leq t \leq 1\}$).

4.5.4 Corollary

There does not exist an admissible Fréchet differentiable norm for ℓ'. (ℓ' is the Banach space of all sequences $\{x_n\}$ such that

$\sum_{n=1}^{\infty} |x_n| < \infty$, with norm $\beta(x) = \sum_{n=1}^{\infty} |x_n|$).

CHAPTER V

DIFFERENTIABLE FUNCTIONS AND ROUGH NORMS

5.1 Introduction

Thus far in this paper we have established that the existence of an admissible Fréchet differentiable norm, on a separable Banach space, is characterized by separability of the dual space. Also, in the previous chapter, we considered more general Banach spaces and proved that the existence of such a norm implies that the density characters of the space and its dual are equal. It follows from the results of this chapter that the converse of the previous statement is not true in general. We will now consider an alternate characterization of the existence of admissible Fréchet differentiable norms, in terms of admissibility of rough norms.

Using the concepts of Leach and Whitfield [13], we will prove, by a construction, that if the dual space X^* is strictly denser than the space X, then there exists an admissible rough norm on X. Also, we will show that the existence of an admissible rough norm on a Banach space X implies that there does not exist any real - valued Fréchet differentiable functions on X with bounded non-empty support. Cons-sequently, we obtain the main conclusion of the previous chapter.

5.2 Rough Norms

In this section a norm which is rough (and thus nowhere Fréchet differentiable) is constructed for a Banach space X, provided X^* is strictly denser than X.

5.2.1 Definition

Let A be any set, then A will be called large (with respect to a Banach space X), if card $A >$ dens X; otherwise A will be called small.

5.2.2 Definition

A norm ρ defined on a Banach space X has a uniformly discontinuous upper Gateaux differential, ρ', if there exists $\mu > 0$ such that for every $x \in X$ and $\delta > 0$, there exists x_1, x_2 and y in X such that $\rho(x_1 - x) < \delta$, $\rho(x_2 - x) < \delta$; $\rho(y) = 1$ and $(\rho' x_2 - \rho' x_1)\, y \geq \mu$.

We will use the adjective rough to mean the property described here.

5.2.3 Theorem

If (X, β) is a Banach space such that dens $X <$ dens X^*, then, given $r > 0$, there exists $\mu > 0$ and a subset E of X^* such that E intersects every open ball of radius r in a large set and if $f, g \in E$ and $f \neq g$ then $\beta^*(f - g) > \mu$.

Proof:

Let E_1 be a subset of X^*, which is maximal subject to:

i) E_1 contains the origin of X^*.

ii) If $f \in E_1$, then $-f \in E_1$.

iii) If $f \neq g \in E_1$, then $\beta^*(f - g) > 1$.

Such a maximal set evidently exists by Zorn's lemma. It is also maximal even if the condition $f \in E$ implies $-f \in E_1$ is omitted.

For any real number t, let $E_t = \{t \cdot f \mid f \in E_1\}$. Since $\bigcup_{n=1}^{\infty} E_{\frac{1}{n}}$ is dense; it follows that card E_1 = dens X^*, hence E_1 is large, so there exists a positive integer n such that E_1 intersects the open ball of radius n about the origin of X^* in a large set. Denote the ball by $B_n(0)$.

Let h be any element of X^* and let $B_{n+\frac{1}{2}}(h)$ be the open ball of radius $n + \frac{1}{2}$ with center h. Let $f \in E_1 \bigcap B_n(0)$. Then, $(f + h) \in B_{n+\frac{1}{2}}(h)$. Now, either $f + h \in E_{\frac{1}{2}}$ or $(f + h) \notin E_{\frac{1}{2}}$. In either case, by the maximality of $E_{\frac{1}{2}}$ there exists $g \in E_{\frac{1}{2}}$ such that $\beta^*((f + h) - g) \leq \frac{1}{2}$ (in the first case, choose $g = f + h$). Further, $\beta^*(h - g) \leq \beta^*((f + h) - g) + \beta^*(f) \leq \frac{1}{2} + n$ and we have that $g \in B_{n+\frac{1}{2}}(h) \bigcap E_{\frac{1}{2}}$.

The above gives a correspondence between the elements of $E_1 \bigcap B_n(0)$ and $B_{n+\frac{1}{2}} \bigcap E_{\frac{1}{2}}$. This correspondence is one-to-one; for, if $f_1, f_2 \in E_1 \bigcap B_n(0)$ and $f_1 \neq f_2$, and they correspond to $g_1, g_2 \in E_{\frac{1}{2}} \bigcap B_{n+\frac{1}{2}}(h)$, respectively, then

$$\beta^*(g_1 - g_2) = \beta^*((f_1 - f_2) + g_1 - (h + f_1) + (h + f_2) - g_2)$$

$$\geq \beta^*(f_1 - f_2) - \beta^*(g_1 - (h + f_1)) - \beta((h + f_2) - g_2)$$

$$> \frac{1}{n} - \frac{1}{2n} - \frac{1}{2n} = 0,$$

and we have that $g_1 \neq g_2$.

This one-to-one correspondence implies that card $(E_1 \cap B_n(0)) \leq$ card $(E_{\frac{1}{2}} \cap B_{n+\frac{1}{2}}(h))$; therefore $E_{\frac{1}{2}}$ intersects every open ball of radius $n + \frac{1}{2}$ in a large set.

Choose a positive integer m such that $r > \frac{1}{m}$, then, by analogous reasoning, we have that $E_{\frac{1}{m(2n+1)}}$ intersects every open ball of radius $\frac{1}{m}$ in a large set. Setting $\mu = \frac{1}{m(2n + 1)}$, proves the theorem.

Under the same hypothesis, define a real-valued function ρ on (X, β) as follows; let $F = \{f \mid f \in E_{\frac{1}{4(2n+1)}}$ and $\beta^*(f) \leq 1\}$ $E_{\frac{1}{4(2n+1)}}$ is as defined in the above proposition, and define ρ on X by

$$\rho(x) = \inf \{t \mid f(x) \leq t \text{ for all except a small set of } f\text{'s in } F\}.$$

The following lemma will prove useful in later theorems concerning the function ρ.

<u>5.2.4 Lemma</u>

For every x in X, the set $\{f \mid f \in F$ and $f(x) > \rho(x)\}$ is small.

Proof:

Let $x \in X$, then, by the definition of the function ρ, it is obvious that, for each positive integer n, the set $\{f \mid f \in F$ and $f(x) > \rho(x) + \frac{1}{n}\}$ is small. It is also easy to see $\{f \mid f \in F$ and $f(x) > \rho(x)\} = \bigcup_{n=1}^{\infty} \{f \mid f \in F$ and $f(x) > \rho(x) + \frac{1}{n}\}$ and is small, since the coutnable union of small sets is small.

5.2.5 Theorem

The function ρ is an equivalent norm on (X, β).

Proof:

Suppose $\rho(x) < 0$ for some $x \in X$, then there exists a constant $t < 0$ and a small subset G of F such that $g(t) > t$ for $g \in G$ and $f(x) \leq t$ for $f \in F - G$. By property (ii) of E_1, we have that $f \in F - G$ implies that $-f \in G$ hence card $(F - G) \leq$ card $G <$ card $(F - G)$ which is a contradiction. Hence, $\rho(x) \geq 0$ for all x in X.

Obviously, for real t, $\rho(tx) = |t| \rho(x)$, since $f(tx) = tf(x)$ for all f in F.

Suppose $\rho(x + y) > \rho(x) + \rho(y)$ for some x, y in X. Then the set $\{f \mid f \in F$ and $f(x + y) > \rho(x) + \rho(y)\}$ is large. This set is a subset of $\{f \mid f(x) > \rho(x)\} \bigcup \{f \mid f(y) > \rho(y)\}$ which must also be large. But this implies that at least one of the set in the union is large, which contradicts the previous lemma. Thus, $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and ρ is a prenorm.

Let f be any element of F, then, for all x in X, $f(x) \leq \beta^*(f) \cdot \beta(x) \leq \beta(x)$ which implies that $\rho(x) \leq \beta(x)$.

Let x be any element of X and f be a linear tangent functional at x, that is, $f \in S_{\beta*}$ and $f(x) = \beta(x)$. The set F intersects the open ball of radius $\frac{1}{4}$ about $\frac{3}{4} f$, in a large set, and for each g in this set, we have,

$$\left(\frac{3}{4} f - g \right) (x) \leq \beta^* \left(\frac{3}{4} f - g \right) \cdot \beta(x) < \frac{1}{4} \beta(x),$$

hence $g(x) > \frac{1}{2}\beta(x)$. It follows, by lemma 5.2.4, that $\rho(x) \geq \frac{1}{2}\beta(x)$ for all $x \in X$ and the theorem is proved.

5.2.6 Lemma

Let $G = \{g \mid g \in F \text{ and } \rho^*(g) > 1\}$, then G is a small subset of F.

Proof:

Let D be a small dense subset of X, and for each $x \in D$ define $P(x) = \{f \mid f \in F \text{ and } f(x) > \rho(x)\}$.

Let g be any element of G, then there exists an element x_0 in D such that $g(x_0) > \rho(x_0)$, hence $g \in P(x_0)$. It follows that $G \subset \bigcup\{P(x) \mid x \in D\}$. But, by lemma 5.2.4, each of the $P(x)$'s is a small set, hence $\bigcup\{P(x) \mid x \in D\}$ is small, which implies that G is small.

5.2.7 Theorem

The norm ρ has a uniformly discontinuous upper Gateaux differential ρ'.

Proof:

Recall from theorem 1.4.3 that, for $x, u \in X$ and any scalar s,

(1) $\rho(x + su) \geq \rho(x) + s \cdot (\rho' x)u$.

Let $x \in X$ and $\delta > 0$ (assume without loss of generality that $\delta < \frac{1}{\delta(2n+1)}$). The subset $\bar{F} = \{f \mid f \in F - G \text{ and } f(x) > \rho(x) - \frac{\delta^2}{2}\}$ is a large set, so we can choose $f, g \in \bar{F}$ such that $f \neq g$. The dual norm ρ^* satisfies $\rho^*(f) \geq \beta^*(f)$ for all $f \in X^*$, so, in particular, we have $\rho^*(f - g) > \frac{1}{4(2n+1)}$ and we can thus choose $u \in X$ $\rho(u) = 1$ and $(f - g)u > \frac{1}{4(2n+1)}$.

Now, if $t > 0$,

$$\rho(x + t \cdot u) \geq \rho^*(f) \cdot \rho(x + t \cdot u) \geq f(x + t \cdot u)$$
$$> \rho(x) + t \cdot f(u) - \frac{\delta^2}{2} .$$

From (1), by letting $s = -t$ and $x = x + t \cdot u$ we get $\rho(x) \geq \rho(x + t \cdot u) - t \cdot (\rho'(x + t \cdot u)u)$

hence

$$\rho(x + t \cdot u) + \rho(x) > \rho(x + t \cdot u) + \rho(x) + t \cdot f(u) - \frac{\delta^2}{2}$$
$$-t \cdot \rho'(x + t \cdot u)u,$$

so

$$\rho'(x + t \cdot u)u \geq f(u) - \frac{\delta^2}{2t}$$

for $t > 0$.

Similarly,

$$- \rho'(x - t \cdot u)u \geq - g(u) - \frac{\delta^2}{2t}$$

for $t > 0$.

Adding, we get

$$(\rho'(x + t \cdot u) - \rho'(x - t \cdot u))u \geq f(u) - g(u) - \frac{\delta^2}{t}$$
$$\geq \frac{1}{4(2n + 1)} - \frac{\delta^2}{t} ,$$

for $t > 0$.

Since $\delta > \frac{1}{8(2n+1)}$ we can choose $t < \delta$ such that $\frac{\delta^2}{t} < \frac{1}{8(2n+1)}$.

Thus we have demonstrated the uniform discontinuity property of ρ', with $\mu = \frac{1}{4(2n+1)}$ and $x_1 = x - t \cdot u$ and $x_2 = x + t \cdot u$.

5.3 Bounded Non-empty Support

The main purpose of this section is to prove that if a Banach space X admits a rough norm, then there does not exist any Fréchet differentiable real-valued functions on X, with bounded non-empty support.

5.3.1 Definition

Let f be a real valued function on a Banach space X. The support of f is the closure of $X - f$ or, equivalently, $\overline{\{x \mid f(x) \neq 0 \text{ and } x \in X\}}$.

The property defined in the following theorem is equivalent to the condition of uniform discontinuity.

5.3.2 Theorem

Let X be a Banach space with norm ρ such that ρ' is uniformly discontinuous, then for every x in X and $\delta > 0$ there exists an element v in X such that $\rho(v) < 2$ and $\rho(x + t \cdot v) > \rho(x) + \mu \cdot \frac{|t|}{2} \delta$, where μ is any number satisfying the uniform discontinuity condition and $|t| < \rho(x)$.

Proof:

Given $x \in X$ and $\delta > 0$, we can choose x_1, x_2 and u from X, so that $\rho(x_1 - x) < \frac{\delta}{4}$, $\rho(x_2 - x) < \frac{\delta}{4}$, $\rho(u) = 1$, and $(\rho' x_1 - \rho' x_2)u \geq \mu$

Let $v = u - ((\rho'x_1 + \rho'x_2)u) \cdot \frac{x}{2(\rho x)}$, then $\rho(v) \leq \rho(u) + \frac{|(\rho'x_1 + \rho'x_2)u|}{2}$. By (1) of the previous theorem,

$$(\rho'x_1)u \leq \rho(x_1 + u) - \rho(x_1) \leq \rho(u) = 1.$$

Similarly $(\rho'x_2)u \leq 1$, hence $0 \leq \rho(v) \leq 2$.

If $\rho(v) = 0$ or 2, then $(\rho'x_1)u = (\rho'x_2)u$, which is a contradiction of the above condition, hence $0 < \rho(v) < 2$.

Suppose t is a real such that $|t| \leq \rho(x)$ and let

$$s = 1 - \frac{t \cdot (\rho'x_1 + \rho'x_2)u}{2(\rho x)},$$

then $t < s < 2$ and we have

$$x + t \cdot v = s \cdot x + t \cdot u$$

(i) $$= s \cdot (x_1 + \frac{t}{s} \cdot u) + s \cdot (x - x_1)$$

(ii) $$= s \cdot (x_2 + \frac{t}{s} \cdot u) + s \cdot (x - x_2)$$

Now, by (1) we have

(iii) $$\rho(x_2 + \frac{t}{s} u) \geq \rho(x_2) + \frac{t}{s} \rho'(x_2)u$$

and, similarly,

(iv) $$\rho(x_1 + \frac{t}{s} u) \geq \rho(x_1) + \frac{t}{s} \rho'(x_1)u.$$

Assume $t \geq 0$, then by (i)

$$\rho(x + t \cdot v) \geq \rho(s \cdot (x_1 + \frac{t}{s} \cdot u)) - \rho(s \cdot (x - x_1))$$

$$\geq s \cdot (\rho(x_1) + \frac{t}{s} \cdot (\rho' x_1)u) - s \cdot \frac{\delta}{4} \quad \text{(by iv)}$$

$$\geq s \cdot (\rho(x) + t\ (\rho' x_1)u) - s \cdot \frac{\delta}{2}$$

$$\geq \rho(x) + \frac{t}{s} \cdot (\rho' x_1 - \rho' x_2)u - s \cdot \frac{\delta}{2}$$

$$> \rho(x) + t \cdot \frac{\mu}{2} - \delta.$$

Assume $t < 0$, then by (ii) and (iii),

$$\rho(x + t \cdot v) \geq s \cdot (\rho x) + t \cdot (\rho' x_2)u - s \cdot \frac{\delta}{2}$$

$$\geq \rho(x) + \frac{(-t)}{2} \cdot (\rho' x_1 - \rho' x_2)u - s \cdot \frac{\delta}{2}$$

$$> \rho(x) + |t| \cdot \frac{\mu}{2} - \delta$$

which is again the desired inequality.

5.3.3 Theorem

Suppose a Banach space X has a norm ρ with a uniformly discontinuous upper Gateaux differential ρ'. If f is a real valued Frechet differentiable function on X and $f(0) = 0$, then there exists x in X such that $1 \leq \rho(x) < 2$ and $f(x) \leq \rho(x)$.

Proof:

Let μ be a number which satisfies the uniform discontinuity condition and choose a sequence $\{x_n\} \subset X$ by induction to satisfy:

(1) $x_0 = 0$

(2) $f(x_n) \leq \rho(x_n)$

(3) $\rho(x_{n+1} - x_n) \leq 1$

(4) $\rho(x_{n+1}) \geq \rho(x_n) + \frac{\mu}{8} \cdot \rho(x_{n+1} - x_n)$

(5) $(x_{n+1} - x_n) \geq \frac{1}{2} M_n$

where $M_n = sup\ \{\rho(y - x_n) \mid y \in X$ and $x_{n+1} = y$ satisfies (2), (3), and (4)$\}$

Assume $\rho(x_n) < 1$, for all n, then $\{\rho(x_n)\}$ is bounded above. Also, by definition, $\{\rho(x_n)\}$ is a monotone increasing sequence. It follows that $\{\rho(x_n)\}$ converges and is thus a Cauchy sequence.

Also, by (4), if $m > n$, then

$$\begin{aligned}\rho(x_m - x_n) &\leq \rho(x_m - x_{m-1}) + \ldots + (x_{n+1} - x_n)\\ &\leq \frac{8}{\mu}((\rho x_m - \rho x_{m-1}) + \ldots + (\rho x_{n+1} - \rho x_n))\\ &\leq \frac{8}{\mu} \cdot (\rho(x_m) - \rho(x_n)),\end{aligned}$$

hence $\{x_n\}$ is also a Cauchy sequence. X is a Banach space, therefore, there exists $\bar{x}$ in X such that $x_n \to \bar{x}$, $\rho(\bar{x}) \leq 1$ and $f(\bar{x}) \leq \rho(\bar{x})$.

Since f is Frechet differentiable at $\bar{x}$, there exists a constant $\delta > 0$ such that,

$$f(\bar{x} + u) - f(\bar{x}) - (f'\bar{x})u \leq \mu \cdot \frac{(\rho u)}{8},$$

$\rho(u) < \delta$. We may also assume that $\delta < 1$ and $\delta < 2\ (\rho\bar{x})$.

By the previous theorem, there exists v in X such that $\rho(v) < 2$ and

$$\rho(\bar{x} + t \cdot v) > \rho(\bar{x}) + \mu \cdot \frac{|t|}{2} - \mu \cdot \frac{\delta}{8},$$

whenever $|t| \leq \rho(\bar{x})$.

Choose $t = (sgn\,(f'\bar{x})v) \cdot \frac{\delta}{2}$, Then

$$\rho(\bar{x} + t \cdot v) > \rho(\bar{x}) + \mu \cdot \frac{\delta}{8}$$

$$> \rho(\bar{x}) + \mu \cdot \frac{\rho(t \cdot v)}{8}$$

and

$$f(\bar{x} + t \cdot v) \leq f(\bar{x}) + (f'\bar{x})\ (t \cdot v) + \mu \cdot \frac{\rho(t \cdot v)}{8}$$

$$\leq f(\bar{x}) + t \cdot (f'\bar{x})v + \mu \cdot \frac{\rho(t \cdot v)}{8}$$

$$< f(\bar{x}) + \mu \cdot \frac{\rho(t \cdot v)}{8}$$

and we obtain the following inequalities,

(2') $f(\bar{x} + t \cdot v) - \rho(\bar{x} + t \cdot v) \leq f(\bar{x}) - \rho(\bar{x}) \leq 0$,

(3') $\rho(\bar{x} + t \cdot v - \bar{x}) < 1$,

and

(4') $\rho(\bar{x} + t \cdot v) - \rho(\bar{x}) > \frac{\mu}{8} \cdot \rho(t \cdot v)$.

But ρ and f are continuous at $\bar{x}$, therefore, $x_{n+1} = \bar{x} + t \cdot v$ satisfies (2), (3) and (4) for large n, hence $M_n \geq (\bar{x} + t \cdot v - x_n)$, for large n. It follows that $M_n > \mu \cdot \frac{\delta}{8}$, for all large n, since,

$$\rho(\bar{x} + t \cdot v - \bar{x}) \geq \rho(\bar{x} + t \cdot v) - \rho(\bar{x})$$

$$> \mu \cdot \frac{\delta}{8},$$

hence $\rho(x_{n+1} - x_n) > \mu \cdot \frac{\delta}{16}$, for all large n, by condition (5). This certainly contradicts the convergence assumed for $\{x_n\}$.

Therefore $\rho(x_n) \geq 1$ for some n; let n_0 be the smallest such integer, then x_{n_0} satisfies the requirements of this theorem.

5.3.4 Theorem

If a Banach space (X, β) admits an equivalent norm ρ such that ρ' is a uniformly discontinuous upper Gateaux differential, then there does not exist a Fréchet differentiable real-valued function f on X, with bounded non-empty support.

Proof:

Suppose f is a Fréchet differentiable real-valued function, with bounded non-empty support, S. Then, since S is non-empty, there exists some element $\bar{x}$ in S, such that $f(\bar{x}) \neq 0$. Also, since S is bounded, there exists a scalar $r > 0$ such that S is a subset of the open ball of radius r and center $\bar{x}$.

Define $g\colon X \to R$ by

$$g(x) = 2 - \frac{2 \cdot f(r \cdot x + \bar{x})}{f(\bar{x})},$$

then g is the composition of f and several C^∞ functions, hence g is Fréchet differentiable and $g(0) = 0$ and $g(x) = 2$, whenever $\rho(x) \geq 1$.

Further $g(x) > \rho(x)$, whenever $1 \leq \rho(x) < 2$ which is a contradiction of the results of the previous theorem.

5.4 Differentiable Norms

The following proposition follows as a consequence of the above theorem and the fact that the composition of a norm with a real function of class C^∞, which has bounded non-empty support, is in the same differentiability class as the norm and has bounded non-empty support. The existence of such a function on the reals is elementary, and the proof is omitted.

5.4.1 Theorem

If a Banach space (X, β) admits an equivalent rough norm, then there does not exist an equivalent Fréchet differentiable norm for X.

This theorem, in conjunction with the construction of the second section of this chapter gives an alternate proof of theorem 4.5.2.

For non-separable Banach spaces, the condition dens X = dens X^*, although necessary, is not sufficient to prove the existence of an admissible Fréchet differentiable norm. For example, using the fact that the usual norm on ℓ^1 is rough, one could take the cartesian product of $\ell^1 \times H$, where H is a Hilbert space with dens $H \geq$ dens $\ell^\infty = (\ell^{1*})$, together with the norm defined as the sum of the component norms in ℓ^1 and in H. Clearly, dens $(\ell^1 \times H)$ = dens $(\ell^1 \times H)^*$, and the norm is rough. Thus, by theorem 5.4.1, $\ell^1 \times H$ does not admit a Frechet differentiable norm.

Certainly any space with ℓ^1 or $C[0, 1]$ as a subspace does not admit a norm of class C^1. As a matter of fact, in order that a Banach space admit a C^1-norm, it is easily seen that not only the space but also all its subspaces must have the same density character as their respective duals.

It is not known whether this condition is sufficient to imply the existence of an admissible C^1-norm on the space. One might reasonably conjecture that such is the case. In particular, it is conjectured that all reflexive spaces admit a C^1-norm. A corollary to the following theorem implies that this conjecture may indeed be true.

5.4.2 Theorem

Let (Y, β) be a Banach space that admits a rough norm, then there is a separable subspace of Y that admits an equally rough norm.

Proof:

Let ρ be a rough norm on Y, and $\mu > 0$ any number satisfying the uniform discontinuity condition of ρ'.

Now, define a sequence of finite - dimensional subspaces of Y as follows:

Let X_1 be any 1-dimensional subspace of Y, then given $X_2, X_3, \ldots X_n$, choose a finite sequence $\{x_i\} \subset X_n$, $i = 1, 2, \ldots, m_n$, such that $\rho(x_i) = 1$, for every i, and for all z in X_n with $\rho(z) = 1$ there exists an i such that $\rho(z - x_i) < \frac{1}{n}$.

Using the rough property of ρ, for each $1 \leq i \leq m_n$, choose x_i', x_i'' and u_i in Y such that $\rho(u_i) = 1$, $\rho(x_i' - x_i) < \frac{1}{n}$, $\rho(x_i'' - x_i) < \frac{1}{n}$, and $(\rho' x_i'' - \rho' x_i'')u_i \geq \mu$. Define X_{n+1} as the subspace generated by X_n, $\{x_1'\}$, $\{x_1''\}$, $\{u_1\}$, $\ldots$, $\{x_{m_n}'\}$, $\{x_{m_n}''\}$, $\{u_{m_n}\}$.

Let $X = \lim X_n = \overline{(\bigcup_{n=1}^{\infty} X_n)}$. Clearly, X is a subspace of Y.

Since finite dimensional vector spaces are separable, we have that for each n there exists a set C_n which is countably dense in X_n. Let $D = \{\bigcup_{n=1}^{\infty} C_n\}$, then D is countable and dense in X, hence X is separable.

To complete the proof, we shall show that ρ restricted to X is a rough norm.

Let y be any non-zero element of X and $\delta > 0$ and set $y_1 = \frac{y}{\rho(y)}$ and $\delta_1 = \frac{\delta}{\rho(y)}$. Now, there exists a positive integer n_0 such that $\frac{3}{n_0} \leq \delta_1$, and further, since $\{X_n\}$ is an increasing sequence, there exists $n \geq n_0$ such that $\rho(y_1 - \bar{x}) < \frac{1}{n_0}$ for some $\bar{x} \in X_n$, with $\rho(\bar{x}) = 1$.

We know, from the inductive process, that there exists $x_{i_0} \in X_n$ such that $\rho(x_{i_0}) = 1$ and $\rho(\bar{x} - x_{i_0}) < \frac{1}{n}$. Associated with x_{i_0}, there are elements x'_{i_0}, x''_{i_0}, and u_{i_0} in $X_{n+1} \subset X$ such that $\rho(x_{i_0} - x'_{i_0}) < \frac{1}{n}$, $\rho(x_{i_0} - x''_{i_0}) < \frac{1}{n}$ and $(\rho' x''_{i_0} - \rho' x'_{i_0}) u_{i_0} \geq \mu$, hence

$$\rho(y_1 - x''_{i_0}) \leq \rho(y_1 - \bar{x}) + \rho(\bar{x} - x_{i_0}) + \rho(x_{i_0} - x''_{i_0})$$

$$< \frac{3}{n_0}$$

$$< \delta_1.$$

Similarly, $\rho(y_1 - x'_{i_0}) < \delta_1$.

Let $y' = x'_{i_0} \cdot \rho(y)$ and $y'' = x''_{i_0} \cdot \rho(y)$ and $u = u_{i_0}$, then

$$\rho(y - y') = \rho(y \cdot \rho(y) - x'_{i_0} \cdot \rho(y)$$
$$= \rho(y) \cdot \rho(y_1 - x'_{i_0})$$
$$\leq \rho(y) \cdot \delta_1$$
$$\leq \delta$$

Similarly, $\rho(y - y'') \leq \delta$. Also, $\rho(u) = \rho(u_{i_0}) = 1$ and we get

$$(\rho' y'' - \rho' y')u = (\rho'(x''_{i_0} \cdot \rho(y)) - \rho'(x'_{i_0} \cdot \rho(y)))u_{i_0}$$
$$= (\rho' x''_{i_0} - \rho' x'_{i_0})u_{i_0} \geq \mu,$$

which concludes the proof.

5.4.3 Corollary

If a Banach space Y admits a rough norm, then Y has a separable subspace with a non-separable dual.

Proof:

By the above theorem, Y has a separable subspace X which admits a rough norm. It follows from theorem 5.4.1 and theorem 3.4 that X^* is non-separable.

It follows that no Banach space for which every separable subspace has a separable dual, admits a rough norm. In particular, we have the following:

5.4.4 Corollary

No reflexive space admits a rough norm.

It is not known whether theorem 5.4.2 can be generalized to show that a Banach space that admits a rough norm has a subspace with any prescribed density character that admits a rough norm.

CHAPTER VI

C^p - SMOOTHNESS

6.1 Introduction

In the latter part of the previous chapter, we related the question of existence of Fréchet differentiable functions, with bounded non-empty support, to that of existence of an admissible norm of class C'. Using the same basic approach, we will develop the concepts of C^p-smoothness, due to Bonic and Frompton [4], and consider the problem of approximating a continuous norm by one of a higher differentiability class.

In this development, we will define the notions of C^p-smooth topologies on Banach spaces and illustrate how they might be useful in characterizing the existence of admissible norms of any differentiability class.

We end this chapter by summarizing some of the conclusions of the paper.

6.2 Smooth Functions

Denote the set of functions $f: X \to R$ which are p times Fréchet differentiable and $f^{(p)}$ is continuous, by $C^p(X, R)$. Clearly, the elements of $C^p(X, R)$ are of class C^p.

6.2.1 Definition

A Banach space X is said to be C^p-smooth, $0 \leq p \leq \infty$, if there is a $C^p(X, R)$ - function on X with bounded non-empty support.

6.2.2 Definition

The C^p-topology on X, denoted Γ^p, is the topology obtained by taking as a sub base all sets of the form

$$\{f^{-1}(a, b) \mid f \in C^p(X, R);\ a, b \in R'\},$$

where R' denotes the extended reals.

The Γ^p-topology of a real Banach space X, is related to the product space topology over the reals. Let $R_f = R$, for each $f \in C^p(X, R)$, and define a mapping $T\colon X \to \Pi\{R_f \mid f \in C^p\}$ by $T(x) = \Pi\{f(x) \mid f \in C^p\}$. Now, since the dual space X^* is total and $X^* \subset C^p(X, R)$, we have that if $T(x) = T(y)$ then $f(x) = f(y)$ for every $f \in C^p$ and, in particular, for every $f \in X^*$ and so $x = y$. Hence T is a one-to-one embedding; consequently, X may be regarded as a subset of $\Pi\{R_f \mid f \in C^p\}$. It is then evident that the Γ^p-topology on X is identical with the relative topology of X as a subset of this product space.

Denote the weak and norm topologies of a Banach space (X, β) by Γ^* and Γ_β respectively.

The following theorem is an immediate result of the fact that $X^* \subset C^\infty(X, R) \subset \ldots C^p(X, R) \subset \ldots C^0(X, R)$ and $\beta \in C^0(X, R)$.

6.2.3 Theorem

Let (X, β) be a Banach space, then the weak, norm and Γ^p topologies have the following relationship:

1) $\Gamma^0 = \Gamma_\beta$;

2) $\Gamma^{p+1} \subset \Gamma^p$, $0 \leq p < \infty$;

3) $\Gamma^\infty \subset \Gamma^p$, for every $p \geq 0$;

and 4) $\Gamma^* \subset \Gamma^\infty$.

The following theorem suggests that a study of Γ^p topologies may be useful in establishing the existence of admissible norms of different differentiability classes.

6.2.4 Theorem

If β is an element of $C^p(X, R)$, then $\Gamma^p = \Gamma_\beta$.

Proof:

By the previous theorem, $\Gamma^p \subset \Gamma_\beta$, hence, we need only show containment in the opposite direction.

Let $U \in \Gamma_\beta$ and let x be some element in U. Then there exists $\delta > 0$, such that the open ball, denoted S, with center x and radius δ is contained in U. Certainly, $\beta^{-1}(-\delta, \delta)$ is a Γ^p-neighbourhood of the origin, so, by proposition 1, p. 9 of Robertson [18], $x + \beta^{-1}(-\delta, \delta)$ is a Γ^p-neighbourhood of x. But $x + \beta^{-1}(-\delta, \delta) = S$, and the theorem is proved.

6.2.5 Corollary

If ρ is an equivalent norm on the Banach space (X, β), then

$\Gamma_\rho = \Gamma_\beta$, hence if $\rho \varepsilon C^p(X, R)$, $\Gamma_\beta = \Gamma^p$.

Since every continuous linear functional is Γ^*-continuous, we have the following:

6.2.6 Theorem

The continuous linear functionals on (X, Γ^p), $0 < \rho \leq \infty$, and (X, Γ^*) are precisely the continuous linear functionals on (X, Γ_β).

We note, without proof, the following important property of the Γ^p topologies.

6.2.7 Theorem

For p, $0 \leq p \leq \infty$, the weakest topology on X which makes every element of $C^p(X, R)$ continuous is the Γ^p topology.

From the definition, a base element of the Γ^p topology, $0 \leq p \leq \infty$, is of the form,

$$\{x \mid t_k < f_k(x) < s_k;\ t_k,\ s_k \ \varepsilon\ R; \text{ and } f_k \ \varepsilon\ C^p(X, R),\ k = 1, 2, \ldots n\},$$

In the following theorem we indicate an equivalent base for the Γ^p topologies.

6.2.8 Theorem

Γ^p has a base of sets of the form $\{x \mid f(x) > 0,\ f \geq 0$, and $f \ \varepsilon\ C^p(X, R)$.

Proof:

Consider a base element of the Γ^p-topology. It is of the form

$$\{x \mid t_k < f_k(x) < s_k;\ t_k,\ s_k \ \varepsilon\ R; \text{ and } f_k \ \varepsilon\ C^p(X, R),\ k = 1, 2, \ldots , n\}.$$

Choose ϕ in $C^{\infty}(R^n, R)$ such that $\phi(r_1, r_2, \ldots, r_n) > 0$ if $t_k < r_k < s_k$, for all k, and $\phi = 0$ otherwise. Let d map X into X^n be defined by $d(x) = (x, x, \ldots, x)$. Clearly d is continuous linear isomorphism, so

$$f = \phi \circ (f_1 \times f_2 \times \ldots \times f_n) \circ d$$

is in $C^p(X, R)$, and

$$\{x \mid t_k < f_k(x) < s_k, \text{ for all } k\} = \{x \mid f(x) > 0\}.$$

The following two theorems are important generalizations of some of the ideas used in the previous chapter and, in the same sense, are used to develop a partial characterization with regards to the existence of admissable norms of higher differentiability classes.

6.2.9 Theorem (Bonic and Frampton)

The norm topology, Γ_β, of a Banach space X is equivalent to the Γ^p topology, induced on X by the functions in $C^p(X, R)$, if and only if X is C^p-smooth.

Proof:

Suppose X is C^p-smooth, U is any open set in X, and x is in U. Let $f \in C^p(X, R)$ be a function with bounded non-empty support. By composing f with C^{∞}-maps, as we did in theorem 5.3.4, we can find $g \in C^p(X, R)$ with $g(x) \neq 0$ and $\{x \mid g(x) \neq 0\} \subset U$. Since $\Gamma^p \subset \Gamma^0$, $C^p(X, R)$ open sets are open and we have that the norm topology is equivalent to the Γ^p-topology.

Conversely, assume that the two topologies are equivalent. Then, using a construction similar to that in the proof of theorem 6.2.8, define a $C^p(X, R)$ function with bounded non-empty support on X. It follows that X is C^p-smooth.

6.2.10 Corollary

Suppose X is a Banach space which admits an equivalent norm contained in $C^p(X, R)$, then X is C^p-smooth.

The contrapositive of this corollary is useful in proving the non-existence of admissible norms of higher differentiability classes. For example, consider the following:

6.2.11 Theorem

If p is not an even integer and $n \geq p$ then L^p and ℓ^p do not admit norms of class C^n.

Proof:

Kurzweil [12] showed that if p is not even and $n \geq p$ then L^p and ℓ^p are not C^n-smooth.

It is not known if the converse of corollary 6.2.10 is true. However, we feel that this is a reasonable conjecture, or, at least, some simple modification of C^p-smoothness might yield the converse. The following theorems certainly support this conjectures.

6.2.12 Theorem

The Banach space c_0 is C^∞-smooth. (This was proven by Bonic and Frampton [3].)

The following theorem was proved by N. H. Kuiper. (See Bonic and Frampton [4; p. 896]).

<u>6.2.13 Theorem</u>

The Banach space c_0 admits an equivalent C^∞-norm.

<u>6.2.14 Theorem</u>

If p is an even integer, then the usual norms of ℓ^p and L^p are of class C^∞. A proof of this result can be found in Bonic and Frampton [4].

The following result follows from corollary 6.2.10.

<u>6.2.15 Theorem</u>

If p is an even integer then ℓ^p and L^p are C^∞-smooth.

As we have illustrated above, corollary 6.2.10 provides an easy way of obtaining smooth functions on a Banach space, via the norm. In the following, we consider further applications of this method.

<u>6.2.16 Theorem</u>

A finite dimensional Banach space is C^∞-smooth.

<u>6.2.17 Theorem</u>

If X^* is separable, X is C^1-smooth. This follows as a consequence of theorem 3.4 and corollary 6.2.10.

<u>6.2.18 Corollary</u>

If (X, β) is a separable Banach space, then $\Gamma_\beta = \Gamma^1$ if and only if X^* is separable.

6.2.19 Theorem

A Hilbert space is C^{∞}-smooth.

Bonic and Reis [4; p. 897] have shown that if a Banach space and its dual space admit C^2-norms, then it is a Hilbert space. In view of the above conjecture, it is likely that if a Banach space and its dual are C^2-smooth then it is a Hilbert space.

6.3 Summary

In this and the previous chapters, we have considered several properties which might be satisfied by a real Banach space. We have used these properties to characterize, at least to some extent, the existence or non-existence of admissible Fréchet differentiable norms.

For a separable space we established that a C^1-norm is admitted when and only when its dual is also separable. There is, however, no characterization of those separable spaces that admit a C^p-norm, $p > 1$. We have also established for separable spaces that an admissible rough norm exists if and only if the dual is non-separable.

The non-separable space is more difficult. However, we have shown that the existence of a Frechet differentiable norm implies that the density character of the space and all its subspaces are equal to the density character of the respective dual spaces. It is also necessary that neither the space nor any of its subspaces admits a rough norm. We do not know if a complete characterization is possible in these terms; although, the evidence seems to infer that it is, or at least, that some modification of these properties might be useful in achieving such a characterization.

We have also established that, if there does not exist a $C^1(X, R)$ function with bounded non-empty support on X, there does not exist an admissible Fréchet differentiable norm for X. It is still an open question whether the existence of a $C^1(X, R)$ function with bounded non-empty support implies the existence of a admissible C^1-norm. However, this seems to be a reasonable conjecture. Similar results hold for functions and norms of higher differentiability classes, as we have seen in our study of C^p-smoothness and we indicated evidence inferring that the above conjecture may, in fact, be true in this situation as well.

BIBLIOGRAPHY

[1] Bishop, E. and Phelps, R. R., A proof that every Banach space is subreflexive, Bull. Amer. Math. Soc. 67 (1961), 97-98.

[2] Bishop, E. and Phelps, R. R., The support functionals of a convex set, Proc. Sympos. Pure Math. Vol. 7, p. 27-35, Amer. Math. Soc., Providence, R. I., 1963

[3] Bonic, R. and Frampton, J., Differentiable functions on certain Banach spaces, Bull. Amer. Math. Soc. 71 (1965), 393-395.

[4] Bonic, R. and Frampton, J., Smooth functions on Banach manifolds, J. of Math. and Mech., 15, 5 (1966), 877-898.

[5] Cudia, D. F., The geometry of Banach spaces. Smoothness, Trans. Amer. Math. Soc. 110 (1964), 284-314.

[6] Dieudonné, J., Foundations of Modern Analysis, Academic Press, New York, 1960.

[7] Dunford, N. and Schwartz, J., Linear Operators, Part I, Interscience, New York, 1966.

[8] James, R. C., Linear functionals as differentials of a norm, Math. Mag., 24 (1950-51), 237-244.

[9] Kadèc, M. I., On weak and norm convergence, Doklady Akad. Nauk SSSR, 122 (1958), 13-16 (Russian).

[10] Kelley, J. L., General Topology, D. van Nostrand, New York, 1955.

[11] Klee, V., Mappings into normed linear spaces, Fund. Math., 49 (1960), 25-34.

[12] Kurzweil, J., On approximation in real Banach spaces, Studia Math., 14 (1954), 213-231.

[13] Leach, E. B. and Whitfield, J. H., Differentiable functions and rough norms on Banach spaces, to appear.

[14] Mazur, S., Uber konvexe Mengen in linearen normierten Raumen, Studia Math., 4 (1933), 70-84.

[15] Phelps, R. R., A representation theorem for bounded convex sets, Proc. Amer. Math. Soc., 11 (1960), 976-983.

[16] Rainwater, J., On a renorming theorem of Klee, unpublished note 1968.

[17] Restrepo, G., Differentiable norms, Soc. Mat. Mexicana Bol., 2, 10 (1965), 47-55.

[18] Robertson, A. P., and Robertson, W. J., Topological Vector Spaces, Cambridge University Press, London, 1964.

[19] Royden, H. L., Real Analysis, Macmillan, New York, 1966.

[20] Whitfield, J. H. M., Certain differentiable functions of Banach spaces and manifolds, Case Inst. of Tech. thesis, 1966.

[21] Wilansky, A., Functional Analysis, Blaisdell, New York, 1964.