The Photon Number Density Operator

BY

Thomas Melde

B.Sc.,Lakehead University (1994)

Submitted to the Department of Physics in partial fulfillment of the requirements for the Degree of

Master of Science

at the

Lakehead University

Thunder Bay, Ontario

Canada

June 1995

ISBN 0-612-09225-9

ACKNOWLEDGMENTS

I would especially like to thank my supervisor Dr. Margaret Hawton, who gave me the opportunity to complete this thesis at Lakehead University. This thesis would not have been possible without her guidance and continued support during the last three years.

I would also like to thank everybody in Graduate Studies, who helped me in the application process to make my MSc. happen. I have to thank especially Mrs. Gwen Wojda and Mrs. Leslie Malcolm in Distance Education and the International Student Advisor Ms. Hedi Lee for their constant help and support during my stay in Thunder Bay.

I was lucky to have so much support from my friends and family when I decided to continue my studies at Lakehead. Thanks Baghira ("Thunder Bay sounds nice"). Thanks Mom, nur noch ein paar Jahre bis zum Abschluss!

Studying at Lakehead was a pleasant experience and I would like to thank everybody in the Department of Physics for making it one.

Finally, I would like to thank Dr. Margaret Hawton, the Physics Department and Graduate Studies for the financial support provided throughout my studies.

In memory of my father Hans Joachim Melde

ABSTRACT

The Photon Number Density Operator

BY

Thomas Melde

A new operator is introduced to represent the density of photons in configuration space. It has some features in common with operators previously introduced by Mandel and Cook but has better transformation properties.

The operator is introduced first in the Coulomb gauge where only transverse photons are necessary to describe physical states. It is the first component of the four vector obtained by contracting the electromagnetic field tensor with the vector potential. It is also shown that in the free field case the corresponding photon current and the photon number density operator satisfy a continuity equation.

In the Lorentz gauge, longitudinal and scalar photons are allowed and the operators are defined in respect to an indefinite metric as proposed by Gupta. The Coulomb gauge operator expressed in the new metric cannot give the right number of ghost photons for arbitrary states and has to be discarded as a valid photon number density operator in the Lorentz gauge. It is shown that the photon number density operator in the Lorentz gauge differs from the one in the Coulomb gauge by a divergence term. The total number of photons for physical states is the same for both operators. The ghost states, which are the longitudinal and scalar photon states, in respect to the old metric are different for the two operators.

The form of the photon number operator in the Lorentz gauge using the new metric can be substantiated by symmetry arguments. The new operator is able to count the ghost state photons in respect to the old metric in a free field and in the case of two fixed charges present.

Table of Contents

Chapter 1
Introduction

Kunst gibt nicht das sichtbare wieder, sondern macht sichtbar.
(Art does not reflect the visible, but makes visible.)

-Paul Klee

1 Out of the darkness

Vision is probably the strongest of our senses and it is no surprise that we are eager to understand the concept that allows us to see and distinguish objects. Early theories thought of the eye as a testing device sending out signals to investigate the world around us. Greek philosophers however concluded that the eye is merely a detector that depends on signals sent by the observed object. Those signals are known as light and objects can either reflect or emit them.

But what is the nature of these signals? One of the first observations was the fact that light appears to travel in straight lines and objects throw shadows. This behavior suggests that light consists of particles travelling the distance between the observed light emitting object and the eye with some velocity. If another object which cannot be traversed by the particle is in the way, it is impossible for our eye to observe the light emitting object. Newton supported this theory because he could not observe with his equipment that light would bend around corners and therefore would follow a curved path.

Robert Hooke, a contemporary of Newton, suggested that light is some kind of oscillatory behavior in a medium that yet had to be found, which made a far more complicated theory necessary to explain the nature of light. In the year 1687 Christiaan Huygens used this idea and formulated indeed a wave theory of light. Newton's reputation and the simplicity of the theory however was reason enough that the corpuscle theory was predominant for some time.

Another question of interest was the velocity with which light can travel. The first attempt to measure the speed of light was probably made by Galileo Galilei, but with his equipment he only could show that light travels extremely fast. He nevertheless predicted a finite value for the speed of light. The first value given for the speed of light is the one of Ole Roemer, who used solar data to obtain a value of $2 \times 10^8 \frac{m}{s}$. In the year 1849 Hippolyte Fizeau conducted the first terrestrial measurement and his result is the value accepted today. The riddle of the speed of light was solved, but the question of the nature of light was still unanswered.

In the early 1800s (approximately at the same time as Fizeau's experiments) the wave theory of light was able to explain the observation that light can in fact bend around corners and Young performed interference experiments that could not be explained with Newton's particle theory. At the end of the century Maxwell formulated his famous set of equations and predicted the existence of electromagnetic waves, which travel with the speed of light. In the same century Heinrich Hertz was able to show the existence of the proposed waves. The nature of light had to be an electromagnetic wave.

2 The dawn of modern physics

It did not take long to destroy this new confidence in the description of the nature of light, because in the year 1900 Max Planck was able to explain the black body radiation problem with an approach that contradicted classical physics. He concluded that the vibrational energy of the oscillators in the black body problem is discrete, or quantized as he called it, with a smallest possible finite unit of energy. Albert Einstein used this revolutionary idea to explain the photo-electric effect in the year 1905 and treated light as a stream of Planck's energy quanta, which he called photons. The concept of photons is more similar to the particle approach and the confusion was evident, sometimes light seems to behave like a wave and sometimes like a particle. The only explanation was that both theories are models of something more complicated and are only valid in special cases.

At the same time Einstein published his theory of special relativity, in which light plays an important role. Photons in this theory have zero rest mass and their finite speed is the fastest speed with which information can be transmitted. Simultaneously the theory of quantum mechanics was developed independently by Erwin Schroedinger and Werner Heisenberg and the basic principles for modern physics were born. The theories are completely abstract models and by far more difficult to understand than the classical theories. It seems that the more completely a model can describe nature the more difficult it is to understand it. Feynman even suggested that "nobody understands quantum mechanics", but because the theory is so powerful most undergraduate students in physics are confronted with it today. In quantum mechanics physical states are described by wave functions in abstract spaces and observable quantities are represented by quantum mechanical operators. This and the fact that certain measurements can only be performed up to a given uncertainty are the main concepts of the first quantized system.

In the year 1927 Dirac published his paper on *The Quantum Theory of the Emission and Absorption of Radiation* which was the foundation of a systematic theory of quantized fields. This is known today as second quantization and once again light played an important role in this development. A year later Dirac coupled his theory of radiation with a relativistic theory of the electron and Quantum electrodynamics was born. Photons could now be interpreted as the excitation modes of the free electromagnetic field in this new theory and the description of the nature of light once again changed.

3 The position of light today

The particle aspect of the photons still constitutes difficulties, because as a particle the photon should be localizable in space and time like every other elementary particle and it would be possible to define a quantum mechanical position operator for photons. But photons have zero mass and Newton and Wigner [1] showed that particles of that kind are not strictly localizable. Modern physics tries to solve this problem, because in experiments photons are detected on a daily basis and experimentalists think of them in a way as point like. Jauch and Piron [2] suggested that photons are weakly localizable particles following an approach to describe photons, which was developed by Mackey [3]. It is based on the so called representation theory, which is an abstract mathematical model describing positions with projection operators. Amrein [4] investigated this approach and its consequences and was able to define operators describing the number of particles in a given small volume of space in relativistic quantum field theory. He also compared this operator with a similar operator given earlier by Mandel [5] for the photon field. Amrein's approach however is not the one investigated in this thesis.

Mandel's photon number operator was the first of its kind and it was followed up by Cook [6][7], who extended the operator and was able to give a continuity equation for photons. Recently Inagaki [8] published a paper reformulating Cook's photon dynamics in conventional quantum mechanics and giving an interpretation why Cook disregarded negative frequency solutions. All those approaches will be briefly discussed in this thesis.

Initially Quantum electrodynamics was studied in the Coulomb gauge where the longitudinal modes of the field could be replaced by the Coulomb interaction. Only transverse photons were necessary to describe the physical system, but the system was not relativistically invariant. The covariant theory introduced a new kind of photons, the scalar photons, and treated the four kind of photons symmetrically. The scalar photon however makes a change in the sign of its commutation relations necessary and it was proposed that the roles of emission and absorption operators for scalar photons should be reversed. This would destroy the symmetry of the theory and leads to difficulties. Suraj N. Gupta [9] used a different approach to solve this problem in a paper published in the year 1950. He introduced an indefinite metric and modified the gauge condition for physical states. The difficulties in the physical interpretation were solved, because in his theory only the transverse photons are observable and the longitudinal and scalar photon states are allowed to have negative norms. As long as no constraints are applied, those unphysical states which Michio Kaku [10] calls Ghosts are allowed to propagate in the theory.

In this thesis a new photon number operator will be introduced in the Coulomb gauge first and in the Lorentz gauge later. The Lorentz description will be more complicated, because longitudinal and scalar photons will be allowed. In the Lorentz gauge an indefinite metric as proposed by Gupta will be used. It also will be shown that the proposed operators will yield the same result as the usual photon number operators in the appropriate limits. It is also hoped that the description in the Lorentz gauge will give a better understanding of the role of scalar and longitudinal photons in physical situations.

Chapter 2
Principles of Quantum Electrodynamics in the Coulomb Gauge

And now for something completely different.

-Monty Python's Flying Circus

In this chapter a brief introduction to the theory of Quantum electrodynamics will be given. The concept of Maxwell's equations and Fourier transformations are the starting point and the quantization of the field will be explained in the second part of this chapter. The quantization will be performed in the Coulomb gauge first and in the Lorentz gauge in a later chapter. Throughout the thesis SI-units will be used.

1 Maxwell's equations and the reciprocal space

This section gives an introduction to the basic principles needed to develop quantum electrodynamics. It starts with the definition of Maxwell's equations and their transformation into reciprocal space. Then the normal variables of the field will be derived, because they are the natural starting point for a quantization of the field.

Maxwell's equations in real space are

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{1}{\varepsilon_0}\rho(\mathbf{r},t) \tag{2.1.1}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 \tag{2.1.2}$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \tag{2.1.3}$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) = -\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) + \frac{1}{\varepsilon_0 c^2}\mathbf{j}(\mathbf{r},t) \tag{2.1.4}$$

As usual $\mathbf{E}(\mathbf{r},t)$ is the electric field, $\mathbf{B}(\mathbf{r},t)$ is the magnetic field, $\rho(\mathbf{r},t)$ is the charge density and $\mathbf{j}(\mathbf{r},t)$ is the current density due to matter. It is possible to show from Maxwell's equations that the charge density and the current density satisfy the continuity equation

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla\cdot\mathbf{j}(\mathbf{r},t) = 0, \qquad (2.1.5)$$

which expresses the global conservation of charge. The fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ can also be written in respect to the vector potential $\mathbf{A}(\mathbf{r},t)$ and the scalar potential $U(\mathbf{r},t)$ as

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) \qquad (2.1.6)$$

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) - \nabla U(\mathbf{r},t). \qquad (2.1.7)$$

With those definitions Eqn.(2.1.2,2.1.3) are satisfied and Eqn.(2.1.1,2.1.4) are given as

$$\triangle U(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\rho(\mathbf{r},t) - \nabla\cdot\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) \qquad (2.1.8)$$

$$\square\mathbf{A}(\mathbf{r},t) = \frac{1}{\varepsilon_0 c^2}\mathbf{j}(\mathbf{r},t) - \left[\nabla\cdot\mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}U(\mathbf{r},t)\right], \qquad (2.1.9)$$

where $\square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle$ and $\triangle = \nabla^2$.

This is a system of coupled second order differential equations, the equations of motion. The potentials are not unique and it is possible to perform transformations that leave the fields invariant. They are known as gauge transformations and they are of the form

$$\mathbf{A}(\mathbf{r},t) \rightarrow \mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \nabla F(\mathbf{r},t) \qquad (2.1.10)$$

$$U(\mathbf{r},t) \rightarrow U'(\mathbf{r},t) = U(\mathbf{r},t) - \frac{\partial}{\partial t}F(\mathbf{r},t), \qquad (2.1.11)$$

where the scalar function $F(\mathbf{r},t)$ is arbitrary. It is therefore possible to choose different gauges to describe the same fields. The most commonly used gauges are the Lorentz gauge

$$\nabla\cdot\mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}U(\mathbf{r},t) = 0 \qquad (2.1.12)$$

and the Coulomb gauge

$$\nabla\cdot\mathbf{A}(\mathbf{r},t) = 0. \qquad (2.1.13)$$

All the equations defined so far are global equations, that means the field at one point in space is dependent on points in its neighborhood and this neighborhood can be extended to the whole space of definition. It is possible to transform the set of Maxwell's equations into a set of strictly localized equations. This is achieved by performing a Fourier transformation and the space of this Fourier transform is called reciprocal or momentum space. The Fourier transformation is defined as

$$\mathcal{E}(\mathbf{k},t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3r \mathbf{E}(\mathbf{r},t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{2.1.14}$$

where

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3r \mathcal{E}(\mathbf{k},t)\, e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{2.1.15}$$

This Fourier transform is of the same form for all fields. The gradient operator ∇ in real space transforms into multiplication by $i\mathbf{k}$ in reciprocal space after performing the Fourier transformation and Maxwell's equations in reciprocal space are

$$i\mathbf{k}\cdot\mathcal{E}(\mathbf{k},t) = \frac{1}{\varepsilon_0}\rho(\mathbf{k},t) \tag{2.1.16}$$

$$i\mathbf{k}\cdot\mathcal{B}(\mathbf{k},t) = 0 \tag{2.1.17}$$

$$i\mathbf{k}\times\mathcal{E}(\mathbf{k},t) = -\frac{\partial}{\partial t}\mathcal{B}(\mathbf{k},t) \tag{2.1.18}$$

$$i\mathbf{k}\times\mathcal{B}(\mathbf{k},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\mathcal{E}(\mathbf{k},t) + \frac{1}{\varepsilon_0 c^2} \jmath(\mathbf{k},\mathrm{t}), \tag{2.1.19}$$

the continuity equation is

$$i\mathbf{k}\cdot\jmath(\mathbf{k},t) + \frac{\partial}{\partial t}\rho(\mathbf{k},t) = 0 \tag{2.1.20}$$

and the relationships between the fields and the potentials are given by the equations

$$\mathcal{B}(\mathbf{k},t) = i\mathbf{k}\times\mathcal{A}(\mathbf{k},t) \tag{2.1.21}$$

$$\mathcal{E}(\mathbf{k},t) = -\frac{\partial}{\partial t}\mathcal{A} - i\mathbf{k}\mathcal{U}(\mathbf{k},t). \tag{2.1.22}$$

This means that for every given wavevector $\mathbf{k}$ the field at this point in the reciprocal space is described by the four Eqn.(2.1.16,2.1.13,2.1.18,2.1.19). For the magnetic field Eq.(2.1.13) yields the fact that the longitudinal component of the magnetic field is identically zero. The longitudinal part of the electric field is zero only in the special case of the free field. In the case of a given charge distribution the longitudinal part of the electric field is due to the Coulomb interaction of the charges[1]. Since this Coulomb interaction is dependent on the positions $\mathbf{r}_\alpha$ of the particles α and therefore is not a dynamical variable of the field, the global system field + particles at time t_0 can be expressed completely in terms of the set of dynamical variables

[1] A rigourous treatment can be found in Ref.[11, p.11]

$\{\mathcal{E}_\perp(\mathbf{k},t_0), \mathcal{B}(\mathbf{k},t_0), \mathbf{r}_\alpha(t_0), \dot{\mathbf{r}}_\alpha(t_0)\}$. It is then only necessary to express Maxwell's equations in terms of the transverse fields and one gets for Eqn.(2.1.18,2.1.19):

$$\frac{\partial}{\partial t}\mathcal{B}(\mathbf{k},t) = -i\mathbf{k}\times\mathcal{E}_\perp(\mathbf{k},t) \tag{2.1.23}$$

$$\frac{\partial}{\partial t}\mathcal{E}_\perp(\mathbf{k},t) = ic^2\mathbf{k}\times\mathcal{B}(\mathbf{k},t) - \frac{1}{\varepsilon_0}j_\perp(\mathbf{k},t) \tag{2.1.24}$$

It is important to notice that only the transverse part of the current appears in Eq.(2.1.24).

2 Normal variables

In the last section Maxwell's equations in reciprocal space were developed. It was shown that only the transverse parts of the field are dynamical variables and that Maxwell's equations in respect to these variables are strictly localized. Maxwell's equations are a set of coupled differential equations and therefore one is tempted to find a new basis in which the equations are independent of each other. This new set of variables is called normal variables. It is possible to rewrite Eqn.(2.1.23,2.1.24) in a different form and to find the eigenfunctions of the system in the case the transverse current equals zero. The equations can be written as

$$\frac{\partial}{\partial t}\mathcal{E}_\perp(\mathbf{k},t) = ic^2\mathbf{k}\times\mathcal{B}(\mathbf{k},t) - \frac{1}{\varepsilon_0}j_\perp(\mathbf{k},t), \tag{2.2.1}$$

$$\mathbf{k}\times\frac{\partial}{\partial t}\mathcal{B}(\mathbf{k},t) = ik^2\mathcal{E}_\perp(\mathbf{k},t). \tag{2.2.2}$$

Sums and differences of Eqn.(2.2.1,2.2.2) yield

$$\frac{\partial}{\partial t}\left(\mathcal{E}_\perp(\mathbf{k},t) \pm c\kappa\times\mathcal{B}(\mathbf{k},t)\right) = \pm i\omega\left(\mathcal{E}_\perp(\mathbf{k},t) \pm c\kappa\times\mathcal{B}(\mathbf{k},t)\right), \tag{2.2.3}$$

with the angular frequency $\omega = ck$ and the unit wave vector $\kappa = \frac{\mathbf{k}}{k}$. The normal variables are then defined as

$$\alpha(\mathbf{k},t) = -\frac{i}{2\mathcal{N}(k)}\left[\mathcal{E}_\perp(\mathbf{k},t) - c\kappa\times\mathcal{B}(\mathbf{k},t)\right] \tag{2.2.4}$$

$$\beta(\mathbf{k},t) = -\frac{i}{2\mathcal{N}(k)}\left[\mathcal{E}_\perp(\mathbf{k},t) + c\kappa\times\mathcal{B}(\mathbf{k},t)\right], \tag{2.2.5}$$

where $\mathcal{N}(k)$ is a normalization coefficient, which has to be chosen later. The real character of the fields in real space makes it necessary for them to satisfy the condition

$$\beta(\mathbf{k},t) = -\alpha^*(-\mathbf{k},t) \tag{2.2.6}$$

and it is now possible to express the fields $\mathcal{E}_\perp(\mathbf{k},t)$ and $\mathcal{B}(\mathbf{k},t)$ in respect to the new complete set of independent variables $\alpha(\mathbf{k},t)$ as

$$\mathcal{E}_\perp(\mathbf{k},t) = i\mathcal{N}(k)\left[\alpha(\mathbf{k},t) - \alpha^*(-\mathbf{k},t)\right] \tag{2.2.7}$$

$$\mathcal{B}(\mathbf{k},t) = i\frac{\mathcal{N}(k)}{c}\left[\kappa \times \alpha(\mathbf{k},t) + \kappa \times \alpha^*(-\mathbf{k},t)\right] \tag{2.2.8}$$

The global system of the fields and the particles at time t_0 is therefore described by the complete set of independent variables $\{\alpha(\mathbf{k},t_0), \mathbf{r}_\alpha(t_0), \dot{\mathbf{r}}_\alpha(t_0)\}$, where $\alpha(\mathbf{k},t_0)$ is complex. The time evolution of the normal variables is described by the equation

$$\dot{\alpha}(\mathbf{k},t) + i\omega\alpha(\mathbf{k},t) = \frac{i}{2\varepsilon_0\mathcal{N}(k)}j_\perp(\mathbf{k},t), \tag{2.2.9}$$

where Maxwell's equations and the definition of the normal variables have been used. For the free field these equations are easily solvable. The normal variables still span a two dimensional space orthogonal to the wavevector k for every given wavevector k. It is usual to simplify the notation and introduce the orthonormal basis $\left\{\varepsilon, \varepsilon', \kappa\right\}$, where ε and ε' describe two unit vectors transverse to the wavevector direction, κ. Expanding $\alpha(\mathbf{k},t)$ in respect to this basis yields

$$\alpha(\mathbf{k},t) = \sum_\varepsilon \varepsilon\alpha_\varepsilon(\mathbf{k},t), \tag{2.2.10}$$

where $\alpha_\varepsilon(\mathbf{k},t) = \varepsilon \cdot \alpha(\mathbf{k},t)$. For an even more concise notation α_i can be defined in the following manner:

$$\alpha_\varepsilon(\mathbf{k},t) \rightarrow \alpha_{\mathbf{k}\varepsilon}(t) \rightarrow \alpha_i. \tag{2.2.11}$$

In respect to this new set of variables the fields in real space can be expanded as

$$\mathbf{A}_\perp = \sum_i \mathcal{A}_{\omega_i}\left[\alpha_i\varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} + \alpha_i^*\varepsilon_i e^{-i\mathbf{k}_i\cdot\mathbf{r}}\right] \tag{2.2.12}$$

$$\mathbf{E}_\perp = i\sum_i \mathcal{E}_{\omega_i}\left[\alpha_i\varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} - \alpha_i^*\varepsilon_i e^{-i\mathbf{k}_i\cdot\mathbf{r}}\right] \tag{2.2.13}$$

$$\mathbf{B} = \sum_i \mathcal{B}_{\omega_i}\left[\alpha_i\kappa_i \times \varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} + \alpha_i^*\kappa_i \times \varepsilon_i e^{-i\mathbf{k}_i\cdot\mathbf{r}}\right], \tag{2.2.14}$$

where the constants $\mathcal{E}_{\omega_i}$, $\mathcal{B}_{\omega_i}$ and $\mathcal{A}_{\omega_i}$ have to be defined later and ω_i is given by the equation $\omega_i = ck_i$. The longitudinal part, $\mathbf{A}_\parallel$, of the vector potential $\mathbf{A}$ is arbitrary and depends on the gauge. It is not a dynamical variable, because it only contributes to the longitudinal part of the electric field, which itself is not a dynamical variable.

The relations between the electric field and the vector potential are

$$\mathbf{E}_\perp = -\frac{\partial}{\partial t}\mathbf{A}_\perp \tag{2.2.15}$$

$$\mathbf{E}_{\parallel} = -\frac{\partial}{\partial t}\mathbf{A}_{\parallel} - \nabla U \tag{2.2.16}$$

In the Coulomb gauge Eq.(2.2.16) simplifies to the expression $\mathbf{E}_{\parallel} = -\nabla U$, because $\mathbf{A}_{\parallel} = 0$ and therefore $\mathbf{E}_{\parallel}$ depends only on U and $\mathbf{E}_{\perp}$ depends only on $\mathbf{A}$ in this gauge. Other gauges are equally possible and in a later chapter the Lorentz gauge will be chosen.

3 Second quantization in the Coulomb gauge

As seen in the preceding section the general system in the Coulomb gauge is defined by the particle variables and the field variables. The first quantization as usual treats the position and momentum variables as operators with certain commutation relations. The second quantization transforms the normal variables to operators with certain commutation relations. For the particle variables the usual quantization will be used, which interprets the variables $r_{\alpha i}$ and $p_{\alpha i}$ as operators with the commutation relations

$$[r_{\alpha i}, r_{\beta j}] = [p_{\alpha i}, p_{\beta j}] = 0 \tag{2.3.1}$$

$$[r_{\alpha i}, p_{\beta j}] = i\hbar\delta_{\alpha\beta}\delta_{ij} \tag{2.3.2}$$

and $i, j = x, y, z$. Second quantization is achieved by interpreting the normal variables defined in the last section as operators, which satisfy commutation relations.

The normal variables α_i and $\alpha_i^\star$ are now replaced by the destruction operator a_i and the creation operator $a_i^\dagger$ with the commutation relations

$$[a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0 \tag{2.3.3}$$

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \tag{2.3.4}$$

With this new interpretation the physical fields can be expanded in terms of the new variables and one obtains

$$\mathbf{A}_{\perp} = \sum_i \mathcal{A}_{\omega_i}\left[a_i\varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} + a_i^\dagger\varepsilon_i e^{-i\mathbf{k}_i\cdot\mathbf{r}}\right], \tag{2.3.5}$$

$$\mathbf{E}_{\perp} = i\sum_i \mathcal{E}_{\omega_i}\left[a_i\varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} - a_i^\dagger\varepsilon_i e^{-i\mathbf{k}_i\cdot\mathbf{r}}\right] \tag{2.3.6}$$

and

$$\mathbf{B} = \sum_i \mathcal{B}_{\omega_i}\left[a_i\kappa_i\times\varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} + a_i^\dagger\kappa_i\times\varepsilon_i e^{-i\mathbf{k}_i\cdot\mathbf{r}}\right] \tag{2.3.7}$$

with

$$\mathcal{E}_{\omega_i} = \left[\frac{\hbar\omega_i}{2\varepsilon_0 L^3}\right]^{\frac{1}{2}}, \quad \mathcal{B}_{\omega_i} = \frac{\mathcal{E}_{\omega_i}}{c} \quad \text{and} \quad \mathcal{A}_{\omega_i} = \frac{\mathcal{E}_{\omega_i}}{\omega_i}. \tag{2.3.8}$$

The longitudinal part of the vector potential $\mathbf{A}_{\parallel}$ however depends on the gauge and since the Coulomb gauge is used it has to satisfy the equation

$$\mathbf{A}_{\parallel} = 0. \tag{2.3.9}$$

The scalar potential U is simply given by the electrostatic potential

$$U = \frac{1}{4\pi\varepsilon_0} \sum_{\alpha} \frac{q_\alpha}{|\mathbf{r} - \mathbf{r}_\alpha|}. \tag{2.3.10}$$

Since we are in the Coulomb gauge the vector potential $\mathbf{A}$ is identical to the perpendicular part of the vector potential $\mathbf{A}_{\perp}$ following the condition given in Eq.(2.3.9). Therefore the index $\perp$ for the vector potential $\mathbf{A}$ will be omitted in the remainder of this chapter. If the theory is restricted to the free field, the Hamiltonian H and the momentum P are given by simple operator expressions. They can be found in Cohen-Tannoudji [11, p.174] and are given as

$$H_R = \frac{\varepsilon_0}{2} \int d^3r \left[\mathbf{E}^2(\mathbf{r}) + c^2\mathbf{B}^2(\mathbf{r})\right] = \sum_i \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2}\right) \tag{2.3.11}$$

and

$$P_R = \varepsilon_0 \int d^3r \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) = \sum_i \hbar\mathbf{k}_i a_i^\dagger a_i \tag{2.3.12}$$

Since the i-th state in the Hamiltonian represents a harmonic oscillator the states of the free field can be expressed by Fock states of the form $|..., n_i, ...\rangle$ and the creation and destruction operators obey the usual relations

$$a_i^\dagger |..., n_i, ...\rangle = \sqrt{n_i + 1}\, |..., n_i + 1, ...\rangle \tag{2.3.13}$$

$$a_i |..., n_i, ...\rangle = \sqrt{n_i}\, |..., n_i - 1, ...\rangle \tag{2.3.14}$$

$$a_i |..., 0, ...\rangle = 0 \tag{2.3.15}$$

If all n_i are equal to zero the state is called the vacuum and every state of the system can be expressed in respect to the vacuum state following the usual rule

$$|..., n_i, ...\rangle = \frac{\left(a_i^\dagger\right)^{n_i}}{\sqrt{n_i!}} |..., 0, ...\rangle. \tag{2.3.16}$$

The so defined states are natural eigenvectors of the Hamiltonian and the momentum operators and the corresponding eigenvalue equations are

$$H_R |..., n_i, ...\rangle = \sum_i \left(n_i + \frac{1}{2}\right) \hbar\omega_i |..., n_i, ...\rangle \tag{2.3.17}$$

$$P_R |..., n_i, ...\rangle = \sum_i n_i \hbar\mathbf{k}_i |..., n_i, ...\rangle. \tag{2.3.18}$$

This completes the short introduction of Quantum Electrodynamics in the Coulomb gauge and in the next chapter this theory will be used to investigate the problem of defining a suitable photon number operator for the quantized field.

Chapter 3
Photon Number Density Operators in the Coulomb Gauge

The answer to the great question of...Life ,the Universe and Everything...Is...Forty-two.

-Douglas Adams, "The Hitch Hikers Guide through the Galaxy"

In the previous chapter the radiation field was quantized and annihilation and creation operators for the field variables were introduced. The excited states of the system are obtained from the vacuum by applying the creation operators for the different modes several times. The state obtained in this manner has the form

$$|n_1, ...n_i, ...\rangle , \tag{3.0.19}$$

with n_i describing the number of times the creation operator of a specific mode and polarization has been applied to the vacuum. The energy of this state is $\sum_i \left(n_i + \frac{1}{2}\right) \hbar\omega_i$ and therefore is raised from the vacuum by the amount $\sum_i n_i \hbar\omega_i$. Similarly the momentum is raised by the amount $\sum_i n_i \hbar \mathbf{k}_i$ and therefore this state can be interpreted as a state with n_i particles, with the energy $\hbar\omega_i$ and the momentum $\hbar\mathbf{k}_i$. These particles are called photons and they describe the excited modes of the quantized fields. Naturally the ground state has no excited modes and therefore is the state with no photon present.

The total number of photons in any state can be found using the number operator

$$N = \sum_i a_i^\dagger a_i. \tag{3.0.20}$$

as given in Loudon [12, p.134]. That this operator indeed counts the photons present in a state can be seen when one lets the operator act on the state given in Eq.(3.0.19). The so defined photons are Bosons, because the field was quantized with commutators and the number of photons in a specific state can exceed one. The question arising is "how many photons are at a given point in space at a given time?". The answer to this question is not trivial, because the photon has zero mass and Newton and

Wigner [1] stated in 1949 that a elementary system of this kind is not localizable. That means there does not exist a probability amplitude for the position operator of the photon. This is a slightly different question than the one investigated in this thesis, but the same problem arises. In this chapter some proposals for photon number density operators, that were given earlier, will be investigated.

1 Mandel's Photon Number Operator for a finite Volume

In 1966 Mandel [5] suggested that the operator

$$\hat{n}_{V,t} = \int_V \hat{\mathbf{A}}^\dagger(\mathbf{r},t) \cdot \hat{\mathbf{A}}(\mathbf{r},t) d^3x \qquad (3.1.1)$$

represents the number of photons in a finite volume V at a given time t. He described this operator in terms of the detection operator $\hat{\mathbf{A}}(\mathbf{r},t)$, which is not equal to the vector potential $\mathbf{A}(\mathbf{r},t)$ used in the last chapter. The detection operator is defined as

$$\hat{\mathbf{A}}(\mathbf{r},t) = \frac{1}{L^{\frac{3}{2}}} \sum_{\{\mathbf{k},\lambda\}} \hat{a}_{\mathbf{k},\lambda} \varepsilon_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-ckt)}. \qquad (3.1.2)$$

As usual the unit polarization vector $\varepsilon_{\mathbf{k},s}$ satisfies the orthogonality relation

$$\varepsilon^*_{\mathbf{k},\lambda} \cdot \varepsilon_{\mathbf{k},\lambda'} = \delta_{\lambda,\lambda'} \qquad (3.1.3)$$

and the integration volume is L^3. The usual photon number operator is

$$\hat{n} = \sum_{\{\mathbf{k}.\lambda\}} \hat{a}^\dagger_{\mathbf{k},\lambda} \cdot \hat{a}_{\mathbf{k},\lambda}, \qquad (3.1.4)$$

as given in Eq.(3.0.20).

First it will be shown that if the integration in Eq.(3.1.1) is over the complete space V it coincides with the number operator defined in Eq.(3.1.4). It is possible to substitute Eq.(3.1.2) into Eq.(3.1.1) and one gets

$$\begin{aligned} \hat{n}_{V,t} &= \int_V \left[\frac{1}{L^{\frac{3}{2}}} \sum_{\{\mathbf{k},\lambda\}} \hat{a}_{\mathbf{k},\lambda} \varepsilon_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-ckt)} \right]^\dagger \\ &\quad \cdot \left[\frac{1}{L^{\frac{3}{2}}} \sum_{\{\mathbf{k},\lambda\}} \hat{a}_{\mathbf{k},\lambda} \varepsilon_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-ckt)} \right] d^3x \end{aligned} \qquad ((3.0.1))$$

Performing the product and rearranging the terms this yields

$$\hat{n}_{V,t} = \frac{1}{L^3}\int_V \sum_{\{\mathbf{k},\mathbf{k}',\lambda,\lambda'\}} \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k}',\lambda'} \times \varepsilon^*_{\mathbf{k},\lambda}\cdot\varepsilon_{\mathbf{k}',\lambda'} e^{i\left[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}-c(k-k')t\right]} d^3r. \quad (3.1.5)$$

Integrating over the whole space L^3 and using periodic boundary conditions give the identity

$$\int_V e^{i\left[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right]} d^3r = \delta_{\mathbf{k},\mathbf{k}'}\cdot L^3, \quad (3.1.6)$$

which gives a Delta function in the above expression and simplifies it to

$$\hat{n}_{L^3,t} = \frac{1}{L^3}\sum_{\{\mathbf{k},\mathbf{k}',\lambda,\lambda'\}} \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k}',\lambda'}\varepsilon^*_{\mathbf{k},\lambda}\cdot\varepsilon_{\mathbf{k}',\lambda'}\delta_{\mathbf{k},\mathbf{k}'}\cdot L^3. \quad (3.1.7)$$

The sum over k and k$'$ reduces to a sum over k and one achieves

$$\hat{n}_{L^3,t} = \sum_{\{\mathbf{k},\lambda,\lambda'\}} \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda'}\varepsilon^*_{\mathbf{k},\lambda}\cdot\varepsilon_{\mathbf{k},\lambda'}, \quad (3.1.8)$$

which is equal to

$$\hat{n}_{L^3,t} = \sum_{\{\mathbf{k},\lambda\}} \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \quad (3.1.9)$$

using Eq.(3.1.3). This is the definition of the number operator as given in Eq.(3.1.4), which completes the proof. It should be mentioned that this result is only valid if the integration space is the total (normalization) volume L^3 and periodic boundary conditions are applied.

Another relation to Eq.(3.1.4) is given by the expectation values. For simple Fock states of the type $|..., n_{k,\lambda}, ...\rangle$ it will be shown that the expectation values are simply related by the expression

$$\langle\hat{n}_{V,t}\rangle = \left(\frac{V}{L^3}\right)\langle\hat{n}\rangle\,. \quad (3.1.10)$$

Substituting Eq.(3.1.1) into this expression

$$\langle\hat{n}_{V,t}\rangle = \langle..., n_{k,\lambda}, ...|\,\hat{n}_{V,t}\,|..., n_{k,\lambda}, ...\rangle \quad (3.1.11)$$

gives

$$\langle\hat{n}_{V,t}\rangle = \langle..., n_{k,\lambda}, ...|\int_V \hat{\mathbf{A}}^\dagger(\mathbf{r},t)\cdot\hat{\mathbf{A}}(\mathbf{r},t)d^3r\,|..., n_{k,\lambda}, ...\rangle\,. \quad (3.1.12)$$

It is possible to take the integral outside and after substituting the detection operators Eq.(12) yields

$$\begin{aligned}\langle \hat{n}_{V,t}\rangle &= \frac{1}{L^3}\int_V \langle ..., n_{k,\lambda}, ...| \sum_{\{\mathrm{k},\mathrm{k}',\lambda,\lambda'\}} \hat{a}^\dagger_{\mathrm{k},\lambda}\hat{a}_{\mathrm{k}',\lambda'}\varepsilon^*_{\mathrm{k},\lambda}\cdot\varepsilon_{\mathrm{k}',\lambda'} \\ &\quad \times e^{i\left[(\mathrm{k}-\mathrm{k}')\cdot\mathbf{r}-c(k-k')t\right]}|..., n_{k,\lambda}, ...\rangle\, d^3r \end{aligned} \tag{3.1.13}$$

Using the fact that only the destruction and creation operators have an effect on the states gives

$$\begin{aligned}\langle \hat{n}_{V,t}\rangle &= \frac{1}{L^3}\int_V \sum_{\{\mathrm{k},\mathrm{k}',\lambda,\lambda'\}} \varepsilon^*_{\mathrm{k},\lambda}\cdot\varepsilon_{\mathrm{k}',\lambda'} e^{i\left[(\mathrm{k}-\mathrm{k}')\cdot\mathbf{r}-c(k-k')t\right]} \\ &\quad \times \langle ..., n_{k,\lambda}, ...|\, \hat{a}^\dagger_{\mathrm{k},\lambda}\hat{a}_{\mathrm{k}',\lambda'}\, |..., n_{k,\lambda}, ...\rangle\, d^3r. \end{aligned} \tag{3.1.14}$$

Allowing the operators to work on the states leaves one with the equation

$$\begin{aligned}\langle \hat{n}_{V,t}\rangle &= \frac{1}{L^3}\int_V \sum_{\{\mathrm{k},\mathrm{k}',s,s'\}} \varepsilon^*_{\mathrm{k},s}\cdot\varepsilon_{\mathrm{k}',s'} e^{i\left[(\mathrm{k}-\mathrm{k}')\cdot\mathbf{r}-c(k-k')t\right]} \\ &\quad \times\sqrt{n_{k',s'}}\cdot\sqrt{n_{k,s}+1}\Big\langle ..., n_{k,s}, ...|..., n_{k,s}+1, ..., n_{k',s'}-1, ...\Big\rangle\, d^3r. \end{aligned} \tag{3.1.15}$$

The orthogonality relation for quantum mechanical states allows one to express this as

$$\begin{aligned}\langle \hat{n}_{V,t}\rangle &= \frac{1}{L^3}\int_V \sum_{\{\mathrm{k},\mathrm{k}',\lambda,\lambda'\}} \varepsilon^*_{\mathrm{k},\lambda}\cdot\varepsilon_{\mathrm{k}',\lambda'} e^{i\left[(\mathrm{k}-\mathrm{k}')\cdot\mathbf{r}-c(k-k')t\right]} \\ &\quad \times\sqrt{n_{k',\lambda'}}\cdot\sqrt{n_{k,\lambda}+1}\,\delta_{\big(\left(..., n_{k,\lambda}, ...\right),\left(..., n_{k,\lambda}+1, ..., n_{k',\lambda'}-1, ...\right)\big)} d^3r \end{aligned} \tag{3.1.16}$$

where the Kronecker delta is equal to one, only if $k = k'$ and $\lambda = \lambda'$. Therefore the equation simplifies to

$$\begin{aligned}\langle \hat{n}_{V,t}\rangle &= \frac{1}{L^3}\int_V \sum_{\{\mathrm{k},\mathrm{k}',\lambda,\lambda'\}} \varepsilon^*_{\mathrm{k},\lambda}\cdot\varepsilon_{\mathrm{k}',\lambda'} e^{i\left[(\mathrm{k}-\mathrm{k}')\cdot\mathbf{r}-c(k-k')t\right]} \\ &\quad \times\sqrt{n_{k',\lambda'}}\cdot\sqrt{n_{k,\lambda}+1}\,\delta_{(\mathrm{k},\mathrm{k}',\lambda,\lambda')} d^3r. \end{aligned} \tag{3.1.17}$$

Because the operators have to work successively this coincides with

$$\langle \hat{n}_{V,t}\rangle = \frac{1}{L^3}\int_V \sum_{\{\mathrm{k},\lambda\}} \sqrt{n_{k,\lambda}}\cdot\sqrt{(n_{k,\lambda}-1)+1}\, d^3r, \tag{3.1.18}$$

which gives the equation

$$\langle \hat{n}_{V,t} \rangle = \frac{V}{L^3} \sum_{\{\mathbf{k},\lambda\}} n_{k,\lambda} = (\frac{V}{L^3}) \langle \hat{n} \rangle \tag{3.1.19}$$

and completes the proof.

The expectation value was calculated in respect to the simple Fock state $|..., n_{k,\lambda}, ...\rangle$. A simple Fock state is obtained by operating on the vacuum state with creation operators as described by Milonni [13, p.333] and he mentions at the same place that those states have a definite photon number.

Difficulties in the theory arise if the commutation relations are investigated. The Mandel operators $\hat{n}$ and $\hat{n}_{V,t}$ always commute, but they nevertheless do not necessarily have the same eigenvectors. Two operators $\hat{n}_{V_1,t_1}$ and $\hat{n}_{V_2,t_2}$ with disjoint space-time regions do not strictly commute, which is due to the difficulty of localizing photons in space-time [5, p.1072]. To further investigate the defined operator $\hat{n}_{V,t}$ Mandel restricted his calculations to a finite volume, which is large compared with the wavelengths of all modes of the set $\{(\mathbf{k}, \lambda)\}$. With this restriction in mind he approximated the orthogonality relation for two different modes $\mathbf{k}, \mathbf{k}'$ as

$$\varepsilon^*_{\mathbf{k},\lambda} \cdot \varepsilon_{\mathbf{k}',\lambda'} \approx \varepsilon^*_{\mathbf{k}',\lambda} \cdot \varepsilon_{\mathbf{k}',\lambda'} = \delta_{\lambda,\lambda'}, \tag{3.1.20}$$

which basically means $\mathbf{k} \approx \mathbf{k}'$. This approximation is only possible if the investigated wave vectors are nearly the same.

2 Cook's Photon dynamics

Mandel was not able to give a continuity equation for his photon number operator, which is expected to be possible, if $\hat{n}_{V,t}$ is a photon number operator.

In the year 1982 Cook [6],[7] presented a different approach with two vector field operators. He introduced them as

$$\mathbf{\Psi}(\mathbf{r}) = (2L^3)^{-\frac{1}{2}} \sum_{\mathbf{k},\lambda} \varepsilon_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} e^{i\mathbf{k}.\mathbf{r}} \tag{3.2.1}$$

and

$$\mathbf{\Phi}(\mathbf{r}) = (2L^3)^{-\frac{1}{2}} \sum_{\mathbf{k},\lambda} (\frac{\mathbf{k}}{k}) \times \varepsilon_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} e^{i\mathbf{k}.\mathbf{r}}. \tag{3.2.2}$$

In contrast to Mandel he used the Heisenberg picture and therefore the time dependence is in the annihilation operator $a_{\mathbf{k}\lambda}$. The first one is identical to Mandel's detection operator, except for a multiplicative constant. The second one is more complicated, but it can be written in a more obvious fashion as

$$\mathbf{\Phi}(\mathbf{r}) = (2L^3)^{-\frac{1}{2}} \sum_{\mathbf{k},\lambda} \varepsilon_{\mathbf{k}\lambda'} a_{\mathbf{k}\lambda} e^{i\mathbf{k}.\mathbf{r}}, \tag{3.2.3}$$

where λ' is always the opposite value of λ and λ is as usual 1 or 2. Therefore it is the same vector operator as in Eq.(3.2.1) , only with the opposite polarization vector $\varepsilon_{\mathbf{k}\lambda'}$.

These operators can be used to construct a detection operator which Cook defined as

$$D(\mathbf{r}) = \mathbf{\Psi}^{\dagger}(\mathbf{r}) \cdot \mathbf{\Psi}(\mathbf{r}) + \mathbf{\Phi}^{\dagger}(\mathbf{r}) \cdot \mathbf{\Phi}(\mathbf{r}). \qquad (3.2.4)$$

This operator is positive definite and hermitian and therefore a good candidate for a quantum mechanical operator, as one would expect. Furthermore integration over the quantization volume gives the number operator $\hat{n}$:

$$\int d^3r D(\mathbf{r}) = \sum_{\mathbf{k},\lambda} a^{\dagger}_{\mathbf{k}\lambda} \cdot a_{\mathbf{k}\lambda} \qquad (3.2.5)$$

Because Cook used the Heisenberg picture it is possible to derive an continuity equation for photons using the Heisenberg equation of motion for $a_{\mathbf{k}\lambda}$, which is given by

$$\dot{a}_{\mathbf{k}\lambda} = \frac{1}{i\hbar}\left[a_{\mathbf{k}\lambda}, H\right], \qquad (3.2.6)$$

where H is the Hamiltonian of the system. Following a similar calculation as in Cohen-Tannoudji [11, p.181] one obtains the relation

$$\dot{a}_{\mathbf{k}\lambda} = -i\omega_k a_{\mathbf{k}\lambda} + i\left(\frac{2\pi}{\hbar\omega_k}\right)^{\frac{1}{2}} J_{\mathbf{k},\lambda}, \qquad (3.2.7)$$

where

$$J_{\mathbf{k},\lambda} = \int \mathbf{u}^{*}_{\mathbf{k},\lambda}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}) d^3r = \langle \mathbf{u}_{\mathbf{k},\lambda}(\mathbf{r}) \mid \mathbf{j}(\mathbf{r})\rangle \qquad (3.2.8)$$

and

$$\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{r}) = L^{-\frac{3}{2}}\varepsilon_{\mathbf{k}\lambda}e^{i\mathbf{k}.\mathbf{r}}.$$

The function $\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{r})$ is normalized and can be regarded as the transverse mode function. Therefore $J_{\mathbf{k},\lambda}$ is the projection of the electric current density on the transverse mode function. With these equations it is possible to derive the field equations.

The first two equations state that the vector field operators $\mathbf{\Psi}(\mathbf{r})$ and $\mathbf{\Phi}(\mathbf{r})$ are transverse

$$\nabla \cdot \mathbf{\Psi}(\mathbf{r}) = 0 \qquad (3.2.9)$$

$$\nabla \cdot \mathbf{\Phi}(\mathbf{r}) = 0. \qquad (3.2.10)$$

Two more equations can be derived, if one calculates the time derivatives $\frac{\partial}{\partial t}\mathbf{\Psi}(\mathbf{r})$ and $\frac{\partial}{\partial t}\mathbf{\Phi}(\mathbf{r})$. It is a straightforward calculation and the result is

$$\nabla \times \mathbf{\Psi}(\mathbf{r}) + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{\Phi}(\mathbf{r}) = \frac{1}{c}\mathbf{S}_1, \qquad (3.2.11)$$

$$\nabla \times \mathbf{\Phi}(\mathbf{r}) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{\Psi}(\mathbf{r}) = \frac{i}{c}\mathbf{S}_2, \tag{3.2.12}$$

where S_1 and S_2 are defined as the hermitian source terms

$$\mathrm{S}_1(\mathbf{r}) = \frac{i\pi^{\frac{1}{2}}}{(2\pi)^3}\int d^3r' \int \frac{d^3k}{(\hbar\omega_k)^{\frac{1}{2}}}\left(\frac{\mathbf{k}}{k}\times \mathbf{j}(\mathbf{r}')\right)e^{i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{k}} \tag{3.2.13}$$

and

$$\mathrm{S}_2(\mathbf{r}) = \frac{\pi^{\frac{1}{2}}}{(2\pi)^3}\int d^3r' \int \frac{d^3k}{(\hbar\omega_k)^{\frac{1}{2}}}\frac{\mathbf{k}}{k}\times\left(\frac{\mathbf{k}}{k}\times \mathbf{j}(\mathbf{r}')\right)e^{i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{k}}. \tag{3.2.14}$$

The hermitian source terms look complicated, but one simple case is in the absence of matter, because then they reduce to zero.

It also can be shown that D satisfies the continuity equation

$$\partial_t D + \nabla\cdot\mathbf{C} = Q, \tag{3.2.15}$$

where

$$Q = \mathrm{S}_1\cdot\mathbf{\Phi} + \mathbf{\Phi}^\dagger\cdot\mathrm{S}_1 + i(\mathrm{S}_2\cdot\mathbf{\Psi} - \mathbf{\Psi}^\dagger\cdot\mathrm{S}_2) \tag{3.2.16}$$

is the operator describing the interaction of the photon field with the sources and

$$\mathbf{C} = c(\mathbf{\Psi}^\dagger\times\mathbf{\Phi} - \mathbf{\Phi}^\dagger\times\mathbf{\Psi}) \tag{3.2.17}$$

is the photon current density.

As one can see, this continuity equation is not equal to zero on the right hand side. Therefore this equation should be called a weak continuity equation for photons in the presence of matter and the extra terms can be interpreted as sources of the photons. For free photons the right hand side vanishes and the regular continuity equation is achieved.

3 Inagaki's Photon Wave function

Recently Inagaki [8] used Cook's photon dynamics and reformulated it in conventional quantum mechanical terms. Only free photons were investigated, therefore the source terms S_1 and S_2 in Eqn.(3.2.11,3.2.12) are equal to zero. Furthermore no statement is made yet, if the negative frequency part in the plane wave expansions of $\mathbf{\Psi}(\mathbf{r})$ and $\mathbf{\Phi}(\mathbf{r})$ should be disregarded like Cook and Mandel did in their detection operators.

First the photon wave function $\vec{\mathbf{\Psi}}$ is defined as

$$\vec{\mathbf{\Psi}} = \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{\Phi} \end{bmatrix}, \tag{3.3.1}$$

which is a six-component column matrix and has the physical meaning of a probability amplitude. The probability density is then expected to be the scalar product of $\vec{\Psi}$ with itself. The scalar product of two functions Θ and Ξ is defined as usual:

$$\langle\Theta|\Xi\rangle = \int \Theta^*(\mathbf{r})\cdot\Xi(\mathbf{r})d^3r \tag{3.3.2}$$

Therefore the probability of finding a free photon at position r and time t is given by

$$\left\langle\vec{\Psi}|\vec{\Psi}\right\rangle = \int \vec{\Psi}^*(\mathbf{r},t)\cdot\vec{\Psi}(\mathbf{r},t)d^3r = \int \left[\mathbf{\Psi}^*(\mathbf{r},t)\mathbf{\Psi}(\mathbf{r},t) + \mathbf{\Phi}^*(\mathbf{r},t)\mathbf{\Phi}(\mathbf{r},t)\right]d^3x. \tag{3.3.3}$$

This defines the probability density as

$$D(\mathbf{r},t) = \mathbf{\Psi}^*(\mathbf{r},t)\mathbf{\Psi}(\mathbf{r},t) + \mathbf{\Phi}^*(\mathbf{r},t)\mathbf{\Phi}(\mathbf{r},t) \tag{3.3.4}$$

and clearly it coincides with the one given in Eq.(3.2.4).

Before the current can be defined it is necessary to find a Schrödinger equation for the given wave function. It is possible to rewrite Eqn.(3.2.9,3.2.10,3.2.11,3.2.12) in respect to the new wave function $\vec{\Psi}(\mathbf{r},t)$ as

$$div\vec{\Psi}(\vec{x},t) = \begin{bmatrix} \nabla\cdot\mathbf{\Psi}(\mathbf{r},t) \\ \nabla\cdot\mathbf{\Phi}(\mathbf{r}) \end{bmatrix} = 0 \tag{3.3.5}$$

and

$$\begin{bmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix}\partial_t\vec{\Psi}(\mathbf{r},t) = c\begin{bmatrix} \mathbf{T}\cdot\nabla & 0 \\ 0 & \mathbf{T}\cdot\nabla \end{bmatrix}\vec{\Psi}(\mathbf{r},t), \tag{3.3.6}$$

where $\mathbf{I}$ equals the 3×3 unit matrix and $\mathbf{T}$ is a three-vector with 3×3 matrices as components,which are given as

$$(T_i)_{jk} = -i\varepsilon_{ijk} \quad \text{with} \quad (i,j,k = 1,2,3). \tag{3.3.7}$$

Multiplying by $-i\hbar\begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}$ from the left on both sides of Eq.(3.3.6) defines the Hamiltonian H for a free photon:

$$H = c\begin{bmatrix} 0 & -\mathbf{T} \\ \mathbf{T} & 0 \end{bmatrix}\cdot\mathbf{p} \tag{3.3.8}$$

where $\mathbf{p} = -i\hbar\nabla$ is the quantum mechanical momentum operator. Another useful definition is the velocity operator

$$v_j = \frac{\partial}{\partial p_i}H = c\begin{bmatrix} 0 & -T_j \\ T_j & 0 \end{bmatrix}. \tag{3.3.9}$$

Now one is able to define the photon current $\mathbf{C}(\mathbf{r},t)$ as

$$\mathbf{C}(\mathbf{r},t)=\left\langle\vec{\boldsymbol{\Psi}}|\mathbf{v}|\vec{\boldsymbol{\Psi}}\right\rangle=c(\boldsymbol{\Psi}^*(\mathbf{r},t)\times\boldsymbol{\Phi}(\mathbf{r})-\boldsymbol{\Phi}^*(\mathbf{r})\times\boldsymbol{\Psi}(\mathbf{r},t)) \tag{3.3.10}$$

and with all this it is straightforward to show that the density $D(\mathbf{r},t)$ and the current $\mathbf{C}(\mathbf{r},t)$ satisfy the continuity equation

$$\partial_t D(\mathbf{r},t)+\nabla\cdot\mathbf{C}(\mathbf{r},t)=0. \tag{3.3.11}$$

Inagaki was able to define a wave function for a free photon with the property that the usual probability density definition leads to the same operator as in the Cook theory[7] and the photon current is defined in the usual way too. He also argued that the negative frequency parts in the plane wave expansions can be disregarded as unphysical, although they are necessary for the completeness relation for the plane wave solutions. Furthermore it is shown that the second quantization of this theory coincides with Cooks theory.

The operators described in this chapter are all fairly complicated constructions and it is difficult to investigate their behavior under Lorentz and gauge transformations. In the next chapter a new operator will be proposed which will be easier to work with and allows a generalization to a covariant form.

Chapter 4
A new Photon Number Density Operator in the Coulomb gauge

I'm not suggesting that the play is without fault; all of my plays are imperfect, I'm rather happy to say- it leaves me something to do.

-Edward Albee

In this chapter a new photon number density operator will be defined with properties similar to the ones defined in the previous chapter. This operator will be written in terms of operators whose equations of motion and properties under transformations are well known. It also will be possible to define a photon number current density and both operators will satisfy a continuity equation in the case of the free field. In a later chapter it also will be shown that those operators define a four vector with interesting properties. But for now the theory will be restricted to the Coulomb gauge and the free field.

1 Some comments about the notation used

As in Cohen-Tannoudji[11, p.171], the fields **E**, **B** and **A** are given in respect to the creation and destruction operators $a_i^\dagger$and a_i as

$$\mathbf{E}_\perp^{(+)}(\mathbf{r}) = \sum_i i \left[\frac{\hbar\omega_i}{2\varepsilon_0 L^3}\right]^{\frac{1}{2}} a_i \varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} \tag{4.1.1}$$

$$\mathbf{A}_\perp^{(+)}(\mathbf{r}) = \sum_i \left[\frac{\hbar}{2\varepsilon_0 L^3 \omega_i}\right]^{\frac{1}{2}} a_i \varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} \tag{4.1.2}$$

$$\mathbf{B}^{(+)}(\mathbf{r}) = \sum_i \frac{i}{c} \left[\frac{\hbar\omega_i}{2\varepsilon_0 L^3}\right]^{\frac{1}{2}} a_i (\mathbf{k}_i \times \varepsilon_i) e^{i\mathbf{k}_i\cdot\mathbf{r}} \tag{4.1.3}$$

$$\mathbf{E}_{\|}(\mathbf{r}) = \mathbf{A}_{\|}(\mathbf{r}) = U(\mathbf{r}) = 0. \tag{4.1.4}$$

In the free field case no matter is present and the fields are purely transverse. In the rest of this section the index $\perp$ will be omitted. Furthermore only the positive frequency parts in the plane wave expansions of the fields are considered.

The commutation relations for $a_i^\dagger$ and a_i are

$$[a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0 \tag{4.1.5}$$

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}. \tag{4.1.6}$$

The task of this chapter is to find a photon number operator density, which is described in respect to already defined quantities. It is desirable to have an equality between the new total photon number operator and the old one, which is

$$N = \sum_i a_i^\dagger a_i. \tag{4.1.7}$$

Before the new operator can be given it is helpful to introduce simple basics of covariant notation, because this notation is needed to motivate the form of the new operator. A more complete introduction will be given in a later chapter when the operator will be introduced in covariant form.

A four vector A^μ in covariant notation uses Greek letters in contrast to the ordinary notation, which uses Latin indices. The contravariant components of the four vector A^μ are defined as

$$A^\mu = \begin{pmatrix} A_s \\ \mathbf{A} \end{pmatrix}, \tag{4.1.8}$$

where $\mathbf{A}$ is the ordinary spatial potential three vector and the scalar A_s is equal to the ordinary scalar potential divided by the speed of light $\frac{U}{c}$. The covariant components A_μ are related to the contravariant components A^μ by the equation

$$A_\mu = g_{\mu\nu} A^\nu, \tag{4.1.9}$$

where $g_{\mu\nu}$ is the diagonal metric tensor ($g_{00} = +1, g_{11} = g_{22} = g_{33} = -1$) and repeated Greek indices are summed over. The standard free field Lagrangian density $\mathcal{L}_R^{st}$ written in this notation is

$$\mathcal{L}_R^{st} = -\frac{\varepsilon_0 c^2}{4} F_{\mu\nu} F^{\mu\nu} \tag{4.1.10}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.1.11}$$

is the electromagnetic field tensor and derivatives are $\partial_\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \nabla \end{pmatrix}$. The electromagnetic field tensor can also be written in the form

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}. \tag{4.1.12}$$

In the rest of the thesis this notation will be used, unless otherwise stated.

2 The photon density operator proposal in the Coulomb gauge

In this section the new photon number density operator will be defined and it will be shown that in the appropriate limits it equals the usual photon number operator. But first it will be shown how the new operator was found. In the Lagrangian theory symmetry transformations and conservation laws are closely related to each other following Noether's theorem. In a recent paper Hawton and Melde showed [14] that a photon number current four vector could be seen as a direct consequence of a hypothetical phase change symmetry of the Lagrangian and can be given by the contraction of the electromagnetic field tensor with the vector potential, $F^{\mu\nu}A_\mu$. This contraction is given by the equation

$$F^{\mu\nu}A_\mu = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \frac{U}{c} \\ -A_x \\ -A_y \\ -A_z \end{pmatrix}, \tag{4.2.1}$$

which equals

$$F^{\mu\nu}A_\mu = \begin{pmatrix} \frac{1}{c}\mathbf{E}\cdot\mathbf{A} \\ \frac{1}{c^2}E_x U + B_z A_y - B_y A_z \\ \frac{1}{c^2}E_y U - B_z A_x + B_x A_z \\ \frac{1}{c^2}E_z U + B_y A_x - B_x A_y \end{pmatrix} \tag{4.2.2}$$

and this result can also be written in the form

$$F^{\mu\nu}A_\mu = \begin{pmatrix} \frac{1}{c}\mathbf{E}\cdot\mathbf{A} \\ \frac{1}{c^2}\mathbf{E}U - \mathbf{B}\times\mathbf{A} \end{pmatrix}. \tag{4.2.3}$$

The achieved four vector gives the form of the new photon number density current four vector χ_μ, but this operator is not quantized. Quantization requires to split this operator in negative and positive frequency parts and to normal order them. It is also necessary to require that the operator is hermitian and normalized and the quantized operator therefore has to be

$$\chi_\mu = -\frac{\varepsilon_0}{i\hbar}\left(\begin{array}{c} c\left(\mathbf{E}^{(-)}.\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot\mathbf{E}^{(+)}\right) \\ c^2\left(\mathbf{B}^{(-)}\times\mathbf{A}^{(+)} + \mathbf{A}^{(-)}\times\mathbf{B}^{(+)} - \frac{1}{c^2}\left(\mathbf{E}^{(-)}U^{(+)} - U^{(-)}\mathbf{E}^{(+)}\right)\right)\end{array}\right). \tag{4.2.4}$$

The photon number density operator ρ_p is therefore defined as

$$\rho_p = -\frac{\varepsilon_0}{i\hbar}\left(\mathbf{E}^{(-)}.\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot\mathbf{E}^{(+)}\right). \tag{4.2.5}$$

It has to be shown now that the integration over the definition space $\int \rho_p\, dx$ is equal to the usual number operator N. Starting with the equation

$$\int \rho_p\, dx = -\frac{\varepsilon_0}{i\hbar}\int\left(\mathbf{E}^{(-)}.\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot\mathbf{E}^{(+)}\right)dx \tag{4.2.6}$$

it is possible to write this as

$$\begin{aligned}\int \rho_p\, dx &= -\frac{\varepsilon_0}{i\hbar}\int\sum_{ji}\frac{i\hbar}{2\varepsilon_0 L^3}\sqrt{\frac{\omega_j}{\omega_i}} \\ &\quad\times\left(-a_j^\dagger\varepsilon_j^* e^{-i\mathbf{k}_j\cdot\mathbf{r}}\cdot a_i\varepsilon_i e^{i\mathbf{k}_i\cdot\mathbf{r}} - a_i^\dagger\varepsilon_i^* e^{-i\mathbf{k}_i\cdot\mathbf{r}}\cdot a_j\varepsilon_j e^{i\mathbf{k}_j\cdot\mathbf{r}}\right)dx\end{aligned} \tag{4.2.7}$$

where Eqn.(4.1.1,4.1.2) were used. Rearranging the terms gives

$$\begin{aligned}\int \rho_p\, dx &= -\frac{1}{2L^3}\sum_{ji}\int\sqrt{\frac{\omega_j}{\omega_i}} \\ &\quad\times\left(-a_j^\dagger a_i\varepsilon_j^*\cdot\varepsilon_i e^{i(\mathbf{k}_i-\mathbf{k}_j\cdot)\mathbf{r}} - a_i^\dagger a_j\varepsilon_i^*\cdot\varepsilon_j e^{i(\mathbf{k}_j-\mathbf{k}_i)\cdot\mathbf{r}}\right)dx\end{aligned} \tag{4.2.8}$$

The only space dependance appears in the exponents and the integration over the whole space gives some delta functions. Performing this integration one yields the expression

$$\int \rho_p\, dx = -\frac{1}{2}\sum_{ji}\sqrt{\frac{\omega_j}{\omega_i}}\left(-a_j^\dagger a_i\varepsilon_j^*\cdot\varepsilon_i\delta_{ji} - a_i^\dagger a_j\varepsilon_i^*\cdot\varepsilon_j\delta_{ji}\right) \tag{4.2.9}$$

which simplifies to

$$\int \rho_p\, dx = -\frac{1}{2}\sum_i\left(-a_i^\dagger a_i\varepsilon_i^*\cdot\varepsilon_i - a_i^\dagger a_i\varepsilon_i^*\cdot\varepsilon_i\right). \tag{4.2.10}$$

This is equal to the photon number operator

$$\int \rho_p\, dx = \sum_i a_i^\dagger a_i = N \tag{4.2.11}$$

which completes the proof.

Therefore it is shown that it is not inconsistent to interpret ρ_p as the photon number density.

3 The continuity equation for photons in the case of a free field

In this section it will be shown that the photon number density current operator satisfies a continuity equation in the case of a free field. Following the form of the four vector the photon current density is naturally defined as the hermitian operator

$$\mathbf{j}_p = \frac{\varepsilon_0 c^2}{i\hbar}\left(\mathbf{B}^{(-)} \times \mathbf{A}^{(+)} + \mathbf{A}^{(-)} \times \mathbf{B}^{(+)}\right). \tag{4.3.1}$$

To show that this operator and the photon density operator ρ_p satisfy a continuity equation of the form

$$\frac{\partial}{\partial t}\rho_p + \nabla \cdot \mathbf{j}_p = 0, \tag{4.3.2}$$

the time derivative of the photon number density will be calculated and compared to the gradient of the photon current density.

The equation for the time derivative of the photon number density operator yields

$$\frac{\partial}{\partial t}\rho_p = \frac{\partial}{\partial t}\left(-\frac{\varepsilon_0}{i\hbar}\left(\mathbf{E}^{(-)}\cdot\mathbf{A}^{(+)} - \mathbf{A}^{(-)} \cdot \mathbf{E}^{(+)}\right)\right). \tag{4.3.3}$$

Using the product rule for the differentiation and keeping the same order of the terms gives

$$\begin{aligned}\frac{\partial}{\partial t}\rho_p &= -\frac{\varepsilon_0}{i\hbar}\left[\left(\frac{\partial}{\partial t}\mathbf{E}^{(-)}\right)\cdot\mathbf{A}^{(+)} + \mathbf{E}^{(-)}\cdot\left(\frac{\partial}{\partial t}\mathbf{A}^{(+)}\right)\right] \\ &\quad +\frac{\varepsilon_0}{i\hbar}\left[\left(\frac{\partial}{\partial t}\mathbf{A}^{(-)}\right)\cdot\mathbf{E}^{(+)} + \mathbf{A}^{(-)}\cdot\left(\frac{\partial}{\partial t}\mathbf{E}^{(+)}\right)\right]\end{aligned} \tag{4.3.4}$$

and substituting Maxwell's equations for the respective terms gives the expression

$$\begin{aligned}\frac{\partial}{\partial t}\rho_p &= -\frac{\varepsilon_0}{i\hbar}\left[c^2\left(\nabla \times \mathbf{B}^{(-)}\right)\cdot\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot c^2\left(\nabla \times \mathbf{B}^{(+)}\right)\right] \\ &\quad -\frac{\varepsilon_0}{i\hbar}\left[\mathbf{E}^{(-)}\cdot\left(-\mathbf{E}^{(+)}\right) - \left(-\mathbf{E}^{(-)}\right)\cdot\mathbf{E}^{(+)}\right].\end{aligned} \tag{4.3.5}$$

But collecting all terms this yields

$$\frac{\partial}{\partial t}\rho_p = -\frac{\varepsilon_0 c^2}{i\hbar}\left[\left(\nabla \times \mathbf{B}^{(-)}\right)\cdot\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot\left(\nabla \times \mathbf{B}^{(+)}\right)\right]. \tag{4.3.6}$$

Next it will be shown that $\nabla \cdot \mathbf{j}_p$ leads to the negative of this expression. Inserting the expression for the current gives

$$\nabla \cdot \mathbf{j}_p = \nabla\cdot\frac{\varepsilon_0 c^2}{i\hbar}\left(\mathbf{B}^{(-)} \times \mathbf{A}^{(+)} + \mathbf{A}^{(-)} \times \mathbf{B}^{(+)}\right) \tag{4.3.7}$$

which is equal to

$$\nabla \cdot \mathbf{j}_p = \frac{\varepsilon_0 c^2}{i\hbar} \left[\nabla \cdot \left(\mathbf{B}^{(-)} \times \mathbf{A}^{(+)}\right) + \nabla \cdot \left(\mathbf{A}^{(-)} \times \mathbf{B}^{(+)}\right)\right]. \tag{4.3.8}$$

Using a well known formula for the gradient operation this can be expressed as

$$\begin{aligned} \nabla \cdot \mathbf{j}_p &= \frac{\varepsilon_0 c^2}{i\hbar} \left[\left(\nabla \times \mathbf{B}^{(-)}\right) \cdot \mathbf{A}^{(+)} - \mathbf{B}^{(-)} \cdot \left(\nabla \times \mathbf{A}^{(+)}\right)\right] \\ &+ \frac{\varepsilon_0 c^2}{i\hbar} \left[\left(\nabla \times \mathbf{A}^{(-)}\right) \cdot \mathbf{B}^{(+)} - \mathbf{A}^{(-)} \cdot \left(\nabla \times \mathbf{B}^{(+)}\right)\right] \end{aligned} \tag{4.3.9}$$

and collecting all terms gives

$$\nabla \cdot \mathbf{j}_p = \frac{\varepsilon_0 c^2}{i\hbar} \left[\left(\nabla \times \mathbf{B}^{(-)}\right) \cdot \mathbf{A}^{(+)} - \mathbf{A}^{(-)} \cdot \left(\nabla \times \mathbf{B}^{(+)}\right)\right] \tag{4.3.10}$$

remembering that $\nabla \times \mathbf{A}^{(\pm)} = \mathbf{B}^{(\pm)}$. Adding the terms in Eqn.(4.2.6,4.2.10) yields

$$\begin{aligned} \frac{\partial}{\partial t}\rho_p + \nabla \cdot \mathbf{j}_p &= \\ & \frac{\varepsilon_0 c^2}{i\hbar} \left[\left(\nabla \times \mathbf{B}^{(-)}\right) \cdot \mathbf{A}^{(+)} - \mathbf{A}^{(-)} \cdot \left(\nabla \times \mathbf{B}^{(+)}\right)\right] \\ & + \frac{\varepsilon_0 c^2}{i\hbar} \left[-\left(\nabla \times \mathbf{B}^{(-)}\right) \cdot \mathbf{A}^{(+)} + \mathbf{A}^{(-)} \cdot \left(\nabla \times \mathbf{B}^{(+)}\right)\right], \end{aligned} \tag{4.3.11}$$

which is obviously equal to zero and it is shown that for the free field the continuity equation given in Eq.(4.2.2) is valid.

It is therefore possible to define a photon number density and a photon current operator, described in respect to well known operators, that satisfy a continuity equation in the case of the free field. The transformation properties of this operator are simple, because only well known field equations were used to define the new photon number density operator. It is believed that this operator is a good alternative to the ones described in the last chapter. In a later chapter this operator will also be given in the Lorentz gauge, but in the next section a simple case of interaction with matter will be investigated.

4 The case of a field with matter interaction

In this section the proposed photon density and photon current will be modified to allow the interaction with matter. It is not possible to use Maxwell's equations in the same way as it was done in the last section, because now the interaction terms have to be included. The annihilation operators now satisfy the differential equations in reciprocal space

$$i\hbar\dot{a}_i = \hbar\omega_i a_i - \frac{\hbar}{2\varepsilon_0\mathcal{N}} j_i. \tag{4.4.1}$$

The general solution of this equation can be derived with the Ansatz

$$a_i \sim e^{i\omega t}, \tag{4.4.2}$$

where ω can be positive and negative, and Eq.(4.4.1) yields

$$\left(i\hbar\left(i\omega\right) - \hbar\omega_i\right) a_i = -\frac{\hbar}{2\varepsilon_0\mathcal{N}} j_i, \tag{4.4.3}$$

which gives for the annihilation operator the expression

$$a_i = \frac{1}{\left(\omega_i + \omega\right) 2\varepsilon_0\mathcal{N}} j_i. \tag{4.4.4}$$

Furthermore it can be assumed that the driving force is of the form

$$j_i \sim e^{i\omega t} \tag{4.4.5}$$

and Eq.(4.4.4) gives

$$a_i = \frac{1}{\left(\omega_i + \omega\right)} A e^{i\omega t}, \tag{4.4.6}$$

where A is a proportionality constant. If the matter is not dense as was argued in a paper by Hawton[15] the frequency shift $\Delta\omega = \omega_i - \omega$ is very small and the driving force only contributes significantly if ω is negative. Thus, if the driving force satisfies $j_i \sim e^{-i|\omega|t}$ the annihilation operator has a big contribution and if it satisfies $j_i \sim e^{i|\omega|t}$ the annihilation operator has a small contribution. In the same way one can show that the creation operator has big contribution if $j_i \sim e^{i|\omega|t}$ and a small contribution if $j_i \sim e^{-i|\omega|t}$ is valid. It is therefore possible to say that the part of the driving force that is proportional to $e^{-i|\omega|t}$ only contributes to the annihilation operator and the part that is proportional to $e^{-i|\omega|t}$ only contributes to the creation operator. In this special case it is therefore possible to split the driving force term in negative and positive frequency parts that solely contribute to the annihilation or creation operators. Now it is possible to investigate the existence of a continuity equation for this special case of matter interaction.

It is necessary to return to Eq.(4.2.4) and the photon current

$$\mathbf{j}_p = \frac{\varepsilon_0 \mathbf{c}^2}{i\hbar}\left(\mathbf{B}^{(-)} \times \mathbf{A}^{(+)} + \mathbf{A}^{(-)} \times \mathbf{B}^{(+)} - \frac{1}{\mathbf{c}^2}\left(\mathbf{E}^{(-)}U^{(+)} - U^{(-)}\mathbf{E}^{(+)}\right)\right). \tag{4.4.7}$$

Next it will be investigated, if the so defined photon density and photon current satisfy a continuity equation. In Maxwell's equations the charge density and the current density are obviously not equal to zero anymore. The separated equations are of the form

$$\nabla \cdot \mathbf{E}^{(+)}(\mathbf{r},t) = \frac{1}{\varepsilon_0}\rho^{(+)}(\mathbf{r},t) \tag{4.4.8}$$

$$\nabla \cdot \mathbf{E}^{(-)}(\mathbf{r},t) = \frac{1}{\varepsilon_0}\rho^{(-)}(\mathbf{r},t) \tag{4.4.9}$$

$$\nabla \times \mathbf{B}^{(+)}(\mathbf{r},t) = -\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^{(+)}(\mathbf{r},t) + \frac{1}{\varepsilon_0 c^2}\mathbf{j}^{(+)}(\mathbf{r},t) \tag{4.4.10}$$

$$\nabla \times \mathbf{B}^{(-)}(\mathbf{r},t) = -\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^{(-)}(\mathbf{r},t) + \frac{1}{\varepsilon_0 c^2}\mathbf{j}^{(-)}(\mathbf{r},t) \tag{4.4.11}$$

and the other Maxwell equations behave in the same manner.

Now it is possible to calculate the continuity equation $\frac{\partial}{\partial t}\rho_p + \nabla \cdot \mathbf{j}_p$. Starting with the first term and substituting the expression for the density in this term one yields

$$\begin{aligned}\frac{\partial}{\partial t}\rho_p &= -\frac{\varepsilon_0}{i\hbar}\left[\left(\frac{\partial}{\partial t}\mathbf{E}^{(-)}\right)\cdot\mathbf{A}^{(+)} + \mathbf{E}^{(-)}\cdot\left(\frac{\partial}{\partial t}\mathbf{A}^{(+)}\right)\right] \\ &\quad +\frac{\varepsilon_0}{i\hbar}\left[\left(\frac{\partial}{\partial t}\mathbf{A}^{(-)}\right)\cdot\mathbf{E}^{(+)} - \mathbf{A}^{(-)}\cdot\left(\frac{\partial}{\partial t}\mathbf{E}^{(+)}\right)\right]\end{aligned} \tag{4.4.12}$$

Following the same calculation as in the last section this is equal to

$$\begin{aligned}\frac{\partial}{\partial t}\rho_p &= -\frac{\varepsilon_0}{i\hbar}\left[c^2\left(\nabla \times \mathbf{B}^{(-)}\right)\cdot\mathbf{A}^{(+)} - \frac{1}{\varepsilon_0}\mathbf{j}^{(-)}\cdot\mathbf{A}^{(+)} + \Lambda\right] \\ &\quad +\frac{\varepsilon_0}{i\hbar}\left[\mathbf{A}^{(-)}\cdot c^2\left(\nabla \times \mathbf{B}^{(+)}\right) - \frac{1}{\varepsilon_0}\mathbf{A}^{(-)}\cdot\mathbf{j}^{(+)}\right]\end{aligned} \tag{4.4.13}$$

where

$$\Lambda = \mathbf{E}^{(-)}\cdot\left(\frac{\partial}{\partial t}\mathbf{A}^{(+)}\right) - \left(\frac{\partial}{\partial t}\mathbf{A}^{(-)}\right)\cdot\mathbf{E}^{(+)} \tag{4.4.14}$$

and rearranging the terms gives

$$\begin{aligned}\frac{\partial}{\partial t}\rho_p &= -\frac{\varepsilon_0}{i\hbar}\left[\frac{1}{\varepsilon_0}\mathbf{A}^{(-)}\cdot\mathbf{j}^{(+)} - \frac{1}{\varepsilon_0}\mathbf{j}^{(-)}\cdot\mathbf{A}^{(+)} + \Lambda\right] \\ &\quad -\frac{\varepsilon_0}{i\hbar}\left[c^2\left(\left(\nabla \times \mathbf{B}^{(-)}\right)\cdot\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot\left(\nabla \times \mathbf{B}^{(+)}\right)\right)\right]\end{aligned} \tag{4.4.15}$$

Now calculate the second term of the continuity equation.

Substituting the new expression for the proposed photon current density operator the second term is equal to

$$\nabla \cdot \mathbf{j}_p = \nabla \cdot \frac{\varepsilon_0 c^2}{i\hbar}\left(\mathbf{B}^{(-)} \times \mathbf{A}^{(+)} + \mathbf{A}^{(-)} \times \mathbf{B}^{(+)} - \frac{1}{c^2}\left(\mathbf{E}^{(-)}U^{(+)} - U^{(-)}\mathbf{E}^{(+)}\right)\right) \tag{4.4.16}$$

and in this expression the first part will yield the same result as it did in the last section. Therefore it is only necessary to calculate the term

$$\Xi = \nabla\cdot\frac{\varepsilon_0 c^2}{i\hbar}\left[-\frac{1}{c^2}\left(\mathbf{E}^{(-)}U^{(+)} - U^{(-)}\mathbf{E}^{(+)}\right)\right] \tag{4.4.17}$$

which is equal to

$$\begin{aligned}\Xi &= -\frac{\varepsilon_0}{i\hbar}\left[\left(\nabla\cdot\mathbf{E}^{(-)}\right)U^{(+)} + \mathbf{E}^{(-)}\cdot\left(\nabla U^{(+)}\right)\right] \\ &\quad +\frac{\varepsilon_0}{i\hbar}\left[\left(\nabla U^{(-)}\right)\cdot\mathbf{E}^{(+)} + U^{(-)}\left(\nabla\cdot\mathbf{E}^{(+)}\right)\right].\end{aligned} \tag{4.4.18}$$

Using Maxwell's equations this can be rewritten as

$$\begin{aligned}\Xi &= -\frac{\varepsilon_0}{i\hbar}\left[\frac{1}{\varepsilon_0}\left(\rho^{(-)}U^{(+)} - U^{(-)}\rho^{(+)}\right) - \mathbf{E}^{(-)}\cdot\mathbf{E}^{(+)} + \mathbf{E}^{(-)}\cdot\mathbf{E}^{(+)}\right] \\ &\quad +\frac{\varepsilon_0}{i\hbar}\left[\mathbf{E}^{(-)}\cdot\left(\frac{\partial}{\partial t}\mathbf{A}^{(+)}\right) - \left(\frac{\partial}{\partial t}\mathbf{A}^{(-)}\right)\cdot\mathbf{E}^{(+)}\right]\end{aligned} \tag{4.4.19}$$

which is equal to

$$\Xi = -\frac{\varepsilon_0}{i\hbar}\left[\frac{1}{\varepsilon_0}\left(\rho^{(-)}U^{(+)} - U^{(-)}\rho^{(+)}\right) - \Lambda\right] \tag{4.4.20}$$

with Λ defined by Eq.(4.3.14).

Combining all these steps yields the final result

$$\frac{\partial}{\partial t}\rho_p + \nabla\cdot\mathbf{j}_p = \frac{1}{i\hbar}\left[\mathbf{j}^{(-)}\cdot\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot\mathbf{j}^{(+)} + U^{(-)}\rho^{(+)} - \rho^{(-)}U^{(+)}\right] \tag{4.4.21}$$

It is easy to see that the free field case is a special case of this equation, because if the charge density and the current equals zero the right hand side vanishes.

It is furthermore possible to simplify the right hand side of Eq.(4.3.21), if it is written as $\left(\mathbf{A}^{(-)}\cdot\mathbf{j}^{(+)} + \rho^{(-)}U^{(+)}\right)^{\dagger} - \mathbf{A}^{(-)}\cdot\mathbf{j}^{(+)} + \rho^{(-)}U^{(+)}$, which shows this term is purely imaginary and therefore the right hand side of Eq.(4.3.21) is purely real. Note that neither the negative, nor the positive frequency parts themselves were assumed to be purely real, or purely imaginary. In the case of matter present the photon number and current operators are not satisfying a continuity equation anymore. A term describing the photon matter interaction has to be included which acts like a source or a sink for photons in physical space. Cook's theory had a similar result and therefore it is not surprising that the extra terms appeared.

Chapter 5
Lagrangian and Hamiltonian formalism in the Lorentz Gauge

There is nothing worse than a brilliant image of a fuzzy concept

-Ansel Adams

In this chapter the covariant formulation of quantum electrodynamics will be introduced. It starts with an introduction to the covariant notation and Lagrangian formalism in this new description. A new Lagrangian density will be introduced and new physical variables will be derived from this Lagrangian density. The theory will also be described in reciprocal space and the normal modes will be extended. Later in the chapter important physical variables will be expressed in respect to a suitable basis and problems like negative energy states will be shown. At the end of the chapter the field will be quantized and the problems solved.

1 The Lagrangian formalism in covariant notation

Before the vector potential **A** can be expressed in the Lorentz gauge and the theory can be quantized, it is necessary to formulate Lagrange's and Hamilton's formalisms in covariant notation. The four vector notation was already introduced in an earlier chapter, but nevertheless it will be included in this chapter, because it is the starting point for a covariant theory.

The potential four vector A^μ in covariant notation uses Greek letters in contrast to the ordinary notation, which uses Latin indices. The contravariant components of the four vector A^μ are defined as

$$A^\mu = \begin{pmatrix} A_s \\ \mathbf{A} \end{pmatrix}, \tag{5.1.1}$$

where **A** is the ordinary spatial potential three vector and the scalar A_s is equal to the ordinary scalar potential divided by the speed of light, $\frac{U}{c}$. The covariant components

A_μ are related to the contravariant components A^μ by the equation

$$A_\mu = g_{\mu\nu} A^\nu, \tag{5.1.2}$$

where $g_{\mu\nu}$ is the diagonal metric tensor ($g_{00} = +1, g_{11} = g_{22} = g_{33} = -1$) and repeated Greek indice are summed over. The standard free field Lagrangian density $\mathcal{L}_R^{st}$ written in this notation is

$$\mathcal{L}_R^{st} = -\frac{\varepsilon_0 c^2}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.1.3}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.1.4}$$

is the electromagnetic field tensor and derivatives are $\partial_\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \nabla \end{pmatrix}$. Now it is possible to formulate the Lagrange formalism in this notation.

In the covariant formulation $U\left(\mathbf{r}\right)$ and $A_j\left(\mathbf{r}\right)$ are treated symmetrically, but the standard Lagrangian does not depend on $\dot{U}(\mathbf{r})$ and therefore $U\left(\mathbf{r}\right)$ does not possess a conjugate momentum. A new Langrangian density for the free field $\mathcal{L}_R$, which depends explicitly on $\dot{U}(\mathbf{r})$ and is manifestly covariant has to be defined. Consider the Lagrangian density

$$\mathcal{L}_R = -\frac{\varepsilon_0 c^2}{4} \left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right), \tag{5.1.5}$$

as defined in Cohen-Tannoudji [11, p.365], which is clearly covariant and depends explicitly on $\dot{U}(\mathbf{r})$. It is interesting to note that this Lagrangian density only involves first order derivatives of the potentials and quadratic terms of A_μ, which insures that the derived Lagrange equations will be linear in the potentials. This Lagrangian density also can be written as

$$\mathcal{L}_R = \frac{\varepsilon_0}{2}\left[\dot{\mathbf{A}}^2 - c^2 \sum_{ij} \left(\partial_i A_j\right)^2 - \left(\frac{\dot{U}}{c}\right)^2 + \left(\nabla U\right)^2\right]. \tag{5.1.6}$$

The spatial and scalar parts in this equation have opposite signs and this situation followed naturally from the definition of the Lagrangian density given in Eq.(5.1.5).

Furthermore one cannot expect the Lagrange equations in respect to this Lagrangian density to be the same as Maxwell's equations, which were derived from the standard Lagrangian, without imposing a subsidiary condition. If interaction of the field with particles is allowed, one has to add the interaction Lagrangian density

$$\mathcal{L}_I = -j_\mu A^\mu \tag{5.1.7}$$

where j_μ is the current four vector $(c\rho, \mathrm{j})$. The Lagrange equations regarding the Lagrangian density $\mathcal{L} = \mathcal{L}_R + \mathcal{L}_I$ are easily calculated and the result is the set of symmetric equations

$$\Box \mathbf{A} = \frac{1}{\varepsilon_0 c^2}\mathrm{j} \tag{5.1.8}$$

$$\Box U = \frac{1}{\varepsilon_0}\rho \tag{5.1.9}$$

where $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle$ is the d'Alembertian.

On the other hand Maxwell's equations in covariant form are

$$\Box \mathbf{A} = \frac{1}{\varepsilon_0 c^2}\mathrm{j} - \nabla\Lambda \tag{5.1.10}$$

$$\Box U = \frac{1}{\varepsilon_0}\rho + \frac{\partial}{\partial t}\Lambda, \tag{5.1.11}$$

where

$$\Lambda = \frac{1}{c^2}\frac{\partial}{\partial t}U + \nabla \cdot \mathbf{A}. \tag{5.1.12}$$

There is a choice of the potentials for which Eqns.(5.1.8,5.1.9) coincide with Maxwell's equations. These are the potentials that satisfy the subsidiary condition

$$\Lambda = \frac{1}{c^2}\frac{\partial}{\partial t}U + \nabla \cdot \mathbf{A} = 0, \tag{5.1.13}$$

which is, written in the covariant notation,

$$\partial_\mu A^\mu = 0. \tag{5.1.14}$$

This defines the Lorentz gauge.

It is possible to derive an equation of evolution for Λ from the Eqns.(5.1.8,5.1.9) and the result is

$$\Box\Lambda = \frac{1}{\varepsilon_0 c^2}\left(\nabla \cdot \mathrm{j} + \frac{\partial}{\partial t}\rho\right) = 0, \tag{5.1.15}$$

where the conservation of charge was used. Furthermore $\dot{\Lambda}$ can be calculated using Eq.(5.1.9) and the following result is achieved

$$\dot{\Lambda} = -\nabla \cdot \mathbf{E} + \frac{\rho}{\varepsilon_0} = 0. \tag{5.1.16}$$

This ensures that if $\dot{\Lambda} = \Lambda = 0$ is true at the initial time, then $\Lambda = 0$ is true at all times. Finally it has to be mentioned that there are still some gauge changes allowed. The potential $A'_\mu = A_\mu - \partial_\mu f$ still satisfies the Lorentz condition, if $\Box f = 0$. In every Lorentz gauge the new Lagrangian density yields the same equations of motion as the standard Lagrangian density, which makes the new Lagrangian density a suitable choice for a covariant theory.

It is also possible to write this Lagrangian density in reciprocal space. To distinguish the Lagrangian densities in the different base representations, the Lagrangian density expressed in respect to the reciprocal space is $\bar{\mathcal{L}}_R$ and $\mathcal{A}$ is the Fourier transform of **A**. The calculation is a Fourier transformation and the result is

$$\bar{\mathcal{L}}_R = \varepsilon_0 \left[\dot{\mathcal{A}}^* \cdot \dot{\mathcal{A}} - \omega^2 \mathcal{A}^* \cdot \mathcal{A} - \dot{\mathcal{A}}_s^* \dot{\mathcal{A}}_s + \omega^2 \mathcal{A}_s^* \mathcal{A}_s\right] \tag{5.1.17}$$

with $\omega = ck$. This Lagrangian density describes four harmonic oscillators, three associated with the spatial components and one with the time component of the potential four vector. The time component oscillator has again the opposite sign of the spatial component oscillators.

The Lagrangian for the radiation field has therefore the two representations

$$L_R = \int d^3r \mathcal{L}_R = \int^* d^3k \bar{\mathcal{L}}_R \tag{5.1.18}$$

where $\int^*$ denotes integration over half the space. This happens, because the potentials A_j and A_s are necessarily real, which implies that their Fourier transforms satisfy the equations

$$\mathcal{A}_j(\mathbf{k}) = \mathcal{A}_j^*(-\mathbf{k}) \tag{5.1.19}$$

$$\mathcal{A}_s(\mathbf{k}) = \mathcal{A}_s^*(-\mathbf{k})\,. \tag{5.1.20}$$

The system is therefore completely defined if the vector potentials are known in one half space.

The conjugate momenta π_j and π_s are defined by the equations

$$\pi_j = \frac{\partial \bar{\mathcal{L}}_R}{\dot{\mathcal{A}}_j^*} = \varepsilon_0 \dot{\mathcal{A}}_j \tag{5.1.21}$$

$$\pi_s = \frac{\partial \bar{\mathcal{L}}_R}{\dot{\mathcal{A}}_s^*} = -\varepsilon_0 \dot{\mathcal{A}}_s \tag{5.1.22}$$

and equations analogous to Eqn.(5.1.21,5.1.22) for the conjugate momenta π_j and π_s have to be true. The sign difference between the spatial equations and the scalar equation is important and will give rise to some difficulties.

The subsidiary condition in reciprocal space is

$$i\mathbf{k}\cdot\mathcal{A} = \frac{1}{\varepsilon_0 c^2}\pi_s. \tag{5.1.23}$$

It is also possible to write the radiation field Hamiltonian density in reciprocal space, $\bar{\mathcal{H}}_R$, as

$$\bar{\mathcal{H}}_R = \varepsilon_0 \left[\frac{1}{\varepsilon_0^2}\pi^* \cdot \pi + \omega^2 \mathcal{A}^* \cdot \mathcal{A} - \frac{1}{\varepsilon_0^2}\pi_s^* \cdot \pi_s - \omega^2 \mathcal{A}_s^* \cdot \mathcal{A}_s\right] \tag{5.1.24}$$

Furthermore it is possible to write Hamilton-Jacobi equations, which give the time evolution of the vector potential and its conjugate momentum. The set of equations for the configuration space parts is given by

$$\dot{\mathcal{A}}_j = \frac{\partial \bar{\mathcal{H}}_R}{\partial \pi_j^*} = \frac{1}{\varepsilon_0}\pi_j \tag{5.1.25}$$

$$\dot{\pi}_j = -\frac{\partial \bar{\mathcal{H}}_R}{\partial \mathcal{A}_j^*} = -\varepsilon_0 \omega^2 \mathcal{A}_j \tag{5.1.26}$$

and for the scalar parts it is given by

$$\dot{\mathcal{A}}_s = \frac{\partial \bar{\mathcal{H}}_R}{\partial \pi_s^*} = -\frac{1}{\varepsilon_0}\pi_s \tag{5.1.27}$$

$$\dot{\pi}_s = -\frac{\partial \bar{\mathcal{H}}_R}{\partial \mathcal{A}_s^*} = \varepsilon_0 \omega^2 \mathcal{A}_s. \tag{5.1.28}$$

Once again it is important to see the sign difference between the right hand sides of the equations in respect to the three space variables and the scalar variables. In the next section the normal modes for this system will be defined.

2 The normal modes in the covariant notation

In chapter two the normal modes were a set of equations that describe the system completely and evolve independent of each other following an equation of the form

$$\dot{\alpha} + i\omega\alpha = 0 \tag{5.2.1}$$

in the absence of sources. For the spatial part one can define the normal variables

$$\alpha_j = \sqrt{\frac{\varepsilon_0}{2\hbar\omega}}\left[\omega\mathcal{A}_j + \frac{i}{\varepsilon_0}\pi_j\right]. \tag{5.2.2}$$

Using Eqn.(5.1.21,5.1.22,5.1.25,5.1.26) it is possible to show that these variables satisfy an equation like the one in Eq.(5.2.1).

Because of the plus sign in $\dot{\pi}_s$ the normal variable for the scalar part is defined slightly differently. It is necessary to define the normal variables for the scalar part as

$$\alpha_s = \sqrt{\frac{\varepsilon_0}{2\hbar\omega}}\left[\omega\mathcal{A}_s - \frac{i}{\varepsilon_0}\pi_s\right] \tag{5.2.3}$$

and this equation satisfies Eq.(5.2.1) too.

Now it is possible to express the vector potentials in respect to this new basis. The potentials A_j and A_s are necessarily real, which implies that their Fourier transforms satisfy the equations

$$\mathcal{A}_j(\mathbf{k}) = \mathcal{A}_j^*(-\mathbf{k}) \tag{5.2.4}$$

$$\mathcal{A}_s(\mathbf{k}) = \mathcal{A}_s^*(-\mathbf{k}) \tag{5.2.5}$$

and analogous equations for the conjugate momenta π_j and π_s. This condition can only be satisfied if the vector potentials expressed in respect to the new basis, are of the form

$$\mathcal{A}_j(\mathbf{k},t) = \sqrt{\frac{\hbar}{2\varepsilon_0\omega}}\left[\alpha_j(\mathbf{k},t) + \alpha_j^*(-\mathbf{k},t)\right] \tag{5.2.6}$$

$$\mathcal{A}_s(\mathbf{k},t) = \sqrt{\frac{\hbar}{2\varepsilon_0\omega}}\left[\alpha_s(\mathbf{k},t) + \alpha_s^*(-\mathbf{k},t)\right] \tag{5.2.7}$$

Therefore the potentials in real space are given in normal variables by

$$A_j(\mathbf{r},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\mathcal{A}_j(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{r}} \tag{5.2.8}$$

which can be expressed as

$$A_j(\mathbf{r},t) = \int d^3k\sqrt{\frac{\hbar}{2\varepsilon_0\omega(2\pi)^3}}\left[\alpha_j(\mathbf{k},t) + \alpha_j^*(-\mathbf{k},t)\right]e^{i\mathbf{k}\cdot\mathbf{r}} \tag{5.2.9}$$

or

$$A_j(\mathbf{r},t) = \int d^3k\sqrt{\frac{\hbar}{2\varepsilon_0\omega(2\pi)^3}}\left[\alpha_j(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{r}} + \alpha_j^*(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \tag{5.2.10}$$

where k was changed to $-\mathbf{k}$, which is possible, because the integral is symmetric. In a similar way the scalar component becomes

$$A_s(\mathbf{r},t) = \int d^3k\sqrt{\frac{\hbar}{2\varepsilon_0\omega(2\pi)^3}}\left[\alpha_s(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{r}} + \alpha_s^*(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \tag{5.2.11}$$

In respect to the free field, $\alpha_j(\mathbf{k},t) = \alpha_j(\mathbf{k})\,e^{-i\omega t}$ follows from Eq.(5.2.1) and the expansion of the potentials in travelling plane waves is given by

$$A_j(\mathbf{r},t) = \int d^3k\sqrt{\frac{\hbar}{2\varepsilon_0\omega(2\pi)^3}}\left[\alpha_j(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \alpha_j^*(\mathbf{k})\,e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \tag{5.2.12}$$

$$A_s(\mathbf{r},t) = \int d^3k\sqrt{\frac{\hbar}{2\varepsilon_0\omega(2\pi)^3}}\left[\alpha_s(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \alpha_s^*(\mathbf{k})\,e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \tag{5.2.13}$$

This appears to represent four degrees of freedom in reciprocal space for every wavevector k, described by the normal variables α_j, with $j = (1, 2, 3)$ and α_s.

No special coordinate system has been chosen yet, but it is usual to choose the following. Assuming the unit vectors ε and ε' form a right-handed coordinate system together with the unit wavevector $\kappa = \frac{\mathbf{k}}{k}$, one can choose as a basis for the spatial components the two transverse normal variables $\alpha_\varepsilon(\mathbf{k}) = \varepsilon \cdot \alpha(\mathbf{k})$ and $\alpha_{\varepsilon'}(\mathbf{k}) = \varepsilon' \cdot \alpha(\mathbf{k})$ and the longitudinal normal variable $\alpha_l(\mathbf{k}) = \kappa \cdot \alpha(\mathbf{k}) = \frac{1}{k}(k_x\alpha_x + k_y\alpha_y + k_z\alpha_z)$. The free potential therefore has for every k four normal modes of vibration described by the set

$$\{\alpha_\varepsilon(\mathbf{k}), \alpha_{\varepsilon'}(\mathbf{k}), \alpha_l(\mathbf{k}), \alpha_s(\mathbf{k})\} \tag{5.2.14}$$

After quantization these four modes will correspond to four different kinds of photons, as will be shown later. As will be seen later it is possible to choose a coordinate system that is even simpler than this one using the form of the subsidiary condition.

Using the subsidiary condition defined in Eq.(5.1.13) and the expansion of the potentials in normal variables one can show that the subsidiary condition for the Lorentz gauge can be expressed as

$$\nabla \cdot \mathbf{A} + \frac{\dot{A}_s}{c} = \int d^3k \sqrt{\frac{\hbar}{2\varepsilon_0\omega(2\pi)^3}} i(\mathbf{k} \cdot \alpha - k\alpha_s) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + c.c. = 0 \tag{5.2.15}$$

where $c.c.$ means complex conjugate and $\mathbf{k} \cdot \alpha = k\alpha_l$. This can only be satisfied, if the coefficient of every exponential is identically zero and the subsidiary condition takes the simple form

$$\alpha_l(\mathbf{k}) - \alpha_s(\mathbf{k}) = 0 \qquad \forall \mathbf{k}. \tag{5.2.16}$$

The very simple form of the subsidiary condition suggests another basis transformation to write this condition in an even more simple form. If the two new normal variables α_d and α_g can be defined as

$$\alpha_d = \frac{i}{\sqrt{2}}(\alpha_l - \alpha_s) \tag{5.2.17}$$

$$\alpha_g = \frac{1}{\sqrt{2}}(\alpha_l + \alpha_s), \tag{5.2.18}$$

the new notation yields for the subsidiary condition the expression

$$\alpha_d = 0. \tag{5.2.19}$$

It should be mentioned that any gauge transformation modifies only the normal variable α_g, which makes this set of basis vectors very useful, because gauge transformations will have no effect on the transverse modes of physical states.

3 The potential four vector and the Hamiltonian in configuration space expressed in the new basis set

Before closing this section the potential four vectors and the Hamiltonian in configuration space will be expressed in this new set of basis vectors. In covariant form the potential four vector of the free field can be written as

$$A_\mu (x^\nu) = \int d^3k \sqrt{\frac{\hbar}{2\varepsilon_0\omega (2\pi)^3}} \left[\alpha_\mu (\mathbf{k}) e^{ik_\nu x^\nu} + \alpha^*_\mu (\mathbf{k}) e^{-ik_\nu x^\nu}\right], \tag{5.3.1}$$

where k^μ is the four vector $\left(\frac{\omega}{c}, \mathbf{k}\right)$ and $\omega = c\,|\mathbf{k}|$, satisfying

$$k^\mu k_\mu = 0. \tag{5.3.2}$$

In Eq.(5.3.1) the coefficients $\alpha_\mu (\mathbf{k})$ are still undefined. If the four vectors

$$\varepsilon^\mu = (0, \varepsilon) \tag{5.3.3}$$

$$\varepsilon'^\mu = \left(0, \varepsilon'\right) \tag{5.3.4}$$

$$k^\mu = (0, \kappa) \tag{5.3.5}$$

$$\eta^\mu = (1, 0) \tag{5.3.6}$$

are considered, the coefficients $\alpha_\mu (\mathbf{k})$ are defined in the new basis system as

$$\alpha_\mu (\mathbf{k}) = \alpha_\varepsilon (\mathbf{k}) \varepsilon_\mu + \alpha_{\varepsilon'} (\mathbf{k}) \varepsilon'_\mu + \alpha_l (\mathbf{k}) \kappa_\mu + \alpha_s (\mathbf{k}) \eta_\mu, \tag{5.3.7}$$

which yields using Eqn.(5.2.17,5.2.18)

$$\alpha_\mu (\mathbf{k}) = \alpha_\varepsilon (\mathbf{k}) \varepsilon_\mu + \alpha_{\varepsilon'} (\mathbf{k}) \varepsilon'_\mu + \frac{1}{\sqrt{2}} \left[\alpha_g (\mathbf{k}) (\kappa_\mu + \eta_\mu) + i\alpha_d (\mathbf{k}) (\eta_\mu - \kappa_\mu)\right] \tag{5.3.8}$$

The potential is therefore represented by three independent parts correlating to the transverse-, the g- and the d-component of the modes. The transverse component $A^T_\mu (x^\nu)$ is given by the equation

$$A^T_\mu (x^\nu) = \int d^3k \sqrt{\frac{\hbar}{2\varepsilon_0\omega (2\pi)^3}} \left\{\left[\alpha_\varepsilon (\mathbf{k}) \varepsilon_\mu + \alpha_{\varepsilon'} (\mathbf{k}) \varepsilon'_\mu\right] e^{-ik_\nu x^\nu} + \text{c.c.}\right\}. \tag{5.3.9}$$

To find the component corresponding to the g-mode it is helpful to observe the following properties.

First it is seen that $\kappa^\mu + \eta^\mu$ is equal to $(1, \kappa) = \frac{1}{k} (k, \mathbf{k})$, which are just the components of the four vector $\frac{k^\mu}{k}$. Now it is possible to define a function $f (x^\nu)$ with the property that

$$\Box f = \partial_\nu \partial^\nu f = 0, \tag{5.3.10}$$

which basically describes gauge transformations. The function f that satisfies Eq.(5.3.10) can be written as

$$f\left(x^{\nu}\right)=-\int d^{3}k\sqrt{\frac{\hbar}{2\varepsilon_{0}\omega\left(2\pi\right)^{3}}}\frac{1}{k\sqrt{2}}\left[i\alpha_{g}\left(\mathbf{k}\right)e^{-ik_{\nu}x^{\nu}}+\text{c.c.}\right]. \tag{5.3.11}$$

Comparing this equation with the Eqn.(5.3.1,5.3.8), the g-component of the vector potential can be expressed as

$$A_{\mu}^{G}\left(x^{\nu}\right)=-\partial_{\mu}f\left(x^{\nu}\right) \tag{5.3.12}$$

and the g-component has the explicit form of a gauge term (which explains the use of the letter G for the component).

Finally the d-component is written as

$$A_{\mu}^{D}\left(x^{\nu}\right)=\frac{i}{\sqrt{2}}\int d^{3}k\sqrt{\frac{\hbar}{2\varepsilon_{0}\omega\left(2\pi\right)^{3}}}\left(\eta_{\mu}-\kappa_{\mu}\right)\left[\alpha_{d}\left(\mathbf{k}\right)e^{-ik_{\nu}x^{\nu}}-\alpha_{d}^{*}\left(\mathbf{k}\right)e^{ik_{\nu}x^{\nu}}\right]=0, \tag{5.3.13}$$

where the subsidiary condition Eq.(5.2.19) is taken into account. The electromagnetic field tensor $F_{\mu\nu}$ is

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}, \tag{5.3.14}$$

which is the same as

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}^{T}-\partial_{\nu}A_{\mu}^{T}=F_{\mu\nu}^{T}, \tag{5.3.15}$$

because the gauge term does not contribute to the field tensor, as can be seen using Eq.(5.3.10). This shows that for states satisfying the Lorentz condition the electromagnetic field without sources is purely transverse, as it should be.

The Hamiltonian in configuration space is

$$\begin{aligned} H_{R} &= \int d^{3}k\frac{\hbar\omega}{2}\left[\left(\alpha_{\varepsilon}^{*}\alpha_{\varepsilon}+\alpha_{\varepsilon}\alpha_{\varepsilon}^{*}\right)+\left(\alpha_{\varepsilon'}^{*}\alpha_{\varepsilon'}+\alpha_{\varepsilon'}\alpha_{\varepsilon'}^{*}\right)\right] \\ &\quad +\frac{\hbar\omega}{2}\left[\left(\alpha_{l}^{*}\alpha_{l}+\alpha_{l}\alpha_{l}^{*}\right)-\left(\alpha_{s}^{*}\alpha_{s}+\alpha_{s}\alpha_{s}^{*}\right)\right]. \end{aligned} \tag{5.3.16}$$

This equation shows that the energy of the radiation field can become negative, because of the opposite sign of the term associated with the time component. This appears to be a problem, because it allows scalar photon states with negative energy. It will be seen later however that it is possible to construct a theory in which the subsidiary condition prevents this from happening for physical states. The introduction of the longitudinal and scalar photons create a new subspace of photons, the so called ls-space. This space does not describe physical photons and therefore the states representing this space will be called ghosts. In physical observables however no ghosts should be present. In the next chapter the field will be quantized and a photon number operator in covariant notation will be given.

Chapter 6
Quantization of the Radiation Field in the Lorentz Gauge

A play should give you something to think about. When I see a play and understand it the first time, then I know it can't be much good.

-T. S. Eliot

Considering the free radiation field, problems arise in the quantization of this field, because of the introduction of scalar and longitudinal photons. In this section the problems will be shown and one way to solve them will be suggested. A new metric will be introduced, which is not positive definite and allows scalar photon states with negative norm. The new class of d- and g- photons will be introduced and properties of the physical states in respect to those new photons will be described. Also a new basis representation for the physical kets in respect to the new metric will be introduced and the subsidiary condition will be investigated in the new and the old metric.

1 The indefinite metric

For the spatial part of the system the three orthogonal unit vectors $\hat{\mathbf{k}}, \varepsilon, \varepsilon'$ are chosen and replacing the normal variables with the creation and absorption operators for the free field the following commutation relations are achieved

$$\left[a_i(\mathbf{k}), a_j^\dagger\left(\mathbf{k}'\right)\right] = \delta_{ij}\delta\left(\mathbf{k}-\mathbf{k}'\right) \tag{6.1.1}$$

$$\left[a_s(\mathbf{k}), a_s^\dagger\left(\mathbf{k}'\right)\right] = -\delta\left(\mathbf{k}-\mathbf{k}'\right). \tag{6.1.2}$$

The minus sign in Eq.(6.1.2) yields some difficulties as will be seen later. It arises from the fact the conjugate momentum of the scalar part of the vector potential has an additional minus sign as was seen in the last chapter. With a commutation relation like this it is not possible in the theory used to construct a basis for scalar photon states with positive definite norms. However, this would be necessary to allow a probability interpretation and one has to construct a theory that is able to define observables in a way that they satisfy the commutation relation given in Eq.(6.2.2) without the minus sign.

The subsidiary condition for physical states cannot be written as

$$a_l(\mathbf{k}) - a_s(\mathbf{k}) = 0 \qquad \forall \mathbf{k}, \tag{6.1.3}$$

because the longitudinal and the scalar part of the given four vectors are in different subspaces, Eq.(6.1.3) can never be true. Here $a_l(\mathbf{k})$ is the longitudinal spatial part $\kappa \cdot \mathbf{a}(\mathbf{k})$ of the annihilation operators. A weaker subsidiary condition has to be used to obtain physical states. For the free field one can use the condition

$$[a_l(\mathbf{k}) - a_s(\mathbf{k})] |\Psi\rangle = 0 \qquad \forall \mathbf{k} \quad \wedge \quad \text{physical} \quad |\Psi\rangle \tag{6.1.4}$$

which states that the operator $[a_l(\mathbf{k}) - a_s(\mathbf{k})]$ working on any physical state $|\Psi\rangle$ must give zero. This procedure was first suggested in a paper by Gupta [9] and later also used by Bleuler[16] . A special state is the vacuum and it seems natural to require that every annihilation operator $a_\mu(\mathbf{k})$ working on the vacuum $|0\rangle$ should give zero. This state satisfies Eq.(6.1.4) and therefore is a physical state.

Next the photon states will be described as states created from the vacuum with the creation operators for the different kind of photons and the problem arising for the scalar photons will be shown and solved. The spatial photon states can be constructed from the vacuum as usual with the creation operators $a_\varepsilon^\dagger, a_{\varepsilon'}^\dagger, a_l^\dagger$ and the obtained state is

$$|n_\epsilon, n_{\varepsilon'}, n_l\rangle = \frac{\left(a_\varepsilon^\dagger\right)^{n_\epsilon} \left(a_{\varepsilon'}^\dagger\right)^{n_{\varepsilon'}} \left(a_l^\dagger\right)^{n_l}}{\sqrt{n_\epsilon! n_{\varepsilon'}! n_l!}} |0\rangle . \tag{6.1.5}$$

If the same procedure would be used for the scalar part every state with an uneven number of scalar photons would have a negative norm. For example the state

$$|\Psi\rangle = a_s^\dagger(\mathbf{k}) |0\rangle \tag{6.1.6}$$

has the norm

$$\langle\Psi|\Psi\rangle = \langle 0| a_s(\mathbf{k}) a_s^\dagger(\mathbf{k}) |0\rangle = -\langle 0|0\rangle , \tag{6.1.7}$$

where the commutation relation Eq.(6.1.2) and the fact that the annihilation operator working on the vacuum gives zero was used. The norm of the state with one scalar photon has therefore the opposite sign to the norm of the vacuum state. This does not allow a probabilistic interpretation in the underlying Hilbert space. The positive definiteness of the norms of the Hilbert space has to be abandoned to allow the first excited state of the scalar part. This is done in Cohen-Tannoudji [11, p.387] by introducing an indefinite metric.

The new scalar product is defined in terms of the unitary operator M as

$$(\Phi \mid \Psi) = \langle \Phi | M | \Psi \rangle , \qquad (6.1.8)$$

with the unitarity condition $M = M^{\dagger} = M^{-1}$. It also will be possible to construct this new scalar product in a way that M will change the sign of the scalar part. All conditions for a scalar product are still valid except the positive definiteness. For a linear operator A the adjoint in the new metric in relation to the one in the old metric is

$$\bar{A} = M A^{\dagger} M, \qquad (6.1.9)$$

where $\bar{A}$ denotes the adjoint of A in the new metric to distinguish it from the old one. The operator A is the same in both metrics, but the notion of a hermitian operator in the new metric is different. An operator A is called hermitian in the new metric, if

$$A = \bar{A}. \qquad (6.1.10)$$

The new mean value in the state Ψ of an operator A is then by definition the quantity

$$(A)_{\Psi} = \frac{(\Psi|A|\Psi)}{(\Psi|\Psi)}. \qquad (6.1.11)$$

If this quantity is real, then the operator A is hermitian.

Physical observables are now required to be hermitian in the new metric. The potential operators A_{μ}, which have to be hermitian in the new sense, are now given by

$$A_{\mu}(\mathbf{r}) = \int d^3k \sqrt{\frac{\hbar}{2\varepsilon_0 \omega (2\pi)^3}} \left[a_{\mu}(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{r}} + \bar{a}_{\mu}(\mathbf{k}, t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \qquad (6.1.12)$$

with the commutation relations

$$\left[a_i(\mathbf{k}), \bar{a}_j\left(\mathbf{k}'\right) \right] = \delta_{ij} \delta\left(\mathbf{k} - \mathbf{k}'\right) \qquad (6.1.13)$$

$$\left[a_s(\mathbf{k}), \bar{a}_s\left(\mathbf{k}'\right) \right] = -\delta\left(\mathbf{k} - \mathbf{k}'\right) \qquad (6.1.14)$$

assumed to be valid in the new metric and all other operators commute.

All the problems arise because of the minus sign in Eq.(6.1.14) and to construct a basis for the scalar photons, with positive norm regarding the old metric, one requires the operator M to satisfy the equations

$$\bar{a}_j = M a_j^{\dagger} M = a_j^{\dagger} \qquad (6.1.15)$$

$$\bar{a}_s = M a_s^{\dagger} M = -a_s^{\dagger}. \tag{6.1.16}$$

The only change in the commutation relations regarding the old metric occurs in the scalar part and the commutation relation for the scalar creation and absorption operators regarding the old metric is now

$$\left[a_s(\mathbf{k}), a_s^{\dagger}\left(\mathbf{k}'\right)\right] = \delta\left(\mathbf{k} - \mathbf{k}'\right). \tag{6.1.17}$$

Now it is possible to construct a basis for the scalar photons, because this commutation relation is well known and the creation and absorption operators act in the usual manner. The basis for the scalar photons is given by

$$|n_s\rangle = \frac{\left(a_s^{\dagger}\right)^{n_s}}{\sqrt{n_s!}} |0_s\rangle, \tag{6.1.18}$$

where $|0_s\rangle$ is the vacuum state of the scalar photons. The basis vectors are normalized in the usual manner and the creation and absorption operators in respect to the old metric yield

$$a_s^{\dagger} |n_s\rangle = \sqrt{n_s + 1} |n_s + 1\rangle \tag{6.1.19}$$

$$a_s |n_s\rangle = \sqrt{n_s} |n_s - 1\rangle \tag{6.1.20}$$

$$a_s |0_s\rangle = 0. \tag{6.1.21}$$

With this new basis it is possible to describe states with one scalar photon and a positive norm in respect to the old metric and the problems mentioned earlier are solved, but not without paying a price.

The vector potential A_s regarding the old metric is now anti-hermitian and therefore would not be considered a physical observable in the old sense. It is a small price to pay remembering that the scalar potential is not a truly physical observable in the first place. The operator M given by the expression

$$M |n_s\rangle = (-1)^{n_s} |n_s\rangle \tag{6.1.22}$$

is suitable to define the new metric. Using Eqn.(6.1.19,6.1.16) the action of $\bar{a}_s$ is found to be

$$\bar{a}_s |n_s\rangle = -\sqrt{n_s + 1} |n_s + 1\rangle. \tag{6.1.23}$$

Now it is possible to calculate the new scalar product between two scalar photon states and the result is

$$\left(n_s | n_s'\right) = \langle n_s | M \left| n_s' \right\rangle = (-1)^{n_s'} \delta_{n_s n_s'}. \tag{6.1.24}$$

As expected the norm for the vacuum is positive and the norm for the states with an uneven number of scalar photons is negative. The space of physical scalar photons expressed in respect to the old metric now behaves nicely. In the new metric the consequences that arise from the negative norms in the scalar states yield no problem, because states with negative norm are allowed in the new metric. The main difference between the metrics is that the scalar operators that are hermitian in respect to the new metric will now be anti-hermitian in respect to the old metric and therefore would not be considered observables in the old sense. For physical states this will yield no problem as will be seen later, because of the subsidiary condition.

2 A new class of photons

The simple form of the subsidiary condition for physical states suggests another basis transformation in the space of photons. In this section the d(ifference) and g(auge)-photons will be defined and their properties in the old and new metric will be investigated. The two orthonormal operators representing the new photons are defined as

$$a_d = \frac{i}{\sqrt{2}} \left(a_l - a_s \right) \tag{6.2.1}$$

and

$$a_g = \frac{1}{\sqrt{2}} \left(a_l + a_s \right). \tag{6.2.2}$$

The first operator is essentially the subsidiary condition for physical states and the physical states are constructed so that they have no d-photons. Using Eqn.(6.2.1,6.2.2) and the definition of the vacuum shows that the vacuum states for the ls-photons and the gd-photons coincide, which gives the identity

$$|0_d 0_g\rangle = |0_s 0_l\rangle . \tag{6.2.3}$$

The basis of the state space for physical photons will as usual be given in respect to the vacuum state in the old metric using the creation operators expressed in the old metric, because the positive definiteness of the scalar product allows a probabilistic interpretation of the theory. The orthonormal basis for the space of physical states is therefore given by the expression

$$|n_\varepsilon n_{\varepsilon'} 0_d n_g\rangle = \frac{\left(a_\varepsilon^\dagger\right)^{n_\varepsilon} \left(a_{\varepsilon'}^\dagger\right)^{n_{\varepsilon'}} \left(a_g^\dagger\right)^{n_g}}{\sqrt{n_\varepsilon ! n_{\varepsilon'} ! n_g !}} |0\rangle \tag{6.2.4}$$

and obviously the transverse and the g-photons can be in any excited state. The g-photon is responsible for the possibility to have gauge transformations in the old metric, which leave the physical states unchanged. The orthonormality relation in regard to the old metric for this basis is given by the equation

$$\left\langle n_\varepsilon n_{\varepsilon'} 0_d n_g | n'_\varepsilon n'_{\varepsilon'} 0_d n'_g \right\rangle = \delta_{n_\varepsilon n'_\varepsilon} \delta_{n_{\varepsilon'} n'_{\varepsilon'}} \delta_{n_g n'_g}. \tag{6.2.5}$$

The scalar product between physical states in the new metric however is different. The transverse part remains the same, but the gd- (or ls-) part is not that simple. It is necessary to explicitly calculate the scalar product $\left(0_d n_g | 0_d n_g'\right)$ in this subspace. The ket $\left|0_d n_g'\right)$ remains basically the same and can be given in respect to the vacuum state as a function of the creation operator $a_g^\dagger$. To be able to give an expression for the physical bra it is necessary to exploit the remaining properties of the new scalar product, which give the relation between the kets and bras as

$$\left|\Psi'\right) = A\left|\Psi\right) \Longleftrightarrow \left(\Psi'\right| = \left(\Psi\right| \bar{A}, \tag{6.2.6}$$

where A is any linear operator. Using this correspondence it is straightforward to show the correspondence between the physical kets and bras in the new metric and the result is

$$\left|0_d n_g'\right) = \frac{\left(a_g^\dagger\right)^{n_g}}{\sqrt{n_g!}} \left|0\right) \Longleftrightarrow \left(0_d n_g'\right| = \left(0\right| \frac{\overline{\left(a_g^\dagger\right)^{n_g}}}{\sqrt{n_g!}}. \tag{6.2.7}$$

The operator $\overline{a_g^\dagger}$ is simply the new adjoint of the creation operator $a_g^\dagger$.

This operator will now be written solely in the notation of the new metric. Using Eqn.(6.2.1,6.2.2) and Eqn.(6.1.15,6.1.16) gives the equations

$$\bar{a}_d = -i a_g^\dagger \tag{6.2.8}$$

and

$$\bar{a}_g = i a_d^\dagger. \tag{6.2.9}$$

Furthermore Eq.(6.2.8) gives the relation

$$\overline{a_g^\dagger} = -i a_d. \tag{6.2.10}$$

Using Eqn.(6.2.1,6.2.2) and Eq.(6.1.13,6.1.14) the commutation relation for d-photons is given as

$$[a_d, \bar{a}_d] = 0. \tag{6.2.11}$$

Now it is possible to calculate the new scalar product and using the definition in Eq.(6.2.7) and the relations given in Eqn.(6.2.8,6.2.10,6.2.11) gives the equation

$$\sqrt{n_g! n_g'!}\left(0_d n_g | 0_d n_g'\right) = \delta_{n_g 0} \delta_{n_g' 0}. \tag{6.2.12}$$

This result is interesting, because it states that the new scalar product between all the basis vectors of the gd-subspace of physical kets is identical to zero, unless the number of g-photons is zero.

Even though it is possible to describe a physical state with a whole class of physical kets, namely the ones with a fixed transverse part , but an arbitrary gd-

(ls-) part, only the vacuum of the g-photons gives a non zero scalar product between physical states. It can be shown that mean values of physical observables in respect to the new metric are the same as the mean values of the transverse parts in respect to the old metric and that the vector potential is the same in the two metrics except an additional gauge term. The physical kets themselves in the new metric describe the gauge arbitrariness, without changing the predictions for physical observables. Those calculations however are not necessary for the theory that will be proposed in the next chapter and are therefore omitted.

3 The subsidiary condition

Because the form of the subsidiary condition will play a crucial part in the remaining chapters, the subsidiary condition as given by Gupta and Bleuler will be investigated and equivalent descriptions in the two metrics will be given in this section. This will be achieved by investigating the expectation value for the operator $\partial_\mu A^\mu$, which is given by the expression

$$(\Psi|\,\partial_\mu A^\mu\,|\Psi) = \int d^3k\sqrt{\frac{\hbar}{2\varepsilon_0\omega\,(2\pi)^3}}\,(\Psi|\left(ika_l + \frac{\dot{a}_s}{c}\right)e^{i\mathbf{k}\cdot\mathbf{r}} + h.c.\,|\Psi)\,. \qquad (6.3.1)$$

The first term in the scalar product is equal to zero, because it naturally gives the subsidiary condition for physical ket states. The hermitian conjugate term however yields

$$(\Psi|\,h.c.\,|\Psi) = (\Psi|\left(-ik\bar{a}_l + \frac{\partial}{c\partial t}\bar{a}_s\right)e^{-i\mathbf{k}\cdot\mathbf{r}}\,|\Psi)\,, \qquad (6.3.2)$$

which can be written as

$$(\Psi|\,h.c.\,|\Psi) = ike^{-i\mathbf{k}\cdot\mathbf{r}}\,(\Psi|\,(\bar{a}_s - \bar{a}_l)\,|\Psi) \qquad (6.3.3)$$

using the equations of motion.

This is the subsidiary condition for physical bra states in the new metric using the correspondence between bras and kets as shown in Eq.(6.2.6). It is also possible to write this subsidiary condition in respect to the old metric when one starts with the equation

$$\left(\Psi'\right| = (\Psi|\,[\bar{a}_s - \bar{a}_l] = \langle\Psi|\,M\,[\bar{a}_s - \bar{a}_l] \qquad (6.3.4)$$

where the correspondence between the old and the new bra was used. Using Eqn.(6.1.15,6.1.16) it can be shown that following Eq.(6.1.24) this is equal to

$$\left(\Psi'\right| = (\Psi|\,[\bar{a}_s - \bar{a}_l] = \langle\Psi|\left[a_s^\dagger - a_l^\dagger\right]M, \qquad (6.3.5)$$

which is the usual subsidiary condition working on the bra. So this term is also zero and the expectation value for the operator $\partial_\mu A^\mu$ is as expected equal to zero.

The subsidiary condition has therefore the same form in the old and in the new metric. This result will be needed in the next chapter, where a new photon number density operator in covariant notation will be defined.

Chapter 7
The photon number density in the new metric

The job is to ask questions -it always was- and to ask them as inexorably as I can. And to face the absence of precise answers with a certain humility.

-Arthur Miller

In the last chapter a new metric was introduced and in this chapter the photon number density operator defined in chapter four will be expressed in this new metric. The operator now has to be hermitian in respect to the new metric and therefore the daggers in the creation operators have to be replaced by bars. It also has to be the first component of the four vector obtained by contracting the electro magnetic field tensor with the vector potential as in chapter four. It will be shown that this operator can be indeed interpreted as a photon number density in respect to the new metric.

In the last part it will be shown however that this operator does not completely act as expected, because of the ghosts that propagate in the theory until the subsidiary condition is applied. In the next chapter the problem will be solved and an explanation will be given why this strange behavior occurs.

1 The Photon Current Four Vector in the new metric

It was stated that in the new metric physical observables have to be hermitian in respect to the new metric. The photon current four vector defined in the Coulomb gauge was hermitian in respect to the old metric and therefore changes have to be made. The proposed density operator then will have to be integrated over the configuration space to obtain the total number operator.

It was mentioned in chapter four that the photon number and current density operators can be found by contracting the electromagnetic field tensor with the potential four vector. The photon current four vector then has to be

$$\Upsilon_{\eta}=\frac{c\varepsilon_0}{i\hbar}\left(F^{-}_{\mu\eta}A^{\mu+}-A^{\mu-}F^{+}_{\mu\eta}\right), \tag{7.1.1}$$

where the positive and negative frequency parts of the electromagnetic field tensor are

$$F^{\pm}_{\mu\eta}=\partial_{\mu}A^{\pm}_{\eta}-\partial_{\eta}A^{\pm}_{\mu} \tag{7.1.2}$$

and the new potential four-vector is

$$A_{\mu}\left(x^{\nu}\right)=\int d^3k\sqrt{\frac{\hbar}{2\varepsilon_0\left(2\pi\right)^3}}\left[a_{\mu}\left(\mathbf{k}\right)e^{-ik_{\nu}x^{\nu}}+\bar{a}_{\mu}\left(\mathbf{k}\right)e^{ik_{\nu}x^{\nu}}\right] \tag{7.1.3}$$

with the usual split in negative and positive frequency parts. This operator differs from the ones used earlier, because the potential four vector is now required to be hermitian in the new sense. It is important to note that Eq.(7.1.3) is only valid in the case of a free field, because time dependance $e^{\pm i\omega t}$ where $\omega=kc$ is assumed. The four-vector k_{ν} is defined as $\left(\frac{\omega}{c},\mathbf{k}\right)$ with $\omega=c\left|\mathbf{k}\right|$ and satisfying the condition $k_{\nu}k^{\nu}=0$. The operator defined in Eq.(7.1.1) is hermitian in respect to the new metric, because $\bar{A}^{+}=A^{-}$.

The scalar component of this four-vector can be split into a transverse part and a part that spans the longitudinal and the scalar components of the photons. This subspace will be denoted as the ls-space. In this notation the scalar component of the derived operator reads

$$\Upsilon_0=\Upsilon_0^T+\Upsilon_0^{LS} \tag{7.1.4}$$

where Υ_0 is scalar component of the operator defined in Eq.(7.1.1) and the right-hand-side of the equation is given by the transverse and the ls-component of the operator. The transverse component is

$$\Upsilon_0^T=\frac{1}{2}\int d^3k\int d^3k'\frac{\left(\omega+\omega'\right)}{\sqrt{\omega}\sqrt{\omega'}}e^{i\left(k_{\nu}-k'_{\nu}\right)x^{\nu}}\bar{a}_T\left(\mathbf{k}\right)a_T\left(\mathbf{k}'\right) \tag{7.1.5}$$

and the ls-component is

$$\begin{aligned}\Upsilon_0^{LS}=\tfrac{-1}{2}\textstyle\int d^3k\int d^3k'\frac{1}{\sqrt{\omega}\sqrt{\omega'}}e^{i\left(k_{\nu}-k'_{\nu}\right)x^{\nu}}\\ \times\left\{-\left(k_0+k'_0\right)\bar{a}_l\left(\mathbf{k}\right)a_l\left(\mathbf{k}'\right)+\bar{a}_s\left(\mathbf{k}\right)\left[\mathbf{k}\cdot\mathbf{a}\left(\mathbf{k}'\right)\right]+\left[\mathbf{k}'\cdot\bar{\mathbf{a}}\left(\mathbf{k}\right)\right]a_s\left(\mathbf{k}'\right)\right\}.\end{aligned} \tag{7.1.6}$$

Integrating over the configuration space the leaves one with the expressions

$$\int d^3x\Upsilon_0^T=\frac{1}{2}\int d^3k'e^{ik_{\nu}x^{\nu}}\bar{a}_T\left(\mathbf{k}\right)a_T\left(\mathbf{k}\right) \tag{7.1.7}$$

and

$$\int d^3x\Upsilon_0^{LS} = -\frac{1}{2}\int d^3k\bar{a}_l(\mathbf{k})\left[a_s(\mathbf{k}) - a_l(\mathbf{k})\right] + \left[\bar{a}_s(\mathbf{k}) - \bar{a}_l(\mathbf{k})\right]a_l(\mathbf{k}). \qquad (7.1.8)$$

In the next section the mean values of those operators will be calculated in the new metric.

2 Mean values of the photon number operator

Now it is possible to calculate the mean values of these two operators in the new metric. For the ls-part the mean value in respect to the new metric is

$$\frac{(\Psi|\int d^3x\Upsilon_0^{LS}|\Psi)}{(\Psi|\Psi)} = \frac{(\Psi_{LS}|\int d^3x\Upsilon_0^{LS}|\Psi_{LS})}{(\Psi_{LS}|\Psi_{LS})}\frac{(\Psi_T|\Psi_T)}{(\Psi_T|\Psi_T)} = \int d^3x\frac{(\Psi_{LS}|\Upsilon_0^{LS}|\Psi_{LS})}{(\Psi_{LS}|\Psi_{LS})}, \qquad (7.2.1)$$

because the ls-part of the operator only works on the LS-subspace and the integration over configuration space can be taken outside the scalar product. Inserting Eq.(7.1.8) yields

$$\begin{aligned}\int d^3x\frac{(\Psi_{LS}|\Upsilon_0^{LS}|\Psi_{LS})}{(\Psi_{LS}|\Psi_{LS})} &= -\frac{1}{2}\frac{1}{(\Psi_{LS}|\Psi_{LS})}\int d^3k \qquad (7.2.2)\\ &\quad(\Psi_{LS}|\bar{a}_l(\mathbf{k})\left[a_s(\mathbf{k}) - a_l(\mathbf{k})\right]|\Psi_{LS})\\ &\quad+(\Psi_{LS}|\left[\bar{a}_s(\mathbf{k}) - \bar{a}_l(\mathbf{k})\right]a_l(\mathbf{k})|\Psi_{LS})\end{aligned}$$

and this is identical to zero, because the terms in the scalar product are simply stating the subsidiary condition for physical states.

The mean value of $\int d^3x\Upsilon_p^T$ in respect to the new metric is given in a similar way by the expression

$$\int d^3x\frac{(\Psi_T|\Upsilon_0^T|\Psi_T)}{(\Psi_T|\Psi_T)} = \int d^3k\frac{(\Psi_T|\bar{a}_T(\mathbf{k})a_T(\mathbf{k})|\Psi_T)}{(\Psi_T|\Psi_T)} \qquad (7.2.3)$$

and because the transverse part is not affected by the new metric this can be written as

$$\int d^3x\frac{(\Psi_T|\Upsilon_0^T|\Psi_T)}{(\Psi_T|\Psi_T)} = \int d^3k\frac{\langle\Psi_T|\bar{a}_T(\mathbf{k})a_T(\mathbf{k})|\Psi_T\rangle}{\langle\Psi_T|\Psi_T\rangle}$$

Therefore the mean value of the new operator in respect to the new metric is equal to the mean value of the operator

$$\int d^3k\hat{n}(\mathbf{k}) = N, \qquad (7.2.4)$$

where $\hat{n}(\mathbf{k}) = \bar{a}_T(\mathbf{k})\, a_T(\mathbf{k})$ in respect to the old metric. It is therefore shown that only the transverse photons contribute to the total number of photons in position-space and that the zeroth component of the hermitian operator

$$\int d^3x \Upsilon_0 = \frac{c\varepsilon_0}{i\hbar}\left(F^-_{\mu\nu}A^{\mu+} - A^{\mu-}F^+_{\mu\nu}\right) \tag{7.2.5}$$

can be interpreted as the photon number density regarding the new metric. In the next section it will be shown that this operator is not able to describe the ghost states in the expected way and therefore cannot be the right photon number density operator in the Lorentz gauge.

3 Mean values of the operators in respect to the old metric

In this section the mean value of a known physical state will be calculated in respect to the old metric and it will be shown that the result does not coincide with the expected value. Problems only arise because of the ghosts in the system, which only propagate in the ls-part, and therefore the transverse part of the operator will always act as expected.

Physical states of the free field always have the same number of l- and s-photons and one special physical state is

$$|1_l, 1_s\rangle = a_s^\dagger a_l^\dagger\, |0_l, 0_s\rangle\,. \tag{7.3.1}$$

The ls-part of the photon number operator was given in Eq.(7.1.8) in respect to the new metric as

$$\int d^3x \Upsilon_0^{LS} = -\frac{1}{2}\int d^3k \bar{a}_l(\mathbf{k})\left[a_s(\mathbf{k}) - a_l(\mathbf{k})\right] + \left[\bar{a}_s(\mathbf{k}) - \bar{a}_l(\mathbf{k})\right] a_l(\mathbf{k})\,. \tag{7.3.2}$$

Using the identities that

$$\bar{a}_l(\mathbf{k}) = a_l^\dagger(\mathbf{k}) \tag{7.3.3}$$

and

$$\bar{a}_s(\mathbf{k}) = -a_s^\dagger(\mathbf{k}) \tag{7.3.4}$$

yields for Eq.(7.3.2)

$$\int d^3x \Upsilon_0^{LS} = \frac{-1}{2}\int d^3k a_l^\dagger(\mathbf{k})\left[a_s(\mathbf{k}) - a_l(\mathbf{k})\right] + \left[-a_s^\dagger(\mathbf{k}) - a_l^\dagger(\mathbf{k})\right] a_l(\mathbf{k})\,, \tag{7.3.5}$$

which can be written as

$$\int d^3x \Upsilon_0^{LS} = \frac{-1}{2}\int d^3k - 2a_l^\dagger(\mathbf{k})\, a_l(\mathbf{k}) + a_l^\dagger(\mathbf{k})\, a_s(\mathbf{k}) - a_s^\dagger(\mathbf{k})\, a_l(\mathbf{k})\,. \tag{7.3.6}$$

Taking the expectation value of this operator in respect to the old metric one finds that the last two terms of the integrand do not contribute to the expectation value, because of the orthogonality relation of the final states. Taking the expectation value of this operator in respect to the old metric yields

$$\frac{\langle 1_s, 1_l | \frac{c\varepsilon_0}{i\hbar} \int d^3x \Upsilon_0^{LS} | 1_s, 1_l \rangle}{\langle 1_s, 1_l | 1_s, 1_l \rangle} = n_l = 1. \tag{7.3.7}$$

This is not the expected value, because the inspected state does not only have a longitudinal photon, but also a scalar photon. This is because scalar photons are not counted using this operator which was defined in the Coulomb gauge originally.

The operator therefore cannot describe the physical state in Eq.(7.3.1) that includes the ghosts in respect to the old metric. It is no surprise however that the right result in respect to the new metric was achieved, because in the new metric the subsidiary condition was applied and the ghosts vanished completely. It is therefore necessary to find an operator that gives the same result in respect to the new metric, but furthermore is also able to describe the system in respect to the old metric correctly. This will be done in the next chapter and an explanation of why the four vector has to have a different form than the one used in the Coulomb gauge will also be given in that chapter.

4 Interaction of the free field with two fixed charges in the covariant notation

In the last section it was shown that the proposed operator cannot describe the ghost states in an expected way, but it is nevertheless helpful to investigate the behavior of this operator in respect to the new metric in the case of two charges present. The operator was able to describe the physical states in the right way and therefore is still useful and investigating this generalized case will give some ideas to what one can expect when using the same case with the operator that will be proposed in the next chapter. This case will be treated analogous to the free field case and very formally. In the next chapter a more direct approach will be used and some of the properties used in this chapter will be shown explicitly.

In the interaction case the Lagrangian will include the interaction term

$$\mathcal{L}_I = -j_\mu A^\mu.$$

In the case of two fixed charges this simplifies to the equation

$$\mathcal{L}_I = -c\rho_e A_s \tag{7.4.1}$$

with

$$\rho_e(\mathbf{r}) = q_1\delta(\mathbf{r}-\mathbf{r}_1) + q_2\delta(\mathbf{r}-\mathbf{r}_2). \tag{7.4.2}$$

and in reciprocal space this equation gives

$$\rho_e(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\left[q_1 e^{-i\mathbf{k}\cdot\mathbf{r}_1} + q_2 e^{-i\mathbf{k}\cdot\mathbf{r}_2}\right]. \tag{7.4.3}$$

The Lagrangian density is therefore

$$\mathcal{L} = -\frac{\varepsilon_0 c^2}{4}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - cq_1\delta(\mathbf{r}-\mathbf{r}_1) - cq_2\delta(\mathbf{r}-\mathbf{r}_2). \tag{7.4.4}$$

In the case of the free field it was possible to obtain the time dependence of the annihilation and creation operators. This is not possible anymore and the proposed photon current operator is now the more general operator

$$\Upsilon_\eta = \frac{\hbar}{2\varepsilon_0(2\pi)^3}\int d^3k\int d^3k'\frac{1}{\sqrt{\omega}\sqrt{\omega'}}\{\Xi\} \tag{7.4.5}$$

where Ξ is now given by the expression

$$\begin{aligned}\Xi \;=\; & \left[\partial_\mu\left(\bar{a}_\eta(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right) - \partial_\eta\left(\bar{a}_\mu(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\right] a^\mu\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} \\ & -\bar{a}^\mu(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\partial_\mu\left(a_\eta\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}}\right) - \partial_\eta\left(a_\mu\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}}\right)\right].\end{aligned} \tag{7.4.6}$$

Since we are only interested in the scalar component of this four vector operator it is possible to restrict the theory to the case $\eta = 0$. The equation can then be written as

$$\begin{aligned}\Xi \;=\; & \left(\frac{\partial}{c\partial t}\bar{a}_s(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right)a_s\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} + \left(\nabla\bar{a}_s(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\mathbf{a}\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} \\ & -\left(\frac{\partial}{c\partial t}\bar{a}_s(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right)a_s\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} + \left(\frac{\partial}{c\partial t}\bar{\mathbf{a}}(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\mathbf{a}\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} \\ & -\bar{a}_s(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\partial}{c\partial t}a_s\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} - \bar{\mathbf{a}}(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\cdot\nabla a_s\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} \\ & +\bar{a}_s(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\partial}{c\partial t}a_s\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} - \bar{\mathbf{a}}(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\partial}{c\partial t}\mathbf{a}\left(\mathbf{k}',t\right)e^{-i\mathbf{k}'\cdot\mathbf{x}}\end{aligned} \tag{7.4.7}$$

and collecting similar terms yields

$$\begin{aligned}\Xi \;=\; & -\bar{\mathbf{a}}(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\cdot\nabla a_s\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} + \left(\nabla\bar{a}_s(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\mathbf{a}\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} \\ & +\left(\frac{\partial}{c\partial t}\bar{\mathbf{a}}(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\mathbf{a}\left(\mathbf{k}',t\right)e^{i\mathbf{k}'\cdot\mathbf{x}} - \bar{\mathbf{a}}(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\partial}{c\partial t}\mathbf{a}\left(\mathbf{k}',t\right)e^{-i\mathbf{k}'\cdot\mathbf{x}}.\end{aligned} \tag{7.4.8}$$

Since the gradient only affects the exponential function and the time derivative only affects the operators the equation is also equivalent to the expression

$$\begin{aligned}\Xi &= -e^{i\left(\mathbf{k}'-\mathbf{k}\right)\cdot\mathbf{x}}\left[\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{k}' a_s\left(\mathbf{k}',t\right)+\bar{a}_s(\mathbf{k},t)\,\mathbf{k}\cdot\mathbf{a}\left(\mathbf{k}',t\right)\right] \quad (7.4.9)\\ &+e^{i\left(\mathbf{k}'-\mathbf{k}\right)\cdot\mathbf{x}}\left[\left(\frac{\partial}{c\partial t}\bar{\mathbf{a}}(\mathbf{k},t)\right)\cdot\mathbf{a}\left(\mathbf{k}',t\right)-\bar{\mathbf{a}}(\mathbf{k},t)\cdot\left(\frac{\partial}{c\partial t}\mathbf{a}\left(\mathbf{k}',t\right)\right)\right].\end{aligned}$$

The equations of motion for the spatial part remain the same as before, because the charges are fixed and therefore no current is present and following result is achieved

$$\begin{aligned}\Xi &= -e^{i\left(\mathbf{k}'-\mathbf{k}\right)\cdot\mathbf{x}}\left[\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{k}' a_s\left(\mathbf{k}',t\right)+\bar{a}_s(\mathbf{k},t)\,\mathbf{k}\cdot\mathbf{a}\left(\mathbf{k}',t\right)\right]\\ &+e^{i\left(\mathbf{k}'-\mathbf{k}\right)\cdot\mathbf{x}}\left[\frac{\omega}{c}\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{a}\left(\mathbf{k}',t\right)+\bar{\mathbf{a}}(\mathbf{k},t)\cdot\frac{\omega'}{c}\mathbf{a}\left(\mathbf{k}',t\right)\right]. \quad (7.4.10)\end{aligned}$$

The only dependance on space coordinates is in the exponential function and integration over the configuration space yields

$$\begin{aligned}\int d^3x\,\Xi &= -(2\pi)^3\delta\left(\mathbf{k}'-\mathbf{k}\right)\left[\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{k}' a_s\left(\mathbf{k}',t\right)+\bar{a}_s(\mathbf{k},t)\,\mathbf{k}\cdot\mathbf{a}\left(\mathbf{k}',t\right)\right]\\ &+(2\pi)^3\delta\left(\mathbf{k}'-\mathbf{k}\right)\left(\frac{\omega+\omega'}{c}\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{a}\left(\mathbf{k}',t\right)\right). \quad (7.4.11)\end{aligned}$$

The integrated form of Eq.(7.4.5) is therefore

$$\begin{aligned}\int d^3x\,\Upsilon_s &= \frac{\hbar}{2\varepsilon_0}\int d^3k\int d^3k'\frac{1}{\sqrt{\omega}\sqrt{\omega'}}\delta\left(\mathbf{k}'-\mathbf{k}\right)\left[\frac{\omega+\omega'}{c}\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{a}\left(\mathbf{k}',t\right)\right]\\ &-\delta\left(\mathbf{k}'-\mathbf{k}\right)\left[\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{k}' a_s\left(\mathbf{k}',t\right)+\bar{a}_s(\mathbf{k},t)\,\mathbf{k}\cdot\mathbf{a}\left(\mathbf{k}',t\right)\right], \quad (7.4.12)\end{aligned}$$

which reduces to the expression

$$\int d^3x\,\Upsilon_s=\frac{\hbar}{2\varepsilon_0}\int d^3k\frac{2}{c}\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{a}(\mathbf{k},t)-\frac{1}{\omega}\bar{\mathbf{a}}(\mathbf{k},t)\cdot\mathbf{k}a_s(\mathbf{k},t)-\frac{1}{\omega}\bar{a}_s(\mathbf{k},t)\,\mathbf{k}\cdot\mathbf{a}(\mathbf{k},t), \quad (7.4.13)$$

because of the delta function. Exploiting the definition for the longitudinal part of the operators and performing the scalar products leaves one with

$$\begin{aligned}\int d^3x\,\Upsilon_s &= \frac{\hbar}{2\varepsilon_0 c}\int d^3k\,2\left(\bar{\mathbf{a}}_T(\mathbf{k},t)\cdot\mathbf{a}_T(\mathbf{k},t)+\bar{a}_l(\mathbf{k},t)\,a_l(\mathbf{k},t)\right) \quad (7.4.14)\\ &-\bar{a}_l(\mathbf{k},t)\,a_s(\mathbf{k},t)-\bar{a}_s(\mathbf{k},t)\,a_l(\mathbf{k},t)\end{aligned}$$

and once again it is possible to split this expression into a transverse and a ls-part. The transverse part is as before given by the expression

$$\int d^3x\,\Upsilon_s^T=\frac{\hbar}{\varepsilon_0 c}\int d^3k\,\bar{\mathbf{a}}_T(\mathbf{k},t)\cdot\mathbf{a}_T(\mathbf{k},t). \quad (7.4.15)$$

The ls-part however is now given by the expression

$$\int d^3x \Upsilon_s^{LS} = \frac{\hbar}{2\varepsilon_0 c}\int d^3k \bar{a}_l(\mathbf{k},t)\left[a_l(\mathbf{k},t) - a_s(\mathbf{k},t)\right] + \left[\bar{a}_l(\mathbf{k},t) - \bar{a}_s(\mathbf{k},t)\right] a_l(\mathbf{k},t) \tag{7.4.16}$$

and adding zero yields

$$\begin{aligned}\int d^3x \Upsilon_s^{LS} &= \frac{\hbar}{2\varepsilon_0 c}\int d^3k \bar{a}_l(\mathbf{k},t)\left[a_l(\mathbf{k},t) - a_s(\mathbf{k},t) + \lambda(\mathbf{k},t)\right] \\ &\quad + \left[\bar{a}_l(\mathbf{k},t) - \bar{a}_s(\mathbf{k},t) + \bar{\lambda}(\mathbf{k},t)\right] a_l(\mathbf{k},t) \\ &\quad - \bar{a}_l(\mathbf{k},t)\lambda(\mathbf{k},t) - \bar{\lambda}(\mathbf{k},t) a_l(\mathbf{k},t),\end{aligned} \tag{7.4.17}$$

where $\lambda(\mathbf{k},t)$ is a constant that will be defined in the next chapter. If the mean value of this term is calculated in respect to the new norm, the terms in the brackets will vanish, because they include the subsidiary condition for the case of the free field interacting with two fixed charges, which will be shown explicitly in the next chapter.

If the mean value is calculated in respect to the new ground state, the two terms with λ's will disappear too, because the λ's are the eigenvalues of the scalar operators acting on the new ground state and acting with a longitudinal annihilation and a scalar creation operator on a physical state will give a state orthogonal to the original one. The number of photons counted is therefore equal to the number of transverse photons.

Chapter 8
Covariant photon number density

To be or not to be: that is the question:

-Shakespeare's Hamlet

In the last previous chapters a Photon Number Density four vector that is Lorentz invariant was introduced and it was shown that the proposed operator indeed counts physical photons in configuration space using the new metric. The operator was not able to describe the ghost state photons and in this chapter that problem will be solved. A new Photon Number Operator which yields the same results in respect to the new metric, but is also able to describe the photons in the ghost states in respect to the old metric, will be introduced. It will be shown in the first section how this modification is achieved and why the same results are expected. In the following section it will be shown that the new operator can be justified using symmetry arguments and at the end of the chapter it then will be shown that the expected results are indeed achieved.

1 Modifying the proposed operator

In chapter four the Photon Number Density Operator was integrated to achieve a result that is comparable to already known operators. This leaves the possibility of changing the density operator without changing the result of the integration, because integrating over the divergence of a term gives a zero contribution to the value of the integral. In this section it will be shown how the operator proposed in the last chapter can be modified to obtain a new operator that should yield the same results in respect to the new metric.

Starting with the scalar component of the operator used in the last chapter, which is given by the expression

$$\Upsilon_s = \frac{c\varepsilon_0}{i\hbar}\left(F^{-}_{\mu s}A^{\mu +} - A^{\mu -}F^{+}_{\mu s}\right) \tag{8.1.1}$$

or in more explicit form

$$\Upsilon_s = \frac{c\varepsilon_0}{i\hbar}\left(\left(\partial_\mu A_s^-\right)A^{\mu+} - \left(\partial_s A_\mu^-\right)A^{\mu+} - A^{\mu-}\partial_\mu A_s^+ + A^{\mu-}\partial_s A_\mu^+\right), \tag{8.1.2}$$

yields after performing the summation and collecting equal and similar terms

$$\Upsilon_s = \frac{c\varepsilon_0}{i\hbar}\left(\nabla A_s^- \cdot \mathbf{A}^+ + \frac{1}{c}\dot{\mathbf{A}}^- \cdot \mathbf{A}^+ - \mathbf{A}^- \cdot \nabla A_s^+ - \frac{1}{c}\mathbf{A}^- \cdot \dot{\mathbf{A}}^+\right). \tag{8.1.3}$$

To change this to the new proposal, which should be just different by a divergence term, it is necessary to investigate the expression

$$\nabla \cdot (\mathbf{A}A_s) = (\nabla \cdot \mathbf{A})\, A_s + \mathbf{A} \cdot \nabla A_s, \tag{8.1.4}$$

which is equal to

$$\mathbf{A} \cdot \nabla A_s = -\left(\frac{1}{c}\dot{A}_s\right) A_s + \nabla \cdot (\mathbf{A}A_s), \tag{8.1.5}$$

where the Lorentz condition was used. The same result can be achieved by using positive and negative frequency parts in the expression and Eq.(8.1.3) can be written as

$$\Upsilon_s = \frac{\varepsilon_0}{i\hbar}\left(\dot{A}_s^- A_s^+ - A_s^- \dot{A}_s^+ + \dot{\mathbf{A}}^- \cdot \mathbf{A}^+ - \mathbf{A}^- \cdot \dot{\mathbf{A}}^+ + c\nabla \cdot \left(A_s^- \mathbf{A}^+\right) - c\nabla \cdot \left(\mathbf{A}^- A_s^+\right)\right). \tag{8.1.6}$$

The divergence terms will have no effect, because they do not contribute to the integral over configuration space that has to be calculated later. It is therefore possible to define the new Photon Number Density operator, which should have the same properties as the old one when integrated, by the expression

$$\Upsilon_s = \frac{\varepsilon_0}{i\hbar}\left(\dot{A}_s^- A_s^+ - A_s^- \dot{A}_s^+ + \dot{\mathbf{A}}^- \cdot \mathbf{A}^+ - \mathbf{A}^- \cdot \dot{\mathbf{A}}^+\right). \tag{8.1.7}$$

This operator has some advantages when symmetry arguments are used and they will be discussed briefly in the next section.

2 Symmetry arguments

The first photon number density operator introduced in this thesis was given in the Coulomb gauge and it was obtained by contracting the electromagnetic field tensor with the vector potential. In the covariant theory this operator was generalized and it was shown that the new operator could describe the transverse photons for physical states, but it was not able to describe the ghost states. In the last section this operator was modified and in this section an argument will be given why this modification is reasonable. The rigorous discussion of those arguments can be found in a recent paper by Hawton and Melde[14].

The theory goes back to Emi Noethers theorem, which states that each symmetry of a given Lagrangian generates a conservation law. In the case of the Coulomb gauge the Lagrangian is defined as

$$\mathcal{L}_R^{st} = -\frac{\varepsilon_0 c^2}{4} F_{\mu\nu} F^{\mu\nu} \tag{8.2.1}$$

and if a phase change symmetry would apply the corresponding conserved density/current four vector would yield

$$\chi_\mu = -\varepsilon_0 c^2 \begin{pmatrix} \frac{1}{c}\mathbf{E}\cdot\mathbf{A} \\ \mathbf{B}\times\mathbf{A} - \frac{1}{c^2}\mathbf{E}U \end{pmatrix}. \tag{8.2.2}$$

However, the same result is achieved contracting the electromagnetic field tensor and the vector potential times $-\varepsilon_0 c^2$, which yields

$$\chi_\mu = -\varepsilon_0 c^2 F_{\mu\nu} A^\nu. \tag{8.2.3}$$

A phase change can only make sense if the fields are not real and it is therefore necessary to separate the fields into negative and positive frequency parts. The hermitian photon density operator that generalizes Eq.(2) after the field were quantized and the operator was normal ordered can be given as

$$\chi_\mu = -\frac{\varepsilon_0}{i\hbar} \begin{pmatrix} c\left(\mathbf{E}^{(-)}\cdot\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot\mathbf{E}^{(+)}\right) \\ c^2\left(\mathbf{B}^{(-)}\times\mathbf{A}^{(+)} + \mathbf{A}^{(-)}\times\mathbf{B}^{(+)} - \frac{1}{c^2}\left(\mathbf{E}^{(-)}U^{(+)} - U^{(-)}\mathbf{E}^{(+)}\right)\right) \end{pmatrix}. \tag{8.2.4}$$

Comparing Eq.(1) and Eq.(3) shows how the photon number/current four vector depends on the given Lagrangian.

In the last section the photon density/current four vector was modified in a way so that the integrated form of both operators give the same mean value in respect to the new metric. The operator themselves however differ by a divergence term and the new operator is

$$\Upsilon_s = \frac{\varepsilon_0}{i\hbar}\left(\dot{A}_s^- A_s^+ - A_s^- \dot{A}_s^+ + \dot{\mathbf{A}}^-\cdot\mathbf{A}^+ - \mathbf{A}^-\cdot\dot{\mathbf{A}}^+\right). \tag{8.2.5}$$

It is possible to rewrite this operator in the form

$$\Upsilon_s = \frac{\varepsilon_0 c}{i\hbar}\left(\partial_t A_\mu^-\right) A^{\mu +} + h.c., \tag{8.2.6}$$

where $h.c.$ means hermitian conjugate. If multiplied by c this is the first component of a four vector and it has a similar relation to the Lagrangian used in this case as the relation that was found between the density/current four vector and the Lagrangian in the Coulomb gauge. The Lagrangian in the Lorentz gauge is

$$\mathcal{L}_R = -\frac{\varepsilon_0 c^2}{4} \left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) \tag{8.2.7}$$

and as before the conserved density/current four vector is obtained by contracting the negative frequency part of the first term in the Lagrangian with the positive frequency part of the vector potential and taking the hermitian conjugate.

This is only a crude comparison of the two cases, but it illustrates why the photon number density operator had to be modified in the Lorentz gauge. In the paper by Hawton and Melde this argument is shown explicitly, but the theory needed to show this argumentation is of a completely different kind than the theory developed in this thesis and therefore omitted. In the next section this new operator and it's properties will be investigated.

3 The number of Photons in the case of two fixed charges present

In this section the proposed Photon Number Density operator will be integrated over the configuration space and the expectation value in respect to the new ground state will be taken. Only the equations of motion and a new subsidiary condition for physical states will be used and the calculations are of the same form as in chapter five. But first the operator has to be expressed in respect to the new vector potential expressed in respect to the new metric, which yields

$$\begin{aligned} \Upsilon_s &= -\frac{1}{2i\,(2\pi)^3}\int d^3k \int d^3k' e^{i\left(\mathbf{k}'-\mathbf{k}\right)\cdot\mathbf{r}} \frac{1}{\sqrt{\omega\omega'}} \\ &\times \left(\frac{\partial}{\partial t}\bar{a}_s(\mathbf{k},t)\right) a_s\left(\mathbf{k}',t\right) - \bar{a}_s(\mathbf{k},t)\left(\frac{\partial}{\partial t}a_s\left(\mathbf{k}',t\right)\right) \\ &- \left(\frac{\partial}{\partial t}\bar{\mathbf{a}}(\mathbf{k},t)\right)\cdot\mathbf{a}\left(\mathbf{k}',t\right) + \bar{\mathbf{a}}(\mathbf{k},t)\cdot\left(\frac{\partial}{\partial t}\mathbf{a}\left(\mathbf{k}',t\right)\right). \end{aligned} \tag{8.3.1}$$

The equations of motion have to be used to simplify this result and to be able to do so it is necessary to solve the differential equation satisfied by the annihilation operators. The equations of motion are

$$\dot{a}_j + i\omega a_j = 0 \tag{8.3.2}$$

and

$$\dot{a}_s + i\omega a_s = \frac{i}{\sqrt{2\varepsilon_0\hbar\omega}} c\rho_e. \tag{8.3.3}$$

The spatial equations are ordinary homogeneous differential equations of the first order and the general solution of an equation of this kind is

$$a_j(\mathbf{k},t) = a_j(\mathbf{k})\, e^{-i\omega t}. \qquad (8.3.4)$$

The scalar part however is an ordinary inhomogeneous differential equation of first order and the general solution is

$$a_s(\mathbf{k},t) = a_s^H(\mathbf{k},t) + a_s^P(\mathbf{k},t), \qquad (8.3.5)$$

where the right hand side is given by the homogeneous solution plus a particular solution. As before the homogeneous solution is of the form

$$a_s^H(\mathbf{k},t) = a_s(\mathbf{k})\, e^{-i\omega t}. \qquad (8.3.6)$$

One solution of this differential equation is given by the expression

$$a_s^P(\mathbf{k}) = \frac{1}{\omega\sqrt{2\varepsilon_0\hbar\omega}} c\rho_e(\mathbf{k}), \qquad (8.3.7)$$

which implies that

$$\dot{a}_s^P(\mathbf{k}) = 0 \qquad (8.3.8)$$

and the particular solution is a constant in time. Using these relations Eq.(8.2.1) is given by the expression

$$\begin{aligned}
\Upsilon_s &= -\frac{1}{2i(2\pi)^3}\int d^3k \int d^3k' e^{i\left(\mathbf{k}'-\mathbf{k}\right)\cdot\mathbf{r}} \frac{1}{\sqrt{\omega\omega'}} \\
&\quad i\omega\bar{a}_s(\mathbf{k})\, e^{i\omega t}\left(a_s\left(\mathbf{k}'\right)e^{-i\omega' t} + \frac{1}{\omega'\sqrt{2\varepsilon_0\hbar\omega'}}c\rho_e\left(\mathbf{k}'\right)\right) \\
&\quad + \left(\bar{a}_s(\mathbf{k})\, e^{i\omega t} + \frac{1}{\omega\sqrt{2\varepsilon_0\hbar\omega}}c\rho_e^*(\mathbf{k})\right) i\omega' a_s\left(\mathbf{k}'\right)e^{-i\omega' t} \\
&\quad - i\left(\omega+\omega'\right)\bar{\mathbf{a}}(\mathbf{k})\cdot\mathbf{a}\left(\mathbf{k}'\right)e^{i\left(\omega-\omega'\right)t}.
\end{aligned} \qquad (8.3.9)$$

Collecting similar terms yields

$$\begin{aligned}
\Upsilon_s &= -\frac{1}{2i(2\pi)^3}\int d^3k \int d^3k' e^{i\left(\mathbf{k}'-\mathbf{k}\right)\cdot\mathbf{r}} \frac{1}{\sqrt{\omega\omega'}} \\
&\quad \times i\left(\omega+\omega'\right)\left[\bar{a}_s(\mathbf{k})\, a_s\left(\mathbf{k}'\right) - \bar{\mathbf{a}}(\mathbf{k})\cdot\mathbf{a}\left(\mathbf{k}'\right)\right] e^{i\left(\omega-\omega'\right)t} \\
&\quad + \frac{i\omega e^{i\omega t}}{\omega'\sqrt{2\varepsilon_0\hbar\omega'}}c\bar{a}_s(\mathbf{k})\,\rho_e\left(\mathbf{k}'\right) + \frac{i\omega' e^{-i\omega' t}}{\omega\sqrt{2\varepsilon_0\hbar\omega}}c\rho_e^*(\mathbf{k})\, a_s\left(\mathbf{k}'\right)
\end{aligned} \qquad (8.3.10)$$

and integrating over the configuration space leaves one with the expression

$$\int d^3x\Upsilon_s = -\frac{1}{2}\int d^3k2\left(\bar{a}_s(\mathbf{k})\,a_s(\mathbf{k}) - \bar{\mathbf{a}}(\mathbf{k})\cdot\mathbf{a}(\mathbf{k})\right) - \frac{c}{\omega\sqrt{2\varepsilon_0\hbar\omega}}\left[\bar{a}_s(\mathbf{k})\,\rho_e(\mathbf{k})\,e^{i\omega t} + \rho_e^*(\mathbf{k})\,a_s(\mathbf{k})\,e^{-i\omega t}\right], \tag{8.3.11}$$

which equals

$$\int d^3x\Upsilon_s = \int d^3k\bar{a}_T(\mathbf{k})\cdot a_T(\mathbf{k}) - \bar{a}_s(\mathbf{k})\,a_s(\mathbf{k}) + \bar{a}_l(\mathbf{k})\,a_l(\mathbf{k}) + \frac{1}{2}\left[\bar{a}_s(\mathbf{k})\,\lambda(\mathbf{k})\,e^{i\omega t} + \lambda^*(\mathbf{k})\,a_s(\mathbf{k})\,e^{-i\omega t}\right], \tag{8.3.12}$$

where the λ's are

$$\lambda(\mathbf{k}) = \frac{c}{\hbar\omega}\sqrt{\frac{\hbar}{2\varepsilon_0\omega}}\rho_e(\mathbf{k}). \tag{8.3.13}$$

The transverse part is the usual number operator for physical photons and the ls-part is given by the expression

$$\int d^3x\Upsilon_s^{LS} = \int d^3k\bar{a}_l(\mathbf{k})\,a_l(\mathbf{k}) - \bar{a}_s(\mathbf{k})\,a_s(\mathbf{k}) + \frac{1}{2}\left[\bar{a}_s(\mathbf{k})\,\lambda(\mathbf{k})\,e^{i\omega t} + \lambda^*(\mathbf{k})\,a_s(\mathbf{k})\,e^{-i\omega t}\right]. \tag{8.3.14}$$

Some fairly simple manipulations allow one to write this equation in the form

$$\int d^3x\Upsilon_s^{LS} = \frac{1}{2}\int d^3k\left(\bar{a}_s + \bar{a}_l\right)\left[a_l - a_s + \lambda\right] + \left[\bar{a}_l - \bar{a}_s + \lambda^*\right]\left(a_s + a_l\right) - \lambda^* a_l - \bar{a}_l\lambda. \tag{8.3.15}$$

The expressions in the square brackets are the subsidiary condition for the free field interacting with two fixed charges as can be seen by the following calculation.

The subsidiary condition follows the expression

$$\partial_\mu A^\mu = \sqrt{\frac{\hbar}{2\varepsilon_0\omega\,(2\pi)^3}}\int d^3k\left(ika_l + \frac{\dot{a}_s}{c}\right)e^{i\mathbf{k}\cdot\mathbf{r}} + h.c. \tag{8.3.16}$$

and can be given as

$$\left[ika_l(\mathbf{k}) + \frac{\dot{a}_s(\mathbf{k})}{c}\right]|\Psi\rangle = 0 \qquad \forall\mathbf{k}. \tag{8.3.17}$$

Using the equations of motion again makes it possible to write the subsidiary condition for physical states of the free field interacting with two fixed charges in the form

$$\left[a_l(\mathbf{k}) - a_s(\mathbf{k}) + \lambda(\mathbf{k})\right]|\Psi\rangle = 0 \qquad \forall\mathbf{k}, \tag{8.3.18}$$

which reduces to the one given for the free field in the absence of sources. Now it is possible to calculate the expectation value of the system of the operator given in Eq.(8.2.15) and the result is

$$\left(\tilde{\Psi}\right| \int d^3x \Upsilon_s^{LS} \left|\tilde{\Psi}\right) = \frac{1}{2} \int d^3k \left(\tilde{\Psi}\right| \left(-\lambda^* a_l - \bar{a}_l \lambda\right) \left|\tilde{\Psi}\right) \tag{8.3.19}$$

where the first two terms vanished because of the subsidiary condition for physical states. The right hand side of the achieved equation can be changed in the following way. By definition of the scalar product the equation results,

$$-\left(\tilde{\Psi}\right| \left(\lambda^* a_l + \bar{a}_l \lambda\right) \left|\tilde{\Psi}\right) = -\left\langle\tilde{\Psi}\right| M \left(\lambda^* a_l + \bar{a}_l \lambda\right) \left|\tilde{\Psi}\right\rangle = 0, \tag{8.3.20}$$

where M is the unitary operator that defines the new metric. The λ's can be taken outside the scalar product, because they do not represent an operator and the annihilation or creation of a longitudinal photon in either the Bra or the Ket will assure that the expression is equal to zero due to the orthogonality relation for different Bras and Kets. Therefore neither the longitudinal nor the scalar photons contribute to the expectation value of the Photon Number operator. This is the same result as the one that was achieved with the operators defined in the last section. In the next section expectation values of some physical and non physical states will be calculated.

4 Mean values of the new operator in the old metric

It was shown in the last section that the new photon number density operator describes physical states in the expected way using the new metric. In the case of physical states the ghost states disappeared in the new metric and only the transverse photons contributed to the number of photons in the system. In this section the operator will be translated in the old metric and mean values in respect to the old metric will be calculated. In the last chapter the proposed operator could not describe certain physical states correctly and it had to be abandoned. It is hoped that the new operator is able to describe those physical states correct and give more information about the ghost states. The case of a free field will be assumed for simplicity.

As before the transverse part yields no problems and it is therefore only necessary to investigate the ls-part. The photon number operator for the ls-photons is

$$\begin{aligned} \int d^3x \Upsilon_s^{LS} &= \int d^3k \bar{a}_l(\mathbf{k})\, a_l(\mathbf{k}) - \bar{a}_s(\mathbf{k})\, a_s(\mathbf{k}) \\ &\quad + \frac{1}{2}\left[\bar{a}_s(\mathbf{k})\, \lambda(\mathbf{k})\, e^{i\omega t} + \lambda^*(\mathbf{k})\, a_s(\mathbf{k})\, e^{-i\omega t}\right]. \end{aligned} \tag{8.4.1}$$

and next, the mean value of this operator in respect to the physical state with one scalar and one longitudinal photon in the case of a free field will be calculated. This expectation value is given by the expression

$$\langle\int d^3x\Upsilon_s^{LS}\rangle = \frac{\langle 1_l,1_s|\int d^3x\Upsilon_s^{LS}|1_l,1_s\rangle}{\langle 1_l,1_s|1_l,1_s\rangle} \tag{8.4.2}$$
$$= \frac{1}{\langle 1_l,1_s|1_l,1_s\rangle}\int d^3k\,\langle 1_l,1_s|\,\bar{a}_l(\mathbf{k})\,a_l(\mathbf{k}) - \bar{a}_s(\mathbf{k})\,a_s(\mathbf{k})\,|1_l,1_s\rangle\,,$$

where the λ terms are zero, because the free field is assumed. The expectation value can be written in respect to the old metric as

$$\langle\int d^3x\Upsilon_s^{LS}\rangle = \frac{1}{\langle 1_l,1_s|1_l,1_s\rangle}\int d^3k\,\langle 1_l,1_s|\,a_l^\dagger(\mathbf{k})\,a_l(\mathbf{k}) + a_s^\dagger(\mathbf{k})\,a_s(\mathbf{k})\,|1_l,1_s\rangle \tag{8.4.3}$$

Each of the two terms counts one photon in respect to this physical state and the final result is

$$\langle\int d^3x\Upsilon_s^{LS}\rangle = 2, \tag{8.4.4}$$

as was expected.

Assuming the free field again, the basis for the ls-subspace of physical states is given by the expression

$$|n_g\rangle = \frac{(a_g)^{n_g}}{\sqrt{n_g!}}\,|0\rangle \tag{8.4.5}$$

and it is useful to investigate the mean values of the operator in respect to these states. The mean value is given as

$$\langle\int d^3x\Upsilon_s^{LS}\rangle = \frac{\langle n_g,0_d|\int d^3x\Upsilon_s^{LS}|n_g,0_d\rangle}{\langle n_g,0_d|n_g,0_d\rangle} \tag{8.4.6}$$
$$= \frac{1}{\langle n_g,0_d|n_g,0_d\rangle}\int d^3k\,\langle n_g,0_d|\,a_l^\dagger(\mathbf{k})\,a_l(\mathbf{k}) + a_s^\dagger(\mathbf{k})\,a_s(\mathbf{k})\,|n_g,0_d\rangle$$

Now the operator has to be expressed in respect to the creation and destruction operators for the gd-photons and using the identities

$$a_l = \frac{1}{\sqrt{2}}\left(a_g - ia_d\right) \tag{8.4.7}$$

and

$$a_s = \frac{1}{\sqrt{2}}\left(a_g + ia_d\right) \tag{8.4.8}$$

gives

$$\langle\int d^3x\Upsilon_s^{LS}\rangle = = \frac{1}{\langle n_g,0_d|n_g,0_d\rangle}\int d^3k\,\langle n_g,0_d|\,\frac{1}{2}\left(a_g^\dagger + ia_d^\dagger\right)\left(a_g - ia_d\right)|n_g,0_d\rangle$$
$$+\langle n_g,0_d|\,\frac{1}{2}\left(a_g^\dagger - ia_d^\dagger\right)\left(a_g + ia_d\right)|n_g,0_d\rangle \tag{8.4.9}$$

Performing the multiplication in the first two terms and collecting equal terms yields

$$\langle\int d^3x\Upsilon_s^{LS}\rangle = \frac{1}{\langle n_g, 0_d|n_g, 0_d\rangle}\int d^3k\,\langle n_g, 0_d|\, a_g^\dagger a_g + a_d^\dagger a_d\,|n_g, 0_d\rangle\,. \tag{8.4.10}$$

For physical states the number of d-photons has to be zero and therefore the second term does not contribute to the expectation value. The final result is then given by the expression

$$\langle\int d^3x\Upsilon_s^{LS}\rangle = \frac{\langle n_g, 0_d|\int d^3x\Upsilon_s^{LS}\,|n_g, 0_d\rangle}{\langle n_g, 0_d|n_g, 0_d\rangle} = n_g \tag{8.4.11}$$

and this is the expected result. The new operator is able to count the gauge photons in the free field and behaves nicely in this case.

Next, one special state in the case of two fixed charges are present will be investigated and the mean value of the ls-part of the photon number operator in respect to this state will be calculated. This mean value is

$$\begin{aligned}\langle\int d^3x\Upsilon_s^{LS}\rangle &= \frac{\left\langle\tilde{\Psi}\right|\int d^3x\Upsilon_s^{LS}\left|\tilde{\Psi}\right\rangle}{\left\langle\tilde{\Psi}|\tilde{\Psi}\right\rangle} \\ &= \frac{1}{\left\langle\tilde{\Psi}|\tilde{\Psi}\right\rangle}\int d^3k\left\langle\tilde{\Psi}\right|\bar{a}_l(\mathbf{k})\,a_l(\mathbf{k}) - \bar{a}_s(\mathbf{k})\,a_s(\mathbf{k})\left|\tilde{\Psi}\right\rangle \\ &\quad+\frac{1}{2}\left\langle\tilde{\Psi}\right|\left[\bar{a}_s(\mathbf{k})\,\lambda(\mathbf{k})\,e^{i\omega t} + \lambda^*(\mathbf{k})\,a_s(\mathbf{k})\,e^{-i\omega t}\right]\left|\tilde{\Psi}\right\rangle,\end{aligned} \tag{8.4.12}$$

which can be written in respect to the old metric as

$$\begin{aligned}\langle\int d^3x\Upsilon_s^{LS}\rangle &= \frac{1}{\left\langle\tilde{\Psi}|\tilde{\Psi}\right\rangle}\int d^3k\left\langle\tilde{\Psi}\right| a_l^\dagger(\mathbf{k})\,a_l(\mathbf{k}) + a_s^\dagger(\mathbf{k})\,a_s(\mathbf{k})\left|\tilde{\Psi}\right\rangle \\ &\quad+\frac{1}{2}\left\langle\tilde{\Psi}\right|\left[-a_s^\dagger(\mathbf{k})\,\lambda(\mathbf{k})\,e^{i\omega t} + \lambda^*(\mathbf{k})\,a_s(\mathbf{k})\,e^{-i\omega t}\right]\left|\tilde{\Psi}\right\rangle.\end{aligned} \tag{8.4.13}$$

The last two terms disappear, because the final states are orthogonal as was seen in the last chapter. Each of the first two terms counts the number of l- and s-photons in respect to this physical state and the final result is

$$\langle\int d^3x\Upsilon_s^{LS}\rangle = n_l + n_s, \tag{8.4.14}$$

as was expected. It has to be noted that in this case n_l is not necessarily equal to n_s anymore, as was the case in the free field. The reason for this is the changed subsidiary condition for physical states and the operator will count all the photons present.

It is therefore shown that the new operator allows one to count physical photons, which are the only ones contributing to the mean value in respect to physical states, and to count photons in the ghost states when the old metric is used. This is a nice and rather unexpected result and in the next chapter the conclusion of the thesis will be given.

Chapter 9
Conclusion

An Artist is someone who produces things that people don't need to have but that he -for some reason- thinks it would be a good idea to give them.

-Andy Warhol

1 Results in the Coulomb gauge

In the first part of the thesis a new photon number density operator was proposed in the Coulomb gauge. The form of this hermitian operator is simple, because it is constructed only using the electromagnetic fields and the vector potential. Every previous attempt to define a photon number density operator had to use more difficult operators and it was hard to describe their behavior under physical operations. The photon number density operator proposed in this thesis on the other hand behaves nicely and it is not complicated to investigate its behavior under Lorentz transformations, because the Lorentz transformations of its constituents are well known.

If the proposed photon number density operator is expressed in respect to the annihilation and creation operators for the field one has to sum over the different modes of the field. Every term in this sum has the weighting factor $\sqrt{\frac{\omega_j}{\omega_i}}$, which shows that the weighting factor depends explicitly on the angular frequencies of the field. In the Mandel operator this factor is missing and the photon number densities in respect to the operator proposed here and the Mandel operator should be different, if polychromatic fields are allowed. After integration over all space however this dependance vanishes and the total photon number operator is the sum of the equally weighted numbers of photons with different wave vectors. The new total photon number operator behaves like every previously defined operator, but it's density operator is different and it's transformation properties are simpler. We believe that this density operator is the physically correct one, because of the following properties.

The proposed photon number density is the scalar part of a four vector, because it was constructed by contracting the electromagnetic field tensor with the vector potential. This is a nice result, because it can be immediately stated that this four vector contracted with another four vector has to be an invariant. One special four vector is $\begin{pmatrix} \partial_t \\ \nabla \end{pmatrix}$ and it follows immediately that the proposed photon number density and the photon current satisfy a weak continuity equation where the invariant was shown to be

$$\frac{1}{i\hbar}\left[\mathbf{j}^{(-)}\cdot\mathbf{A}^{(+)} - \mathbf{A}^{(-)}\cdot\mathbf{j}^{(+)} + U^{(-)}\rho^{(+)} - \rho^{(-)}U^{(+)}\right].$$

In the case of the free field this invariant is equal to zero, which means that for the free field the operators satisfy a continuity equation. It is not surprising that in the case of matter interaction the invariant in not zero anymore, because it is possible that sources or sinks of photons are present in the configuration space. The invariant depends on the charge density\current four vector, which explicitly manifests the possibility of photon sources present.

2 Results in the Lorentz gauge

Before a photon number density operator could be given in the Lorentz gauge it was necessary to solve problems arising in the second quantization for the fields in the Lorentz gauge. It was shown that following the usual quantization scheme the norm for the states with an uneven number of scalar photons will have a different sign than the norm for states with an even number of scalar photons. This is a problem, because it allows states with negative energy eigenvalues for the Hamiltonian and it was necessary to introduce a new theory that allowed states with negative and zero norms. The indefinite metric following an approach by Gupta was introduced and a new "scalar product" was defined that allowed the possibility to have negative norms. Strictly speaking this is not a scalar product, but in this new metric it plays the same role as the usual scalar product plays in the Hilbert space. The beauty of the theory is that all operations and conditions are defined in the new metric, where negative norms are allowed, but every physical entity described by an operator defined in this manner has a positive mean value in the old metric.

The condition to be hermitian is now given in respect to this new metric and physical operators like observables and the vector potential have to be hermitian in respect to the new metric. In respect to the old metric those operators are not hermitian anymore, because their scalar parts are now anti-hermitian in respect to the old metric. Thus the commutation relations for the annihilation and creation operators for the scalar photons expressed in the old metric are the usual ones (i.e. no minus signs) and it was possible to construct a basis for the scalar photon states using the linear combinations of creation operators. These new basis vectors always have positive norms in respect to the old metric, but in respect to the new metric the norms can also be negative or zero.

The mean values of operators can be expressed in respect to the new metric now. The vector potential describing the system in this way would classically not be an observable, but this is no problem, because the vector potential is not an observable. However, mean values of every observable in respect to the new metric for physical states are equal to the mean values of the transverse parts in respect to the old metric. The physical states are the ones satisfying a new subsidiary condition, which assures that the mean value of the operator $\partial_\mu A^\mu$, which basically represents the Lorentz gauge condition, is zero.

Because of the simple form of the subsidiary condition, another base transformation could be defined and the new classes of d-photons and gauge-photons were introduced. Physical states require that no d-photon is present, but the number of gauge-photons is arbitrary.

Physical states can now be constructed from the vacuum using the creation operators for transverse and gauge-photons and physical systems can be represented by a class of states, i.e. the states with the same number of transverse photons, but an arbitrary number of gauge-photons. This means that the gauge arbitrariness is included in the state vectors themselves. The orthonormality relations for physical states are the usual ones in respect to the old metric, but scalar products between states in respect to the new metric are always zero, unless the number of g-photons is zero. In the new metric the gauge-photons cannot contribute to expectation values in respect to physical states, but in the old metric they can.

The photon number operator was then expressed in respect to the new metric and it was shown that although this operator could describe physical states in respect to the new metric it failed to describe the states in respect to the old metric correctly. The ghost states, which drop out for physical states in respect to the new metric, could not be counted correctly, because the number operator for scalar photons dropped completely out of the total photon number operator.

An interpretation following symmetry arguments was given, which explained that the operator had to be modified, because the Lagrangian of the system has to be modified when using the Lorentz gauge. A number operator appropriate in the Lorentz gauge was obtained and the two photon number density operators only differ by a gradient term (as do the Lagrangians) and therefore the two total number operators in respect to the new metric count the same number of photons.

Inspecting mean values of different states in respect to the two metrics showed that this new operator describes physical systems as expected. The mean value of the operator in respect to the new metric for physical states gives the same result as the mean value of the transverse parts of the operator in respect to the old metric. That means that in the new metric for physical states only transverse photons are counted.

The same operator in respect to the old metric, where it is not a hermitian operator, counts the same number of transverse photons, but also counts the g-photons. It therefore automatically counts the ghost photons that appear in respect to the old metric, because of the gauge arbitrariness.

It also was shown that in the case of two fixed charges the operator can be generalized and additional terms, depending on the charges, appear. The subsidiary condition changes and the therefore the basis of physical states has to be modified.

In summary it was shown in the thesis that a mathematical model can be defined that allows one to introduce new photon number and current density operators in the Coulomb and in the Lorentz gauges that can count physically relevant photons and gauge photons of physical states.

References

[1] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

[2] J. M. Jauch, C. Piron, Helv. phys. Acta **40**, 559 (1967), *Generalized Localizability*

[3] G. W. Mackey, *Induced representations of groups and quantum mechanics* (W.A. Benjamin, New York, 1968).

[4] W. O. Amrein, Helv. phys. Acta **42**, 149 (1969), *Localizability for Particles of Mass Zero.*

[5] L. Mandel, Phys. Rev. **144**, 1071 (1966), *Configuration-Space Photon Number Operators in Quantum Optics.*

[6] R.·J. Cook, Phys. Rev. A **25,** 2164 (1982), *Photon Dynamics.*

[7] R. J. Cook, Phys. Rev. A **26**, 2754 (1982), *Lorentz covariance of photon dynamics.*

[8] T. Inagaki, Phys. Rev. A **49**, 2839 (1994), *Quantum-mechanical approach to a free photon.*

[9] S. N. Gupta, Proc. Phys. Soc. London A **63**, 681 (1950), *Theory of Longitudinal Photons in Quantum Electrodynamics.*

[10] M. Kaku, *Quantum Field Theory* (Oxford University press, New York, 1993).

[11] C. Cohen Tannoudji, J. Dupont-Roc and G. Grynberg, *Photons and Atoms* (John Wiley and Sons,Inc., New York, 1989).

[12] R. Loudon, *The Quantum Theory of Light* (Oxford University Press, New York, 2nd ed.,1983)

[13] P. W. Milonni, *The Quantum Vacuum* (Academic Press, Inc., San Diego, 1994).

[14] M. Hawton and T. Melde, Phys. Rev. A **51,** 4186 (1995)

[15] M. Hawton, Phys. Rev. A **48**, 1824 (1993).

[16] K. Bleuler,Helv. phys. Acta **23**, 5 (1950), *A new method for the treatment of longitudinal and scalar photons.*