

ABSTRACT CONVEXITY IN METRIC SPACES

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by

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T.K.Saha.

ABSTRACT

Convexity in metric space is the main topic of discussion in this thesis. To undertake the study we have studied extensively the means introduced by Doss and included the results concerning means derived by Gahler and Murphy. We use this definition of a mean to define a new notion of convexity on a metric space, called B-convexity. B-convexity has been compared with other notions of convexity on a metric space. Finally following a construction given by Machado, we show that a B-convex metric space, satisfying certain properties, is essentially a convex subset of a normed space and the space is unique.

BASIC NOTATIONS

The following terminology will be used in this thesis. Symbols other than these will be defined individually.

X	any arbitrary nonempty set.
ϕ	empty set.
A, B, \dots	capital letters will usually denote subsets of X .
a, b, c, x, y, \dots	small letters will usually denote elements of X .
\subseteq	usual set containment.
\cup	usual set union.
\cap	usual set intersection.
\setminus	usual set difference operation.
e.g.	for example.
i.e.	that is.
\in	belongs to ($x \in A$ means x belongs to A).
\mathbb{R}	set of real numbers.
\mathbb{Q}	set of rational numbers.
\mathbb{Z}	set of all integers.
\mathbb{N}	set of natural numbers .
\mathbb{R}^N	N -dimensional vector space.

$S(x,r)$ open ball with centre x , and radius r .

$\bar{S}(x,r)$ closed ball with centre x , and radius r .

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Chapter I

Introduction

1.1. Historical Review:

Convexity is a broad geometrical concept studied by different mathematicians since early in the nineteenth century. The scope and applicability of convexity led different authors to investigate and extend the notion of convexity in different spaces. Our main topic of discussion will be convexity in metric space.

Perhaps the earliest notion of convexity in a metric space was given by Menger [17] in 1928. He defined a betweenness condition: z is between x and y if $d(x,z) + d(z,y) = d(x,y)$. C is convex if for every $x,y \in C$, then $z \in C$, for z between x and y . Busemann [4] defined convexity in the Menger sense as follows: A set H in a metric space is convex in the Menger sense if for every pair of distinct points x, z in H , there exists a point y in H such that $d(x,y) + d(y,z) = d(x,z)$. In a similar way P.S. Soltan [18] introduced d -convexity as follows: a set M of a metric space X is called d -convex if $d(x_1,x_2) = d(x_1,x_3) + d(x_3,x_2)$ with $x_1, x_2 \in X$, then $x_3 \in M$.

In 1969 W. Takahashi [19] introduced another notion of convexity in a metric space X by an operator W from $X \times X \times [0,1]$ into X satisfying $d(z, W(x,y;\alpha)) \leq (1-\alpha)d(z,x) + \alpha d(z,y)$, for all $x,y,z \in X$ and $\alpha \in [0,1]$.

Further a subset $K \subseteq X$ is convex iff $W(x,y;\alpha) \in K$ for $x,y \in K$ and $0 \leq \alpha \leq 1$.

Machado [15] in 1973 introduced conditions under which a convex metric space (using Takahashi's definition) is isomorphic to a convex subset of some normed linear space E . Although, Takahashi's convexity structure produced a reasonably rich notion of convexity for a metric space, however, in general, W need not be continuous and the permutations of the order of repeated convex combination may not be related. Louis A. Talman [20] addressed those problem and he complemented Takahashi's notion of convexity by assuming compactness of the metric space or continuity of W . Secondly he extended the definition of Takahashi to a higher dimension and defined a strong convex structure on X . Using this he proved a fundamental property of convex hulls.

In this thesis we will use Doss's [9] definition of mean in a metric space and the properties of the mean as given by Gahler and Murphy[10]. Some of the proof presented in this thesis are different from Gahler and Murphy. We relate the definition of mean to a new definition of convexity . We call it B-convexity. The results derived by W.Takahashi [19], H.V.Machado [15] and L.A.Talman [20] regarding convexity structure in their papers can be derived from B-convexity.

We begin by giving some necessary definitions in the next section.

1.2. Basic Definitions:

The following definitions will be often referred to in this thesis. They can be easily found in any standard text book.

1.2.1. Definition: A linear space over the field \mathbf{R} is a non empty set V of elements with two operations $+$ and $*$ is called additions and multiplications,

respectively, satisfying the following axioms with respect to the elements of V and \mathbf{R} .

(1) To every pair, x and $y \in V$, there correspond an element $x + y$, called the sum of x and y , in such a way that:

- (i) addition is commutative, $x + y = y + x$;
- (ii) addition is associative, $x + (y + z) = (x + y) + z$;
- (iii) there exist in V a unique element "O"(called the origin) such that $x + O = x$ for every $x \in V$;
- (iv) to every vector there corresponds a unique element $-x$ such that $x + (-x) = O$.

(2) To every pair $\alpha \in \mathbf{R}$ and $x \in V$, there correspond an element $\alpha \cdot x$ in V , called the product of α and x in such a way that

- (v) multiplication is associative, i.e., $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$;
- (vi) there is a $1 \in \mathbf{R}$ so that $1 \cdot x = x = x \cdot 1$, for every $x \in V$.

(3) (i) Multiplication is distributive with respect to addition,

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y;$$

(ii) Multiplication by vectors is distributive with respect to scalar addition,

$$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x.$$

The elements of V are called vectors whilst the elements of \mathbf{R} are scalars.

1.2.2. Definition : Two linear spaces X and X^* over the same field are said to be

isomorphic if there is one-to-one corresponding $x \longleftrightarrow x^*$ between X and X^* which preserves the operations in the sense that :

$$x \longleftrightarrow x^*, y \longleftrightarrow y^* \text{ implies } x + y \longleftrightarrow x^* + y^* \text{ and}$$

$$\alpha x \longleftrightarrow \alpha x^*, \text{ where } \alpha \text{ is an arbitrary scalar and } x, y \in X \text{ and } x^*, y^* \in X^*.$$

1.2.3. Definition: Let X be any set. A function $d(x,y)$ on the set $X \times X$ is a metric provided:

- (i) $d(x,y)$ is a non negative real number for every pair (x,y) of $X \times X$;
- (ii) $d(x,y) = d(y,x)$;
- (iii) $d(x,y) = 0$ iff $x = y$;
- (iv) $d(x,z) \leq d(x,y) + d(y,z)$.

The pair (X,d) is called a metric space.

1.2.4. Definition: Let (X,d) be a metric space. For $p \in X$ and $\delta > 0$,

$S(p, \delta) = \{x : d(p,x) < \delta\}$ and $\bar{S}(p, \delta) = \{x : d(p,x) \leq \delta\}$ are the open and closed balls centered at x with radius δ .

1.2.5. Definition: A metric space (X,d) is isometric to a metric space (Y,e)

iff there exists a one-one and onto function $f : X \longrightarrow Y$ which preserves the distance, i.e., for all $a,b \in X$, $d(a,b) = e(f(a),f(b))$.

1.2.6. Definition: Let X be a linear space. A function which is associates with each $x \in X$, a real number $\|x\|$ is called a norm on X , provided :

- (i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

(iii) $\|\lambda x\| = |\lambda| \|x\|$ where $\lambda \in \mathbf{R}$.

A linear space X with a norm is called a normed linear space or simply a normed space.

1.2.7. Definition: A subset S of a linear space is called a convex set iff for all

$a, b \in S$ and $0 \leq \lambda \leq 1$ then $\lambda x + (1 - \lambda) y \in S$.

1.2.8. Definition: A set C is compact if every infinite subset of C has an accumulation point in C .

1.2.9. Fact: A bounded infinite set has at least one accumulation point. If the accumulation point lies in the set, then the set is finitely compact.

Chapter II

Axiomatic Convexity Space.

2.1. Introduction:

Convexity is a broad geometric concept which has been studied for a long time. Usually when we consider convexity we consider convex subsets of linear spaces. Also convexity has been defined in several ways. In this chapter, an axiomatic setting for the theory of convexity is provided, following the approach of Kay and Womble [14].

2.2. Axiomatic Convexity Space:

2.2.1. Definition: Let X be a set and \mathcal{C} be a family of subsets of X . Then (X, \mathcal{C}) is an axiomatic convexity space if

- (i) $\phi, X \in \mathcal{C}$;
- (ii) $\cap F \in \mathcal{C}$, for $F \subseteq \mathcal{C}$.

The sets in \mathcal{C} are called \mathcal{C} -convex (or convex) sets.

2.2.2. Example: Let G be a group and \mathcal{C} consists of all subgroups of G including ϕ and G . Then (G, \mathcal{C}) is an axiomatic convexity space, since intersection of subgroups of G is a subgroup of G .

2.2.3. Example: Let (X, Y) be a topological space, where Y is a collection of closed subsets of X . Therefore (X, Y) is an axiomatic convexity space.

2.2.4. Example: Let V be a linear space. Let \mathcal{C} be the usual collection of convex sets in V . Then (V, \mathcal{C}) is an axiomatic convexity space.

2.2.5 Example: Let (P, \leq) be a poset. Let $\mathcal{C} = \{ A: A \subseteq P \text{ with the property that for } a, b \in A, a \leq x \leq b \text{ implies } x \in A \}$. It can be shown that (P, \mathcal{C}) is an axiomatic convexity space.

2.3. The convex hull:

2.3.1. Definition: Let (X, \mathcal{C}) be a convexity space. For $A \subseteq X$, the convex hull of A is defined as $\mathcal{C}(A) = \bigcap \{ C: C \in \mathcal{C}, A \subseteq C \}$.

2.3.2. Proposition:

The hull operator possesses the following properties:

- (i) $\mathcal{C}(\emptyset) = \emptyset$;
- (ii) $A \subseteq \mathcal{C}(A)$ for each $A \subseteq X$;
- (iii) $A \subseteq B$ implies $\mathcal{C}(A) \subseteq \mathcal{C}(B)$;
- (iv) $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$;
- (v) $A \in \mathcal{C}$ iff $\mathcal{C}(A) = A$.

Proof: (i) and (ii) are trivial by definition of $\mathcal{C}(A)$.

(iii) $\mathcal{C}(A) = \bigcap \{ C: C \in \mathcal{C}, A \subseteq C \}$. Also, by (ii), we have $A \subseteq B \subseteq \mathcal{C}(B) = \bigcap \{ C \in \mathcal{C}, B \subseteq C \}$. But $\mathcal{C}(A) \subseteq C$ for all $A \subseteq C$. Therefore $\mathcal{C}(A) \subseteq \mathcal{C}(B)$.

(iv) By (ii) and (iii), $\mathcal{C}(A) \subseteq \mathcal{C}(\mathcal{C}(A))$.

Now suppose $a \in \mathcal{C}(\mathcal{C}(A)) = \bigcap \{ C: C \in \mathcal{C}, \mathcal{C}(A) \subseteq C \}$ which now implies

$a \in C$ for all $C \in \mathcal{C}$ such that $\mathcal{C}(A) \subseteq C$. Therefore $a \in \mathcal{C}(A)$.

Thus $\zeta(A) = \zeta(\zeta(A))$.

(v) If $A \in \zeta$, by Definition 2.3.1., $\zeta(A) \subseteq A$. Thus $A = \zeta(A)$.

2.3.3. Remarks: It may be noted that the hull operator ζ is not necessarily a topological closure operator since $\zeta(A \cup B) \neq \zeta(A) \cup \zeta(B)$ in general.

2.4. Join hull-commutative (JHC):

For any members $a, b \in X$ we denote the convex hull of a and b by $\zeta(a, b)$, and call it a segment with end points a and b . The convex hull of finite sets $\{a, b, c, d, \dots\}$ will be denoted by $\zeta(a, b, c, d, \dots)$. If $a = b$ it is not necessary that the open segment be empty.

2.4.1. Definition: The join of x and A in a convexity space (X, ζ) is the set defined by $xJA = \cup\{\zeta(x, a) : a \in A\}$ where $x \in X, A \subseteq X$. For two sets $A, B \subseteq X$, their join will be $AJB = \cup\{\zeta(a, b) : (a, b) \in (A \times B)\}$.

2.4.2. Lemma: Let (X, ζ) be an axiomatic convexity space. For any

$a, x \in X, xJ\zeta(A) \subseteq \zeta(x \cup A)$.

Proof: Suppose $y \in xJ\zeta(A)$. Then $y \in \zeta(x, a)$ for some $a \in \zeta(A)$. Now $\zeta(A) \subseteq \zeta(x \cup A)$. So $\zeta(x, a) \subseteq \zeta(x \cup A)$. Hence $xJ\zeta(A) \subseteq \zeta(x \cup A)$.

The reverse inclusion does not always hold.

2.4.3. Example: Let X be the 3-dimensional euclidean space (\mathbb{R}^3) and let ζ be the collection of one and two dimensional convex subsets (in the usual

sense) including ϕ and X . Let A be a two dimensional subset of X and x , be a point not on the plane generated by A . Then $x \vee \zeta(A)$ is a convex cone (in the usual sense with vertex x) whilst $\zeta(x \cup A)$ is $X(=R^3)$ which is the smallest member of ζ containing x and A .

2.4.4. Definition: (X, ζ) is said to be join hull-commutative (JHC) iff $\cup\{\zeta(x, b) : b \in \zeta(A)\} = \zeta(\cup\{\zeta(x, a) : a \in A\}) = \zeta(x \cup A)$ for each $A \subseteq X$ and for every $x \in X$.

2.5. Domain Finite (DF)

2.5.1. Definition: (X, ζ) is said to be domain finite (DF) iff for each $A \subseteq X$, $\zeta(A) = \cup\{\zeta(F) : F \subseteq A \text{ and } F \text{ is finite}\}$.

2.5.2. Lemma: If (X, ζ) is domain finite and join hull commutative, then a subset A of X is convex iff $\zeta(a, b) \subseteq A$ for all $a, b \in A$.

Proof: It is easy to see that if A is convex then by definition $\zeta(a, b) \subseteq \zeta(A) = A$, $\forall a, b \in A$. Conversely, it is sufficient to show that $\zeta(A) \subseteq A$. We prove this part by induction. Take $|F| = 1$ i.e. $F = \{x\}$, then by assumption, $\zeta(x) = \zeta(x, x) \in A$. Hence $\zeta(F) \subseteq A$. Now, suppose the above relation is true for $|F| \leq n - 1$ and $F \subseteq A$. We shall prove that the relation is true when $|F| = n$. Let $t \in \zeta(F)$ then $t \in \zeta(x_1, x_2, \dots, x_n)$. By join-hull commutativity there exists $b \in \zeta(x_1, x_2, \dots, x_n)$ such that $t \in \zeta(x_1, b) \subseteq A$. Hence $\zeta(F) \subseteq A$ for any finite set $F \subseteq A$. Again by

definition of domain finiteness, $\zeta(A) = \cup \{ \zeta(F) : F \subseteq A, F \text{ is finite} \} \subseteq A$.

2.5.3 Lemma: If (X, ζ) is a convexity space which is domain finite and join hull commutative, then $\zeta(A \cup B) = \zeta(A) \text{ J } \zeta(B) = \zeta(B) \text{ J } \zeta(A)$.

Proof: By definition, $\zeta(A) \text{ J } \zeta(B) = \cup (\zeta(a, b) : a \in \zeta(A), b \in \zeta(B))$. Let

$a \in \zeta(A)$ and $b \in \zeta(B)$, then $a, b \in \zeta(A) \cup \zeta(B) \subseteq \zeta(A \cup B)$. Therefore $\zeta(a, b)$

$\subseteq \zeta(A \cup B)$. So, $\cup (\zeta(a, b)) \subseteq \zeta(A \cup B)$. Hence $\zeta(A) \text{ J } \zeta(B) \subseteq \zeta(A \cup B)$ Again by

definition of domain finiteness for $x \in \zeta(A \cup B)$, we have

$x \in \zeta(a_1, a_2, \dots, a_n)$ where $a_1, a_2, \dots, a_n \in A \cup B$. If a_1, a_2, \dots, a_n belongs to

either A or B, then $x \in \zeta(A) \text{ J } \zeta(B)$. Consequently $\zeta(A \cup B) \subseteq \zeta(A) \text{ J } \zeta(B)$. Next

we consider $a_1, a_2, \dots, a_m \in A$, and $a_{m+1}, a_{m+2}, \dots, a_n \in B$, then by definition of the

join hull commutative property we have $\zeta(a_1, a_2, \dots, a_m, a_{m+1}, \dots, a_n) = a_1 \text{ J } a_2 \text{ J}$

$a_3 \dots \text{ J } a_m \text{ J } \zeta(a_{m+1}, \dots, a_n) \subseteq \zeta(a_1, a_2, \dots, a_m) \text{ J } \zeta(a_{m+1}, \dots, a_n) \subseteq \zeta(A) \cup \zeta(B)$.

2.5.4. Lemma: Let (X, ζ) be a DF and JHC convexity space. If A and B are

convex subsets of X, then for each $x \in \zeta(A \cup B)$, there exists $a \in A$ and $b \in B$

such that $x \in \zeta(a, b)$.

Proof: Follows directly from the lemma 2.5.3.

2.6. Regularity (REG):

2.6.1. Definition: (X, ζ) is regular (REG) iff each segment in (X, ζ) satisfies the

following properties:

(i) non-discrete : $\forall x, y \in X, x \neq y$, then $\zeta(x, y) \setminus \{x, y\} \neq \phi$.

(ii) decomposable: $\forall z \in \zeta(x, y)$. $\zeta(x, z) \cap \zeta(x, y) = \{z\}$ and $\zeta(x, z) \cup \zeta(z, y) = \zeta(x, y)$.

(iii) extendable: $\forall x \neq y \exists u, v \in X$ such that $\zeta(x, y) \subseteq \zeta(x, v) \setminus \{v\}$ and $\zeta(x, y) \subseteq \zeta(u, y) \setminus \{u\}$.

2.6.2. Lemma: If (X, ζ) is REG then (i) $\zeta(x, x) = x, \forall x \in X$; (ii) If $a \in \zeta(b, c)$ and $b \in \zeta(a, c), a \neq c, b \neq c$ then $a = b$; (iii) For any $a, b, c \in X, a \neq b \neq c$, if $a \in \zeta(b, c)$, then $b \notin \zeta(a, c)$ and $c \notin \zeta(a, b)$.

Proof: (i) Since $x \in \zeta(x, x)$ implies $\zeta(x, x) \neq \phi$.

Also by definition of the non discrete property, we have $\zeta(x, y) \setminus \{x, y\} \neq \phi$ for $x \neq y$. Since $x \in \zeta(x, y)$, then we have $\zeta(x, x) \cap \zeta(x, y) = \{x\}$ and $\zeta(x, x) \cup \zeta(x, y) = \zeta(x, y)$. So $\zeta(x, x) \subseteq \zeta(x, y)$. Therefore $\zeta(x, x) = \{x\}$.

(ii) Let $a \in \zeta(b, c)$. Therefore $\zeta(b, c) = \zeta(b, a) \cup \zeta(a, c)$ and $\{a\} = \zeta(b, a) \cap \zeta(a, c)$. Again $b \in \zeta(a, c)$. Therefore $\{b\} = \zeta(a, b) \cap \zeta(b, c)$, and $\{b\} = \zeta(a, b) \cap \{\zeta(b, a) \cup \zeta(a, c)\} = \zeta(a, b) \cap \{\zeta(a, b) \cup \zeta(a, c)\} = \zeta(a, b) \cup \{\zeta(a, b) \cap \zeta(a, c)\} = \zeta(a, b) \cup \{a\} = \zeta(a, b)$

Similarly, we can show that $\{a\} = \zeta(a, b)$. Hence $a = b$.

(iii) On the contrary let us suppose $b \in \zeta(a,c)$. Then $\zeta(b,c) \subseteq \zeta(a,c)$.

Also since $a \in \zeta(b,c)$ then $\zeta(a,c) \subseteq \zeta(b,c)$ and $\zeta(b,c) = \zeta(a,c)$. By (ii) we have $a = b$ which is contradictory to the hypothesis. Therefore $b \notin \zeta(a,c)$.

Similarly we can show that $c \notin \zeta(a,b)$.

2.6.3. Remarks: The segments in a regular space can be given a natural linear ordering, because the decomposability relation essentially yields a betweenness relation.

2.7. Straight (STR):

2.7.1. Definition: An axiomatic space (X, ζ) is said to be straight iff the union of two segments having more than one point in common is a segment.

2.7.2. Theorem: Let (X, ζ) be a straight regular space. For $a, b \in X$, $a \neq b$, then (a, b) the line determined by a and b , is uniquely determined.

Proof: Omitted [22].

2.8. Line Spaces: The idea of line space was first introduced by Cantwell [5] in 1974. Later in 1978, Cantwell & Kay [6] proved that straight line spaces of dimension three or higher were isomorphic to an open convex subset of a real vector space. The approach used by Cantwell and Kay was classical and descriptive geometry and self contained. However, in 1976, Doignon [7] had also essentially obtained the same result that line spaces of dimension three or greater or of dimension two and Desarguesian are linearly open convex subsets of a real affine space using a different technique than Cantwell and

Kay. But the result of Doignon [7] depends on a 1938 theorem of Spencer. In 1981 Whitfield and Yong [22] used Doignon's result to prove the following linearization theorem:

2.8.1. Theorem: If (X, ζ) be convexity space of dimension 2 and desarguesian or of dimension >2 , then (X, ζ) is isomorphic to a linearly open convex subset of a real affine space iff (X, ζ) is DF, CMP, JHC, REG, and STR.

Chapter III

Metric Characterization of Normed Linear Space

3.1. Introduction: In this chapter a metric characterization of an arbitrary normed linear space is given by following the work of Gähler and Murphy [10].

In section 2, the fundamental notion of a mean $B_\rho(a,b)$ as given by Doss [8] is considered. Frechet [9] called $B_\rho(a,b)$ a generalized mean (T), and indicated that $B_\rho(a,b)$ may be thought of as the points that divide the segment joining a and b in the ratio $\rho : (1-\rho)$. That this holds is shown in Lemma 3.2.10.

In section 3, $B_\rho(a,b)$ is shown to be a singleton for every $\rho \in \mathbb{R}$ and for every a,b in an arbitrary normed linear space.

In section 4, we consider a metric space X with property GM (given by Gähler and Murphy in their paper as C). In that case also means are singletons. The property GM holds in every normed linear space.

In section 5, it is shown that a metric space with property GM will generate a linear structure iff it has a certain property A which means associativity of addition.

In section 6, assuming the same conditions for X , we show there exists a norm on X such that the corresponding metric is equal to the given metric on X . The resulting normed linear space is unique up to an isometric

isomorphism.

3.2. Means:

3.2.1 Definition: Let (X,d) be a metric space. For any $a,b \in X$ and any real ρ ,

the set $B_\rho(a,b)$ is defined as follows:

$$B_\rho(a,b) = \begin{cases} p \in X, d(p,x) \leq (1-\rho)d(a,x) + \rho d(b,x), \forall x \in X, \rho \in [0,1]. \\ p \in X, d(p,x) \geq (1-\rho)d(a,x) + \rho d(b,x), \forall x \in X, \rho \notin [0,1]. \end{cases}$$

Frechet [9] calls an element of $B_\rho(a,b)$ a generalized mean. As seen below

(Lemma 3.2) the elements of $B_\rho(a,b)$ divide the segment from a to b in the

ratio $\rho: (1-\rho)$. We shall refer to the elements of $B_\rho(a,b)$ as means. If $A,B \subseteq X$,

we define $B_\rho(A,B) = \cup\{ B_\rho(a,b) : a \in A, b \in B\}$. The set $B_\rho(a,b)$ may be

empty, a singleton or have more than one element as the following examples will illustrate.

3.2.2. Example: For a discrete metric space $X = \{a,b,c,d\}$, $B_\rho(a,b) = \phi$ for

$\rho \neq 0,1$.

3.2.3. Example: Let $X = \{a,b,c,d\}$ be a discrete metric space with metric d

except $d(a,b) = 2$. Then $B_\rho(a,b) = \{c,d\}$.

When the set $B_\rho(a,b)$ consists of one element p then we will write

$B_\rho(a,b) = p$ instead of $B_\rho(a,b) = \{p\}$. The following lemmas list several properties of means.

3.2.4. Lemma: $B_0(a,b) = a$, $B_1(a,b) = b$ for any $a,b \in X$.

Proof: Clearly $a \in B_0(a,b)$, follows from the definition. Next if

$p \in B_0(a,b)$, then $d(p,x) \leq d(a,x)$ for all $x \in X$. So, $0 \leq d(p,a) \leq d(a,a) = 0$ for $x = a$. This implies $p = a$.

Again $b \in B_1(a,b)$ since $d(b,x) \leq d(b,x)$, for all $x \in X$. Next if

$p \in B_1(a,b)$ then $d(p,x) \leq d(b,x)$ for all $x \in X$ and $0 \leq d(p,b) \leq d(b,b) = 0$ for $x = b$. This implies $p = b$.

3.2.5. Lemma: For any real ρ and any $a \in X$, $B_\rho(a,a) = a$.

Proof: From definition we have

$$B_\rho(a,b) = \begin{cases} p \in X, d(p,x) \leq d(a,x) \text{ for all } x \in X, \rho \in [0,1]. \\ p \in X, d(p,x) \geq d(a,x) \text{ for all } x \in X, \rho \notin [0,1]. \end{cases}$$

In each case the inequality implies $p = a$.

3.2.6. Lemma: For any $\rho \neq 0$, $a,b \in X$ and $c \in B_\rho(a,b)$ then $b \in B_{1/\rho}(a,c)$,

and conversely.

Proof: For $\rho \in (0,1)$, we have, $d(c,x) \leq (1-\rho)d(a,x) + \rho d(b,x)$ for all x if and

only if $d(b,x) \geq (1-\rho^{-1})d(a,x) + \rho^{-1}d(b,x)$.

3.2.7. Lemma: For any $\rho < 0$, and any $a, b \in X$, $c \in \beta_\rho(a, b)$ implies

$a \in B_{\rho/(\rho-1)}(c, b)$ and conversely.

Proof: $\rho < 0$ implies $0 < \rho(\rho-1)^{-1} < 1$. Now $c \in B_\rho(a, b)$ implies

$d(c, x) \geq (1 - \rho) d(a, x) + \rho d(b, x)$ for all x , if and only if

$$d(a, x) \leq \rho(\rho-1)^{-1} d(b, x) - (\rho-1)^{-1} d(c, x) = (1 - \rho(\rho-1)^{-1}) d(c, x) + \rho(\rho-1) d(b, x)$$

implies $a \in B_{\rho/(\rho-1)}(c, b)$.

3.2.8. Lemma. For any $a, b \in X$, $\rho \in \mathbb{R}$, $B_\rho(a, b) = B_{1-\rho}(b, a)$.

Proof: Follows immediately from the definition of mean.

3.2.9. Corollary : $B_{1/2}(a, b) = B_{1/2}(b, a)$.

3.2.10 Lemma: If an element $p \in B_\rho(a, b)$ then $d(a, p) = |\rho| d(a, b)$ and

$d(b, p) = |1 - \rho| d(a, b)$, for any $a, b \in X$ and for any $\rho \in \mathbb{R}$.

Proof: Case (i). $\rho \in [0, 1]$. By definition, $d(p, x) \leq (1 - \rho) d(a, x) + \rho d(b, x)$,

for all $x \in X$. So $d(p, a) \leq (1 - \rho) d(a, a) + \rho d(a, b) = \rho d(a, b)$. In a similar way ,

$d(b, p) \leq (1 - \rho) d(a, b)$. Now, $d(a, b) \leq d(a, p) + d(p, b) \leq \rho d(a, b) + (1 - \rho) d(a, b) =$

$d(a, b)$. Thus $d(a, p) + d(p, b) = d(a, b)$, $d(a, p) = \rho d(a, b)$ and $d(b, p) = (1 - \rho) d(a, b)$.

Case (ii). Let $\rho \in (1, \infty)$, then by Lemma 3.2.4. and using case (i)

$p \in B_{1/\rho}(a,b)$, $d(a,b) = \rho^{-1}d(a,p)$, i.e., $d(a,p) = \rho d(a,b)$ and $d(b,p) = (1 - \rho^{-1})$

$$d(a,p) = (\rho-1)\rho^{-1}$$

$$d(a,p) = (\rho-1)\rho^{-1}\rho d(a,b) = (\rho - 1) d(a,b). \quad \text{Thus } d(b,p) = |1 - \rho| d(a,b).$$

Case (iii). $p \in (-\infty,0)$. By Lemma 3.2.5. $a \in B_{\rho/(\rho-1)}(p,b)$ for $p \in B_{\rho}(a,b)$. By

case (i) $d(a,p) = \rho(\rho-1) d(p,b)$ and $d(b,a) = (1 - \rho(\rho-1)^{-1}) d(p,b) = -(\rho-1)^{-1}d(p,b)$.

Therefore, $d(p,b) = (1 - \rho) d(a,b)$

$$\text{Also, } d(a,p) = \rho(1-\rho)^{-1}(\rho-1) d(a,b) = -\rho d(a,b).$$

3.2.11. Lemma: For $\rho, \rho', \rho^* \in [0,1]$ ($\rho, \rho', \rho^* \notin [0,1]$ with $\rho + \rho'\rho^* - \rho\rho^* \notin [0,1]$)

and $a,b \in X$,

$$B_{\rho^*}(B_{\rho}(a,b), B_{\rho'}(a,b)) \subseteq B_{\rho + \rho'\rho^* - \rho\rho^*}(a,b).$$

Proof: Case (i). Consider first $\rho, \rho', \rho^* \in [0,1]$.

Let $p^* \in B_{\rho^*}(B_{\rho}(a,b), B_{\rho'}(a,b))$ then $p^* \in B_{\rho^*}(p,p')$ for some $p \in B_{\rho}(a,b)$ and

$p' \in B_{\rho'}(a,b)$. Therefore $d(p^*,x) \leq (1-\rho^*)d(p,x) + \rho^*(p',x)$ for all $x \in X$

$$\leq (1 - \rho^*)((1 - \rho) d(a,x) + \rho d(b,x)) + \rho^*((1-\rho') d(a,x) + \rho' d(b,x))$$

$$= (1-(\rho + \rho'\rho^* - \rho\rho^*)) d(a,x) + (\rho + \rho'\rho^* - \rho\rho^*) d(b,x)$$

As $\rho + \rho'\rho^* - \rho\rho^* \in [0,1]$, therefore $p^* \in B_{\rho + \rho'\rho^* - \rho\rho^*}(a,b)$.

Case(ii). If $\rho, \rho', \rho^* \in [0,1]$ and $\rho + \rho'\rho^* - \rho\rho^* \in [0,1]$ then for $p^* \in B_{\rho^*}(B_{\rho}(a,b),$

$$B_{\rho'}(a,b)), \quad d(p^*,x) \geq (1 - (\rho + \rho'\rho^* - \rho\rho^*)) d(a,x) + (\rho + \rho'\rho^* - \rho\rho^*) d(a,b)$$

Therefore $p^* \in B_{\rho + \rho'\rho^* - \rho\rho^*}(a,b)$.

3.2.12. Theorem: If the means are singleton on the metric space (X,d) , then

$$B_{\rho^*}(B_{\rho}(a,b), B_{\rho'}(a,b)) = B_{\rho + \rho'\rho^* - \rho\rho^*}(a,b), \text{ for any } x,y \in X \text{ and } \rho, \rho', \rho^* \in [0,1].$$

Proof: Follows from lemma 3.2.11.

3.2.13. Corollary: $B_{\rho^*}(B_0(b,a), B_{\rho'}(b,a)) = B_{\rho'\rho^*}(b,a)$.

3.2.14. Lemma: For any $a,b \in X$ and any real $\rho', \rho^* > 1$ and $\rho = 0$,

$$B_{\rho^*}(B_0(a,b), B_{\rho'}(a,b)) \subseteq B_{\rho + \rho'\rho^* - \rho\rho^*}(a,b).$$

Proof: We need to show $B_{\rho^*}(a, B_{\rho'}(a,b)) \subseteq B_{\rho'\rho^*}(a,b)$

Let $p \in B_{\rho^*}(a, B_{\rho'}(a,b))$, so $p \in B_{\rho^*}(a,q)$ for some $q \in B_{\rho'}(a,b)$

Then it follows that for all $x \in X$, $d(p,x) \geq (1 - \rho^*) d(a,x) + \rho^* d(q,x) \geq (1 - \rho^*) d(a,x)$

$$+ \rho^* \{(1-\rho') d(a,x) + \rho' d(b,x)\} = (1-\rho^*\rho') d(a,x) + \rho^*\rho' d(b,x).$$

As $\rho^*\rho' > 1$, $p \in B_{\rho^*\rho'}(a,b)$.

3.2.15. Theorem: If the means are singletons on the metric space (X,d) , then

for every pair $x,y \in X$ with $x \neq y$, then $\rho \in [0,1] \longrightarrow B_{\rho}(x,y)$ is a continuous

injection of $[0,1]$ into X .

Proof: Let $\alpha, \beta \in [0,1]$ and suppose without loss of generality $\alpha > \beta$. By

theorem 3.2.12. , $d(B_\alpha(x,y), B_\beta(x,y)) = d(B_{(\alpha-\beta)/(1-\beta)}(B_\beta(x,y), B_1(x,y)) ,$

$$B_\beta(x,y)) = (\alpha-\beta)/(1-\beta) d(B_\beta(x,y), y) = (\alpha-\beta)(1-\beta)^{-1} (1-\beta) d(x,y) = (\alpha-\beta) d(x,y).$$

3.2.16. Remarks: The above argument proves that the map $B_\alpha(x,y) \longrightarrow \alpha d(x,y)$

gives an isometry of the subspace $\{B_\alpha(x,y) : \alpha \in [0,1]\}$ of X onto a closed

interval $[0, d(x,y)]$. In particular $\{B_\alpha(x,y) : \alpha \in [0,1]\}$ is homeomorphic with $[0,1]$

if $x \neq y$ and is a singleton if $x = y$.

3.3. The mean $B_\rho(a,b)$ in a normed linear space:

Frechet [9] asked if Doss' [8] mean would give a metric characterization of algebraic betweenness in normed linear spaces. Gahler and Murphy [10] answered Frechet's question in the affirmative. Their proof of this will be the main topic of discussion in this section.

3.3.1. Lemma: If d is a translation invariant metric in a linear metric space X

then $B_\rho(a+c, b+c) = B_\rho(a,b) + c$, for any $\rho \in \mathbf{R}$ and $a, b, c \in X$.

Proof: Let $p \in B_\rho(a+c, b+c)$, then for every $x \in X$, $d(p-c, x) = d(p, x+c) \leq$

$(1-\rho) d(a+c, x+c) + \rho d(b+c, x+c) = (1-\rho) d(a, x) + \rho d(b, x)$. Therefore $p-c \in$

$B_\rho(a,b)$ that is, $p \in B_\rho(a, b) + c$. Thus $B_\rho(a + c, b + c) \subseteq B_\rho(a, b) + c$.

Again let $q + c \in B_\rho(a, b) + c$, then for any $x \in X$, $d(q + c, x) = d(q, x - c) \leq (1 - \rho) d(a, x - c) + \rho d(b, x - c) = (1 - \rho) d(a + c, x) + \rho d(b + c, x)$. Therefore $q + c \in B_\rho(a + c, b + c)$. Hence $B_\rho(a + c, b + c) \subseteq B_\rho(a, b) + c$.

3.3.2. Corollary : For any normed linear space X , $B_\rho(a + c, b + c) = B_\rho(a, b) + c$.

In the remainder of the section X will be considered as a normed linear space (n.l.s.).

3.3.3. Lemma: If $\rho \in [0,1]$, then for any $a, b \in X$, $p \in B_\rho(a,b)$, where $p = (1 - \rho)a + \rho b$.

Proof: Let $v(x)$ denote the norm for each $x \in X$, then for any $x \in X$, $d(p, x) = v(p - x) = v((1 - \rho)a + \rho b - x) = v((1 - \rho)(a - x) + \rho(b - x)) \leq (1 - \rho)v(a - x) + \rho v(b - x) = (1 - \rho) d(a - x) + \rho d(b, x)$. Thus $p \in B_\rho(a,b)$.

The above lemma fails in a linear metric space .

3.3.4. Example: Consider $X = \ell^2_{1/2}$ be the set of pairs of realnumbers, with the metric defined by:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|^{1/2} + |y_1 - y_2|^{1/2}.$$

Let $a = (1,0)$ and $b = (0,1)$. By definition $B_\rho(a,b) = \cap\{S(x, r(x)): x \in X\}$, where

$r(x) = (1 - \rho) d(a, x) + \rho d(b, x)$, $\rho \in [0, 1]$. But $B_\rho(a, b) = \emptyset$.

3.3.5. Lemma: For any $\rho \in \mathbf{R}$ and $a, b \in X$, the set $B_\rho(a, b)$ is unaltered when the norm v is replaced by μv , where $\mu > 0$.

Proof: The proof is immediate from the definition of $B_\rho(a, b)$ and $d(p, x) = v(p-x)$.

3.3.6. Lemma: For any $\rho \in [0, 1]$ and for any $a, b \in X$, $B_\rho(a, b) = p$, where $p = (1 - \rho) a + \rho b$.

Proof: If the dimension of $X \leq 1$, the above relation clearly holds. Assume that the dimension of X is at least two. Suppose there is a $p \in B_\rho(a, b)$ such that $p \neq \sigma a + \rho b$, where $\sigma = 1 - \rho$. By Corollary 3.3.2. we may assume, without loss of generality that $\sigma a + \rho b = 0$. We know by Lemma 3.3.5., the set $B_\rho(a, b)$ is unaltered if we replace v by μv , when $\mu > 0$. So we assume $v(b-a) = 1 = d(a, b)$.

Let X' be a two dimensional subspace of X containing a, b, p and coordinatized such that $a = (-\rho, 0)$, $b = (\sigma, 0)$ and $p = (0, \alpha)$ where $\alpha > 0$.

Therefore, $d(a, 0) = \rho d(a, b) = \rho$, $d(b, 0) = \sigma d(a, b) = \sigma$ and $d(p, 0) = \alpha$, where $v(0, 1) = 1$. Now $p \in B_\rho(a, b)$ implies $d(p, 0) \leq (1 - \rho) d(a, 0) + \rho d(b, 0) = \sigma \rho d(a, b)$

$+ \sigma \rho d(a, b) = 2\sigma\rho$. So $\alpha \leq 2\sigma\rho$. Either $\rho \leq 1/2$ or $\sigma \leq 1/2$ so $\rho\sigma \leq 1/2$, and $\alpha \leq 1$. For any $x, y \in X$, the connecting segment $\{(1 - \epsilon)x + \epsilon y : \epsilon \in [0, 1]\}$ is denoted by $\langle x, y \rangle$. Now $0 \in B_\rho(a, b)$ and $p \in B_\rho(a, b)$ so, by lemma 3.2.11., $\langle 0, p \rangle \subseteq B_\rho(a, b)$. Let (σ, α') be any point of $\langle (\sigma, 0), (\sigma, \alpha) \rangle$.

Therefore $d((0, 0), (\sigma, \alpha')) = d((0, -\alpha'), (\sigma, 0)) = d(p', b)$. Further, since $p' \in B_\rho(a, b)$ and, by lemma 3.2.10, $d((0, 0), (\sigma, \alpha')) = \sigma d(a, b) = \sigma$. Thus any point of $\langle (\sigma, 0), (\sigma, \alpha) \rangle$ has norm σ . Therefore it follows that $s_1 = (\sigma, \sigma)$ has norm $\geq \sigma$.

Let $q = (0, \sigma)$ and S denote the line through b and s_1 . S_1 is the line through q and the point s'_1 of $\langle 0, s_1 \rangle$ such that $v(s'_1) = \sigma$. The point intersection of S_1 and S is $s^*_1 = (\sigma, \sigma_1)$. Now $v(q) = \sigma$, also $v(s'_1) = \sigma$, by construction. We claim all points of $\langle q, s'_1 \rangle \setminus \{q, s'_1\}$ have norm less than σ . To see this let us assume that there is one point on $\langle q, s'_1 \rangle \setminus \{q, s'_1\}$ which has norm $\geq \sigma$. Now any point $r = (\rho_1, \rho_2)$ of $[0, \infty) \setminus \langle q, s'_1 \rangle \setminus \{q, s'_1\}$ can be written as $(\rho_1, \rho_2) = \mu(\rho'_1, \rho'_2)$ where μ is a suitable positive number and (ρ'_1, ρ'_2) is a point on $\langle q, s'_1 \rangle \setminus \{q, s'_1\}$.

Let us assume $v(\rho'_1, \rho'_2) = \sigma$, therefore $v(\rho_1, \rho_2) = v \mu(\rho'_1, \rho'_2) = \mu\sigma$.

For some suitable λ , $(\rho'_1, \rho'_2) = (1-\lambda)(0, \sigma) + \lambda(\sigma, \sigma_1) = (\lambda\sigma, (1-\lambda)\sigma + \lambda\sigma_1)$.

$$\rho'_1 = \lambda\sigma, \quad \rho'_2 = (1-\lambda)\sigma + \lambda\sigma_1 = (1-\rho'_1\sigma^{-1})\sigma + \rho'_1\sigma^{-1}\sigma_1 = (\sigma - \rho'_1) + \rho'_1\sigma^{-1}\sigma_1$$

$$= \sigma - \rho'_1(1 - \sigma_1\sigma^{-1}) = \sigma - \rho'_1((\sigma - \sigma_1)\sigma^{-1}). \text{ Now } \rho_1 = \mu\rho'_1 \text{ and } \rho_2 = \mu\rho'_2 = \mu(\sigma - \rho'_1(\sigma - \sigma_1)\sigma^{-1})$$

implies $\mu\sigma = \rho_2 + \mu\rho'_1(\sigma - \sigma_1)\sigma^{-1} = \rho_2 + \rho_1(\sigma - \sigma_1)\sigma^{-1}$. Therefore $v(\rho_1, \rho_2) = \rho_2 +$

$\rho_1(\sigma - \sigma_1)\sigma^{-1}$. Now if we apply the definition $p \in B_\rho(a, b)$ with $x = (\sigma, 1)$ and then

using the translation invariance of d , we get, $d((0, -\alpha), (\sigma, 1)) \leq (1-\rho)d((- \rho, 0), (\sigma, 1))$

+ $\rho d((\sigma, 0), (\sigma, 1))$, i.e., $d((0, 0), (\sigma, 1+\alpha)) \leq \sigma d((- \rho, \alpha), (\sigma, 1+\alpha)) + \rho d((\sigma, \alpha), (\sigma, 1+\alpha))$.

Then, $v(\sigma, 1+\alpha) \leq \sigma d((- \rho + \rho, \alpha - \alpha), (\sigma + \rho, 1 + \alpha - \alpha)) + \rho d((\sigma - \sigma, \alpha - \alpha), (\sigma - \sigma, 1 + \alpha - \alpha))$

$= \sigma d((0, 0), (1, 1)) + \sigma d((0, 0), (0, 1)) = \sigma v(1, 1) + \rho$. Therefore, $(\sigma - \sigma_1)\sigma^{-1}\sigma + (1+\alpha) \leq$

$\sigma(\sigma - \sigma_1)\sigma^{-1} + \sigma + \rho$ yielding $\alpha \leq 0$ which is impossible. Thus any points of

$\langle q, s_1 \rangle \setminus \{q, s_1\}$ must have norm less than σ .

Again let $s_2 = (\sigma, 1+\alpha)$ and s'_2 be points of $\langle 0, s_2 \rangle$ such that $v(s'_2) = \sigma$.

S_2 is the line containing q and s'_2 and $s^*_2 = (\sigma, \sigma_2)$ is the intersection of S_2 and S .

An argument analogous to that above shows, by using $x = (\sigma, 1 + \alpha + \rho)$ and

$p = (0, -\alpha)$ that the points of $\langle q, s'_2 \rangle \setminus \{q, s'_2\}$ can never have norm σ . Now s_2 has

norm $(\sigma - \sigma_2)\sigma^{-1}\sigma + 1 + \alpha$. Now using the defining relation $p \in B_\rho(a, b)$, $x = (\sigma, 1)$

$(\sigma - \sigma_2) \sigma^{-1} \sigma + 1 + \alpha \leq \sigma(\sigma - \sigma_1) \sigma^{-1} + \rho + \sigma$ implies $\alpha \leq \sigma_2 - \sigma_1$. By repeated application of

this procedure we obtain $\{s_i\}$, $\{s'_i\}$ and $\{s^*_i\}$, where $s_i = (\sigma, \xi_i)$ and $\xi_i = \xi_{i-1} + \rho + \alpha$,

$i > 1$, s'_i is the point of $\langle 0, s_i \rangle$ with norm σ , and $s^*_i = (\sigma, \sigma_i)$ is the intersection of S_i

(line through q and s'_i) and S . By application of the defining relation

$p \in B_\rho(a, b)$, as above, with co-ordinates $x = (\sigma, \xi_i + \rho)$ we can show that all points

on $\langle q, s'_i \rangle \setminus \{q, s'_i\}$ have norm $< \sigma$. Another application of the defining inequality

for $p \in B_\rho(a, b)$ with $x = (\sigma, \xi_i - \alpha)$ shows that $\sigma_i \geq \sigma_{i-1} + \sigma$. Choose j such that $\sigma_j > 2\sigma$.

Since $-b, q$ both have norm σ , the norm of s_j must be greater than σ , which is a

contradiction. Thus $p = \sigma a + \rho b$ and $B_\rho(a, b) = p$.

3.3.7. Lemma: $B_\rho(a, b) = p$, for $\rho \in (1, \infty)$ and for all $a, b \in X$, where

$$p = (1 - \rho)a + \rho b.$$

Proof: Here $p = (1 - \rho)a + \rho b$ implies $b = (1 - \rho^{-1})a + \rho^{-1}p$. Hence for any $x \in X$,

by lemma 3.3.3., we have, $d(b, x) \leq (1 - \rho^{-1})d(a, x) + \rho^{-1}d(p, x)$, so, $d(p, x) \geq (1 - \rho)$

$d(a, x) + \rho d(b, x)$. Therefore $p \in B_\rho(a, b)$.

Next let p and $p' \in B_\rho(a, b)$, then by lemma 3.2.4,

$b \in B_{\rho^{-1}}(a, \rho) \cap B_{\rho^{-1}}(a, \rho')$. Since $\rho^{-1} \in [0, 1]$, by lemma 3.3.3. and 3.3.6.,

$b = (1 - \rho^{-1})a + \rho^{-1}b$. Therefore $\rho = \rho'$. Hence $B_{\rho}(a, b) = p$, where $p = (1 - \rho)a + \rho b$.

3.3.8. Lemma: $B_{\rho}(a, b) = p$, for $\rho \in (-\alpha, 0)$ and for any $a, b \in X$, where

$$p = (1 - \rho)a + \rho b.$$

Proof: Here $p = (1 - \rho)a + \rho b$ for $-\alpha < \rho < 0$. Then $a = (1 - (\rho - 1)^{-1}\rho)p + (\rho - 1)^{-1}\rho b$. Now

$$0 < r = \rho(1 - \rho)^{-1} < 1. \text{ Therefore, } d(a, x) = v(a - x) = v\{(1 - r)p + rb - x\}$$

$$= v\{(1 - r)(p - x) + r(b - x)\} \leq (1 - r)v(p - x) + v r(b - x). \text{ So, } d(p, x) \geq (1 - \rho)d(a, x) + \rho d(b, x),$$

proving $p \in B_{\rho}(a, b)$. To prove uniqueness let us consider

$$p, p' \in B_{\rho}(a, b). \text{ Then it follows by lemma 3.2.7., } a \in B_{(\rho/(\rho-1))}(p, b) \cap$$

$$B_{(\rho/(\rho-1))}(p', b). \text{ Hence by lemma 3.3.3. and 3.3.6., } a = (1 - r)p + rb = (1 - r)p'$$

$$+ rb, \text{ which implies } p = p'. \text{ Thus } B_{\rho}(a, b) = p, \text{ where } p = (1 - \rho)a + \rho b.$$

The results of this section can be summarized in the following :

3.3.9. Theorem: In a normed linear space X , $B_{\rho}(a, b) = p$, for any real $\rho \in \mathbf{R}$ and

for any $a, b \in X$, where $p = (1 - \rho)a + \rho b$.

Proof: The proof of the theorem follows from the lemmas 3.3.3., 3.3.5., 3.3.7., and 3.3.8.

3.4. Metric Space with Property GM.

In the rest of the section X is considered to be a metric space with property GM, which was introduced by Gähler and Murphy as a sufficient condition for means to be singletons.

Property GM:

(1) $B_{1/2}(a,b)$ and $B_2(a,b)$ are singletons.

(2) The subset $\cup\{B_\rho(a,b): \rho \in [0,1]\}$ is complete, for all $a,b \in X$.

3.4.1. Lemma: $B_{-1}(a,b)$ is singleton, for all $a,b \in X$.

Proof: Since $B_\rho(a,b) = B_{1-\rho}(b,a)$, by lemma 3.2.6.,

$B_{-1}(b,a) = B_2(b,a)$ is a singleton.

3.4.2. Lemma: If ρ is a dyadic rational of the form $m/2^n$ such that $\rho \in [0,1]$

where m and n are natural numbers, then $B_\rho(a,b)$ contains at least one element.

Proof: By lemma 3.2.9., we have $B_{1/2}(a, B_{1/2}(a,b)) \subseteq B_{1/4}(a,b)$. By property

GM(1), $B_{1/2}(a, B_{1/2}(a,b))$ is a singleton. Therefore $B_{1/4}(a,b)$ is nonempty. Again

by lemma 3.2.9., $B_{1/2}(B_0(a,b), B_{1/4}(a,b)) \subseteq B_{1/8}(a,b)$, and $B_{1/8}(a,b)$

$= B_{1/2^3}(a,b)$ is nonempty. Similarly, $B_{1/2}(B_{1/2}(a,b), B_1(a,b)) \subseteq B_{3/4}(a,b)$, by

lemma 3.2.9. and $B_{3/4}(a,b) = B_{3/2^2}(a,b)$ is nonempty. Continuing in the same

manner we can show that $B_{m/2^n}(a,b)$ is nonempty.

3.4.3. Lemma: For any $\rho \in \mathbf{R}$ and $a, b \in X$, $B_\rho(a, b)$ is closed.

Proof: Let $\{p_j\}$ be a sequence which converges to p such that $p_j \in B_\rho(a, b)$.

Therefore, for every $x \in X$, $d(x, p_j) \leq (1-\rho)d(a, x) + \rho d(x, b)$. As $p_j \rightarrow p$, then

$$d(x, p) \leq (1-\rho)d(a, x) + \rho d(x, b) \quad \text{and} \quad p \in B_\rho(a, b).$$

3.4.4. Lemma: If ρ is a non dyadic rational such that $\rho \in [0, 1]$, then $B_\rho(a, b)$ is nonempty.

Proof: When ρ is not a dyadic rational, there exists a monotone decreasing

sequence of dyadic rationals $\{p_i\}$, $i \in \mathbf{N}$, converging to ρ and by lemma 3.4.2.,

there is $p_i \in B_{\rho_i}(a, b)$. Now by lemma 3.2.8., we get $d(p_i, p_j) = |p_i - p_j| d(a, b)$ for any

i, j . So the sequence $\{p_i\}$ is a Cauchy sequence and must converge to

$p \in B_\rho(a, b)$ and, consequently, $B_\rho(a, b)$ is nonempty.

3.4.5. Lemma: For $\rho \in [0, 1]$ and $a, b \in X$, $B_\rho(a, b)$ is a singleton.

Proof: If $\rho = 1/2$ then $B_\rho(a, b)$ is a singleton, by GM(1). If $\rho \neq 1/2$, we consider

$\rho < 1/2$. Let $a' \in B_\rho(a, b)$, $b' \in B_{1-\rho}(a, b)$ and $p' \in B_{1/2}(a, b)$. By lemma 3.2.9., we get

$$B_{1/2}(a', b') \subseteq B_{(\rho + (1-\rho)/2 - \rho/2)}(a, b) = B_{1/2}(a, b) = p'. \quad \text{So} \quad p' = B_{1/2}(a', b') = B_{1/2}(b', a'),$$

then by lemma 3.4.1., $a' = B_{-1}(p', b')$. Hence $B_p(a, b)$ must be singleton. The case $p > 1/2$ follows by symmetry.

3.4.6. Lemma: For any natural number m , $B_{2^m}(a, b) \neq \emptyset$.

Proof: By Lemma 3.2.14., $B_2(B_0(a, b), B_2(a, b)) \subseteq B_4(a, b)$ and by GM(1),

$B_2(a, B_2(a, b))$ is a singleton. Therefore $B_{2^2}(a, b) \neq \emptyset$. Similarly by lemma

3.2.14., $B_2(B_0(a, b), B_4(a, b)) \subseteq B_8(a, b)$ and $B_{2^3}(a, b) \neq \emptyset$. In a similar way it can

be shown that $B_{2^m}(a, b) \neq \emptyset$.

3.4.7. Lemma: If $\rho \notin [0,1]$, $B_\rho(a,b)$ is a singleton, for every $a,b \in X$.

Proof: It suffices to consider the case $\rho \in (1,\infty)$, since $B_\rho(a,b) = B_{1-\rho}(a,b)$.

Let m be a natural number such that $\rho < 2^m$. Then by lemma 3.4.6., $\rho \in$

$B_{2^m}(a,b)$. Let $\rho' = (\rho-1)/(2^m-1)$, then by lemma 3.4.5., $B_{\rho'}(b,p) = p'$ and

$B_{1/\rho}(a,p') = b'$ are singletons. Thus since $b \in B_{1/2^m}(a,p)$, for any $x \in X$,

$$d(b',x) \leq (1-\rho^{-1})d(a,x) + \rho^{-1}d(p',x)$$

$$\leq (1-\rho^{-1})d(a,x) + \rho^{-1}((1-\rho')d(b,x) + \rho'd(p,x))$$

$$\leq (1-\rho^{-1})d(a,x) + (1-\rho')\rho^{-1}\{(1-2^{-m})d(a,x) + 2^{-m}d(p,x)\} + \rho'\rho^{-1}d(p,x)$$

$$= (1-\rho^{-1})d(a,x) + (1-\rho')\rho^{-1}(1-2^{-m})d(a,x) + (1-\rho')\rho^{-1}2^{-m}d(p,x) + \rho'\rho^{-1}d(p,x)$$

$$= d(a,x)\{(\rho-1)\rho^{-1} + (1-\rho')\rho^{-1}(2^m-1)2^{-m}\} + d(p,x)\{(1-\rho')\rho^{-1}2^{-m} + \rho'\rho^{-1}\}$$

$$= (1-2^{-m})d(a,x) + 2^{-m}d(p,x).$$

Hence $b' \in B_{2^m}(a,p)$, and since $B_{2^m}(a,p)$ is a singleton, therefore $b = b'$.

Thus $b \in B_{1/\rho}(a,p')$ which implies $p' \in B_\rho(a,b)$. Therefore the set $B_\rho(a,b)$

is nonempty.

To show that $B_\rho(a,b)$ is a singleton, let $r = B_{1/2}(a,b)$ and $q \in B_{1-\rho}(a,b)$

$= B_\rho(b,a)$ and $p^* \in B_\rho(a,b)$. Thus for any $x \in X$, $d(q,x) \geq \rho d(a,x) + (1-\rho)d(b,x)$

and $d(p^*,x) \geq (1-\rho)d(a,x) + \rho d(b,x)$. Therefore $d(q,x) + d(p^*,x) \geq d(a,x) + d(b,x)$.

Since $r \in B_{1/2}(a,b)$, therefore for any $x \in X$, $d(r,x) \leq (1/2) d(a, x) + (1/2) d(b,x) \leq (1/2)\{d(q,x) + d(p^*,x)\}$. Therefore $r \in B_{1/2}(q,p^*)$, so $p^* \in B_2(q,r)$ which is a singleton.

3.4.8. Theorem: Let X be a metric space with property GM. Then, for any real ρ and $a,b \in X$, $B_\rho(a,b)$ is a singleton.

Proof: The proof of this theorem follows from the above lemmas.

3.5. Linearization of X: In this section using a construction given by Frechet [9], a linear structure is defined on a metric space X which satisfies property GM and some conditions given below

First scalar multiplication is defined. In order to do this choose a fixed point in X which will be the origin and designated as "O". For any $\alpha \in \mathbf{R}$ and any $x \in X$, there exist one and only one point $\alpha x \in X$ defined by $\alpha x = B_\alpha(O,x)$.

Secondly addition of $a,b \in X$ is defined as the uniquely determined point $2m_{a,b}$ where $m_{a,b} = B_{1/2}(a,b)$ and is denoted $a + b$.

In order to show that X together with these operations is a linear space, a sequence of lemmas will be proved.

3.5.1. Lemma: $\alpha O = O$ and $1a = a$, for all $\alpha \in \mathbf{R}$ and for all $a \in X$.

Proof: By lemma 3.2.5., $\alpha O = B_\alpha(O,O) = O$. Also by lemma 3.2.4., $1a = B_1(O, a) = a$.

3.5.2. Lemma: If $a \in X$ and $\alpha \neq 0 \in \mathbf{R}$, then $\alpha a = 0$ implies $a = 0$.

Proof: $0 = \alpha a = B_\alpha(O, a)$. By lemma 3.2.10., $d(O, \alpha a) = \alpha d(O, a)$ implies $d(O, 0) = \alpha d(O, a)$, i.e., $\alpha d(O, a) = 0$. But $\alpha \neq 0$ implies $d(O, a) = 0$. Therefore $a = 0$.

3.5.3. Lemma: For every $a \in X$, there exists $b \in X$ such that $a + b = 0$.

Proof: If $b = B_{-1}(O, a)$ then, for all $x \in X$, $d(b, x) \geq 2d(O, x) - d(a, x)$. Thus, $d(O, x) \leq (1/2)[d(a, x) + d(b, x)]$ and $O \in B_{1/2}(a, b)$. Therefore $a + b = 0$.

3.5.4. Lemma: If $a + b = 0$, then $b = B_{-1}(O, a)$.

Proof: If $a + b = 0$, then $2B_{1/2}(a, b) = 0$. Therefore $B_{1/2}(a, b) = 0$. Thus $O \in B_{1/2}(a, b)$. By lemma 3.2.6., $b \in B_2(a, 0) = B_{-2}(O, a) = B_{-1}(O, a)$.

3.5.5. Remarks: From the above lemmas it follows that for every $a \in X$ there exists a unique additive inverse $b (= -a)$ such that $b = -a = B_{-1}(O, a)$ and $a + b = 0$.

3.5.6. Lemma: For all $a \in X$, $d(O, a) = d(O, -a)$.

Proof: By remarks 3.5.5., $B_{-1}(O, a) = -a$. Therefore by lemma 3.2.10., we get $d(O, a) = |-1|d(O, a) = d(O, a)$.

3.5.7. Lemma: For all $x \in X$, $\alpha \neq 0$, $\alpha\{(1/\alpha)b\} = b$.

Proof: By definition $(1/\alpha)b = B_{1/\alpha}(O, b)$. Therefore by lemma 3.2.6., we get

$$b = B_{\alpha}(O, \alpha^{-1}b) = \alpha\{(1/\alpha)b\}.$$

3.5.8. Lemma: For all $\alpha \in \mathbf{R}$ and for all $a \in X$, $(-\alpha)a = \alpha(-a) = -(\alpha a)$.

Proof: For $\alpha = 0$ or 1 , the result is immediate.

For $\alpha > 1$, let $p \in B_{\alpha}(O, -a)$, therefore for all $x \in X$, $d(p, x) \geq (1 - \alpha) d(O, x) +$

$\alpha d(-a, x)$. But $-a \in B_{-1}(O, a)$, so, $d(-a, x) \geq 2d(O, x) - d(O, a)$. Thus, $d(p, x) \geq (1 - \alpha)$

$d(O, x) + \alpha[2d(O, x) - d(O, a)] = (1 + \alpha)d(O, x) - \alpha d(O, a)$. Therefore $p \in B_{-\alpha}(O, a)$

and $\alpha(-a) = (-\alpha)a$.

Next we prove $(-\alpha)a = -(\alpha a)$. Let $q_1 \in B_{-\alpha}(O, a)$ and $q_2 \in B_{\alpha}(O, a)$,

therefore for all $x \in X$, $d(q_1, x) \geq (1 + \alpha) d(O, x) - \alpha d(a, x)$ and $d(q_2, x) \geq (1 - \alpha)$

$d(O, x) + \alpha d(a, x)$. Therefore $d(q_1, x) + d(q_2, x) \geq 2d(O, x)$, which implies

$d(O, x) \leq (1/2)\{d(q_1, x) + d(q_2, x)\}$. Therefore $O = B_{1/2}(B_{-\alpha}(O, a), B_{\alpha}(O, a))$ which

implies $(-\alpha)a + \alpha a = O$. Thus $(-\alpha)a = -(\alpha a)$, by lemma 3.5.4.

If $0 < \alpha < 1$, then $(1/\alpha) > 1$ and $(1/\alpha)\{-(\alpha a)\} = -\{(1/\alpha)(\alpha a)\} = -a$, by

above. Thus $\alpha\{(1/\alpha)\{-(\alpha a)\}\} = \alpha(-a)$ implies $-(\alpha a) = \alpha(-a)$, by lemma 3.5.7.

Again, $(1/\alpha)\{(-\alpha)a\} = a$. Therefore $(1/\alpha)\{(-\alpha)a\} = a$ which implies $\alpha\{(1/\alpha)(-\alpha)a\} =$

$\alpha(-a)$. Hence $(-\alpha)a = -(\alpha a)$.

Finally let $\alpha < 0$, put $\beta = -\alpha > 0$. Then $\alpha a = \{-(-\alpha)\}a = (-\beta)a = -(\beta a) =$

$-((-α)a)$ implies $-(αa) = (-α)a$. Again $α(-α) = \{-(-α)\}(-a) = (-β)(-a) = -\{β(-a)\} =$
 $-\{βa\} = βa = (-α)a$.

3.5.9. Lemma: For every $α, β ∈ ℝ$ and $a ∈ X$, $α(βa) = (αβ)a$.

Proof: The case for $α = 0$ or $β = 0$ and $α = 1$ or $β = 1$ are immediate. To complete the proof four cases are considered.

Assume $0 < α, β < 1$. By lemma 3.2.11., $B_α(O, B_β(O, a)) ⊆ B_{αβ}(O, a)$.

Now each of the sets $B_α(O, B_β(O, a))$ and $B_{αβ}(O, a)$ are singletons. Therefore

$B_α(O, B_β(O, a)) = B_{αβ}(O, a)$; consequently $α(βa) = (αβ)a$.

Assume $α > 1, β > 1$. Then by lemma 3.2.14., $B_α(O, B_β(O, a)) ⊆ B_{αβ}(O, a)$

and again by the same argument $α(βa) = (αβ)a$.

Assume $0 < α < 1, β > 1$ or $0 < β < 1, α > 1$. It is sufficient to consider

$0 < α < 1, β > 1$. Then two subcases may arise $αβ > 1$ or $αβ < 1$. In the first case, $αβ > 1, β^{-1} < 1, α^{-1} < 1$. By lemma 3.5.7., $(1/β)((1/α)[αβ]a) = (1/β)(α^{-1}αβ)a = (1/β)(βa) = a$. Now multiplying both sides by $β$, then by $α$, $(αβ)a = α(βa)$.

Now consider $αβ < 1, (1/α)((1/β)[αβ]a) = (1/α)(β^{-1}αβ)a = (1/α)(αa) = a$, by lemma 3.5.7. Now multiplying both sides by $α$, then by $β$,

we obtain $(\alpha\beta)a = \beta(\alpha a)$.

Assume $\alpha < 0, \beta < 0$ or $\alpha > 0, \beta < 0$. It suffices to prove one of these cases, because the other case can be proved in a similar way.

Let $\alpha < 0, \beta < 0$, i.e., $-\alpha > 0$ and $-\alpha\beta < 0$. Therefore $[-\alpha\beta]a = [(-\alpha)\beta]a = (-\alpha)(\beta a)$ implies $-[\alpha\beta]a = -\alpha(\beta a)$ by lemma 3.5.8., and $(\alpha\beta)a = \alpha(\beta a)$.

3.5.10. Lemma: $a + b = b + a$, for all $a, b \in X$.

Proof: $a + b = 2B_{1/2}(a, b) = 2B_{1/2}(b, a) = b + a$.

3.5.11. Lemma: $a + O = a$, for all $a \in X$.

Proof: $m_{a,0} = B_{1/2}(a, O) B_{1/2}(O, a)$. By lemma 3.2.6., $a = B_2(O, m_{a,0}) = 2m_{a,0} = a + O$.

The main results of the lemmas in this section thus far can be summarized as follows :

3.5.12. Summary : Let X be a metric space with property GM . Given $O \in X$, two operations addition and scalar multiplication over \mathbf{R} have been defined on X which satisfy the following:

- (1) Addition is commutative.
- (2) $a + O = O$, for all $a \in X$.
- (3) To every "a" there exists a "b"(= -a) such that $a + b = O$.
- (4) $1a = a$.
- (5) Multiplication by scalars is associative, i.e., $\alpha(\beta a) = (\alpha\beta)a$.

We will have a linear structure in X provided the associative property of addition and the two distributive properties hold. In the remainder of this section the following notations will be used.

A: Addition is associative.

D₁: Multiplication is distributive with respect to scalar addition ,i.e., $(\alpha + \beta)a = \alpha a + \beta a$.

D₂: Multiplication by scalars is distributive with respect to vector addition, i.e. , $\alpha(a + b) = \alpha a + \beta a$.

The properties A,D₁,D₂ are not independent. As seen below, D₁ and D₂ follow from A. Thus scalar multiplication and addition generate a linear structure on X provided X has the property A.

3.5.13. Lemma: If X has property A, then for any $a, b \in X$, $-(a + b) = -a - b$.

Proof: By associativity and commutativity: $a + b + (-a - b) = (a - a) + (b - b) = 0$
Therefore $-(a + b) = -a - b$.

3.5.14. Lemma: If X has property A, then for any $a \in X$ and $\alpha, \beta \in [-1, 1]$,

$$(\alpha + \beta)a = \alpha a + \beta a.$$

Proof: First consider $\alpha, \beta > 0$. Therefore for any $x \in X$, $d(m_{\alpha a, \beta a}, x) \leq$

$$(1/2) d(\alpha a, x) + (1/2) d(\beta a, x) \leq (1/2) (1 - \alpha) d(0, x) + \alpha/2 d(a, x) + ((1 - \beta)/2) d(0, x) +$$

$$(\beta/2) d(a, x) = (1 - (\alpha + \beta)/2) d(0, x) + ((\alpha + \beta)/2) d(a, x). \text{ Therefore } m_{\alpha a, \beta a} = B_{(\alpha + \beta)}$$

$\beta)/2$ (O,a). Thus $\alpha a + \beta a = 2m_{\alpha a, \beta a} = 2B_{(\alpha + \beta)/2}(O, a) = 2((\alpha + \beta)/2)a = (\alpha + \beta)a$.

Next consider $\alpha < 0$, $\beta < 0$. Let us put $\alpha' = -\alpha > 0$ and $\beta' = -\beta > 0$. Now $(\alpha' + \beta')a = \alpha'a + \beta'a = (-\alpha)a + (-\beta)a$. Therefore, $(-\alpha - \beta)a = -(\alpha a) - (\beta a) = -(\alpha a + \beta a)$, by lemma 3.5.11., which implies, $-(\alpha + \beta)a = -(\alpha a + \beta a)$, i.e., $(\alpha + \beta)a = (\alpha a + \beta a)$.

Now let $\alpha > 0$, $\beta < 0$, and $|\beta| < \alpha$. $\alpha a + \beta a = ((\alpha + \beta) - \beta)a + \beta a = ((\alpha + \beta)a - \beta a) + \beta a = (\alpha + \beta)a + (-\beta a + \beta a) = (\alpha + \beta)a$.

If $|\beta| > \alpha$, then also $\alpha a + \beta a = (\alpha + \beta)a$.

3.5.15. Lemma: For every $n \in \mathbf{N}$, $na = a + a + a + \cdots + a$.

Proof: $n = 1$ is trivial.

$n = 2$ is also trivial, as it follows immediately from the definition, for,

$a + a = 2m_{a,a}$. But $m_{a,a} = B_{1/2}(a,a) = a$, implies $a + a = 2a$.

Assume the result is true for n . We need to prove the result for $n + 1$.

Let $p = m_{na,a}$. By definition $na \in B_n(O,a)$ implies $a \in B_{1/n}(O,na)$.

Therefore for any $x \in X$, $d(p,x) \leq (1/2)d(na,x) + (1/2)d(a,x)$

$\leq (1/2)d(na,x) + (1/2) \{ (1 - (1/n))d(O,x) + (1/n)d(na,x) \}$

$= ((n-1)/(2n)) d(O,x) + ((n+1)/2n) d(na,x)$.

Therefore $m_{na,a} = B_{(n+1)/2n}(O,na) = ((n+1)/2n)na = ((n+1)/2)a$

So, $2m_{na,a} = (n+1)a$.

3.5.16. Lemma: For any $n \in \mathbf{Z}$, and any $a, b \in X$, where X has property A, then $n(a + b) = na + nb$.

Proof: The case $n = 0, 1$ are trivial. Let n be a positive integer, then by lemma 3.5.15, associativity and commutativity,

$$\begin{aligned} n(a+b) &= (a+b) + (a+b) + \cdots + (a+b) \\ &= \cdots \cdots \cdots \\ &= (a+a+\cdots+a) + (b+b+\cdots+b) \\ &= na + nb. \end{aligned}$$

Let n be a negative integer, then for $m = -n$, $m(a + b) = ma + mb = (-n)a + (-n)b = -(na) - (nb) = -(na+nb)$ by lemma 3.5.9. Therefore $-n(a + b) = -(na+nb)$ and $n(a + b) = (na+nb)$.

3.5.17. Lemma: If $q \in \mathbf{Q}$, and for any $a, b \in X$ (with property A), then $q(a + b) = qa + qb$.

Proof: If $q \in \mathbf{Q}$, then $q = (n/m)$ where $n \in \mathbf{Z}$ and $m > 0$. Now, $m(m^{-1}a + m^{-1}b) = mm^{-1}a + mm^{-1}b = a+b$ implies $m^{-1}a + m^{-1}b = m^{-1}(a + b)$. Therefore $q(a+b) = (n/m)(a+b) = n(m^{-1}(a+b)) = n(m^{-1}a+m^{-1}b) = (n/m)a + (n/m)b$.

3.5.18. Lemma: If $\alpha \in \mathbf{R}$ and $q \in \mathbf{Q}$ such that $(\alpha/q) \in [-1, 1]$, and for any $a \in X$ (having property A), $(\alpha + q)a = \alpha a + qa$.

Proof: By lemma 3.5.14., $((\alpha/q)+1)a = (\alpha/q)a + a$. Therefore $q((\alpha/q)+1)a = q((\alpha/q)a + a)$ implies $(\alpha + q)a = \alpha a + qa$.

3.5.19. Lemma: If $\alpha \in [1, -1]$ and $a, b \in X$ (having property A) then

$$\alpha(a + b) = \alpha a + \alpha b.$$

Proof: By lemma 3.5.17., if α is rational then $\alpha(a + b) = \alpha a + \alpha b$.

If α is an irrational, let $\{\alpha_i\}$, $i \in \mathbf{N}$, be a monotone decreasing sequence of rationals which converges to α , and by Lemma 3.5.14, $\alpha(a + b) = [(\alpha - \alpha_i) + \alpha_i]$

$$(a+b) = (\alpha - \alpha_i)(a + b) + \alpha_i(a+b) = (\alpha - \alpha_i)(a+b) + (\alpha_i - \alpha)b + (\alpha_i - \alpha)a + \alpha a + \alpha b.$$

Noting that, for arbitrary $c, x \in X$ and $\gamma \in [0, 1]$, then the relation $d(\gamma c, x) \leq$

$(1 - \gamma)d(o, x) + \gamma d(c, x)$ holds, we observe that the sequences $\{(\alpha - \alpha_i)(a+b)\}$,

$\{(\alpha_i - \alpha)a\}$ and $\{(\alpha_i - \alpha)b\}$, $i \in \mathbf{N}$ converge to zero. Therefore $\alpha(a + b) = \alpha a + \alpha b$.

3.5.20. Lemma: If $\alpha \notin [-1, 1]$ and $a, b \in X$, then $\alpha(a+b) = \alpha a + \alpha b$.

Proof: Here $\alpha^{-1} \in [-1, 1]$. Therefore $\alpha^{-1}(\alpha a + \alpha b) = \alpha^{-1}(\alpha a) + \alpha^{-1}(\alpha b) = a + b$

implies $\alpha a + \alpha b = \alpha(a+b)$.

3.5.21. Lemma: Let X be a metric space with property A, then $\alpha(a+b) = \alpha a + \alpha b$,

for any $a, b \in X$ and any $\alpha \in \mathbf{R}$.

Proof: The proof follows immediately from the lemmas 3.5.19 and 3.5.20.

3.5.22. Lemma: For any $a \in X$ and any $\alpha, \beta \in \mathbf{R}$, then $(\alpha + \beta)a = \alpha a + \beta a$, where X is a metric space with property A.

Proof: If $\alpha, \beta \in [-1, 1]$, then by lemma 3.5.14., $(\alpha + \beta)a = \alpha a + \beta a$.

Now consider $\alpha > 1, \beta \in [-1, 1]$, then $-1 < (\beta/\alpha) < 1$. Therefore $(\alpha/\beta)a + a = (\alpha/\beta)a + (\alpha/\beta)\{(\beta/\alpha)a\} = (\alpha/\beta)(a + (\beta/\alpha)a) = (\alpha/\beta)\{(1+\beta/\alpha)a\} = ((\alpha/\beta)+1)a$.

Therefore $\alpha a + \beta a = \beta[(\alpha/\beta)a + a] = \beta[(\alpha/\beta)+1]a = (\alpha+\beta)a$.

Next consider $\alpha, \beta > 1$. Therefore $1/\alpha\beta < 1$. It follows that

$(1/\alpha\beta)(\alpha a + \beta a) = (1/\alpha\beta)\alpha a + (1/\alpha\beta)\beta a = (1/\beta)a + (1/\alpha)a = ((1/\alpha) + (1/\beta))a$ implies

$\alpha a + \beta a = (\alpha+\beta)a$.

Finally, the cases $\alpha < -1$ and $\beta \in [-1, 1]$ and $\alpha, \beta < -1$ can be proved in a similar way.

3.5.23. Theorem: For the metric space (X, d) having the property GM, the addition in X is associative if and only if the consistent mid point property holds i.e. for any arbitrary $a, b, c, d \in X$,

$$m_{a,b} ; m_{c,d} = m_{a,c} ; m_{b,d}$$

Proof: Assuming addition is associative in X , then scalar multiplication and addition generate a linear structure in X , since D_1, D_2 holds.

Therefore $(1/2)[(a+b)/2 + (c+d)/2] = (1/2)[(a+c)/2 + (b+d)/2]$

Thus $m_{a,b} ; m_{c,d} = m_{a,c} ; m_{b,d}$.

If the consistent midpoint property holds then $m_{m} ; m =$

$m_{2e,o} ; m_{2f,o}$ implies $m_{(2e+2f)/2 ; o} = m_{e,f}$. Therefore $2e+2f = 2(e+f)$.

Thus $m_{2a,2b} ; m_{o,2c} = m_{a,o} ; m_{2b,2c}$ and

$$m_{(2a+2b)/2, (o+2c)/2} = m_{(2a+o)/2, (2b+2c)/2}$$

Hence $(a + b) + c = a + (b + c)$.

The results of this section can be summarized in the following linearization theorem:

3.5.24. Theorem : Let (X,d) be a metric space with the property GM:

Property GM:

(1) $B_{1/2}(a,b)$ and $B_2(a,b)$ are singleton

(2) $\cup\{ B_\rho(a,b) : \rho \in [0,1] \}$ is complete for all $a,b \in X$.

Then a linear structure is generated in X iff the consistent midpoint property

holds, i.e., for any arbitrary $a,b,c,d \in X$, $m_{a,b}; m_{c,d} = m_{a,c}; m_{b,d}$.

3.6. Norming of X : In this section we assume that a linear structure is generated in X with the scalar multiplication, addition and associativity of addition with respect to a fixed point o , as introduced in the previous section. Under this condition, there exists a norm on X such that the corresponding metric is equal to the given metric of X .

3.6.1. Lemma: For any $a,b,c \in X$, $d(a, b) = d(a + c, b + c)$, i.e., the metric is translation invariant.

Proof: First we shall prove $d(a + c, b + c) \leq d(a, b)$. Assume $d(a + c, b + c) >$

$d(a, b)$, $\tau = d(a + c, b + c) - d(a, b)$ and choose a natural number m such that j

$= 2^m$ and $2j\tau \geq 5d(a, a + c)$ holds. Set $b_i = a + i(b - a)$, $c_i = b_i + c$, for $i = 0, 1, 2, \dots$ and

$c' = a + 2c$. Now $m_{b_{2j}, c'} = (1/2)(a + 2j(b - a) + a + 2c) = a + j(b - a) + c = c_j$. So it

follows that $d(b_{2j}, c') = 2d(c_j, c') \geq 2d(c_j, c_0) - 2d(c_0, c')$. Now $c_j = b_j + c = a + j(b$

$- a) + c = c_0 + j(b - a) = c_0 + j(c_1 - (c + a)) = c_0 + j(c_1 - c_0) = (1 - j)c_0 + c_1$. Therefore

by lemma 3.2.10., we get $d(c_0, c_j) = j d(c_0, c_1) = jd(a + c, b + c)$. Also we have

$d(c_0, c') = d(a + c, a + 2c) = d(a, a + c)$. These relation give: $d(b_{2j}, c') \geq 2d(c_j,$

$c_0) - 2d(c_0, c') = 2jd(a + c, b + c) - 2d(a, a + c) = 2jd(a, b) + 2j\tau - 2d(a, a + c) \geq$

$2jd(a, b) + 3d(a, a + c)$. (i)

Now $b_{2j} = a + 2j(b - a) = (2j - 1)a + 2jb = 2jb + (2j - 1)b_0$. By lemma 3.2.10.,

$d(b_0, b_{2j}) = 2jd(b_0, b_1) = 2jd(a, b)$. Moreover, $d(b_0, c') = 2d(a, a + c)$. Therefore

$d(b_{2j}, c') \leq d(b_{2j}, b_0) + d(b_0, c') = 2jd(a, b) + 2d(a, a + c)$, which contradicts (i).

Therefore $d(a + c, b + c) \leq d(a, b)$.

Then $d(a, b) = d((a + c) - c, (b + c) - c) \leq d(a, b)$ and the lemma is proved.

3.6.2. Lemma: The function $v(a) = d(O, a)$ defines a norm on X .

Proof: That $v(a) = 0$ iff $a = O$ is trivial. Now $\alpha a = B_\alpha(O, a)$ and by lemma 3.2.10.,

$v(\alpha a) = d(O, \alpha a) = |\alpha| d(O, a) = |\alpha|v(a)$. Therefore for any $a, b \in X$, we have

$d(O, a + b/2) \leq 1/2 d(O, a) + 1/2 d(O, b)$ and $d(O, (a + b)/2) \leq (1/2) d(O, a + b) =$

$(1/2) v(a + b)$. Therefore $(1/2) v(a + b) \leq (1/2) v(a) + (1/2) v(b)$ which implies

$$v(a + b) \leq v(a) + v(b)$$

The main result of this section can be summarized in the following:

3.6.3. Theorem: Let (X, d) be a metric space satisfying property GM and having consistent midpoint property. Then X is a normed linear space with norm $v(a) = d(O, a)$.

3.7. Isomorphism of generated normed linear spaces:

The above linearization and norming of X are dependent on the choice of the point O . Since the definition of $m_{a,b}$ is independent of " O ", it follows by theorem 3.5.23 that whether or not X has the consistent midpoint property is independent of the special choice of " O ".

For any $O \in X$ ($O' \in X$) denote by M (N) the normed linear space generated from X for which O (O') is the origin. The operations in M will be denoted $\alpha \cdot a$, $a + b$ while in N they will be denoted by $\alpha \bullet a$, $a \oslash b$, v and μ will be the corresponding norms.

3.7.1. Theorem: M and N are isometrically isomorphic. An isometric iso-

morphism $f : M \longrightarrow N$ is given by $f(a) = a + O'$

Proof: By lemma 3.3.1., $f(\alpha a) = \alpha a + O' = B_{\alpha}(O, a) + O' = B_{\alpha}(O + O', a + O')$

$$= B_{\alpha}(O', a + O') = B_{\alpha}(O', f(a)) = \alpha \bullet f(a). \quad (i)$$

$$\text{Also, } f(B_{1/2}(a,b)) = B_{1/2}(a, b) + O' = B_{1/2}(a + O', b + O') = B_{1/2}(f(a), f(b))$$

$$\text{Now, by (i), } f(a + b) = f(2B_{1/2}(a, b)) = 2 \bullet f(B_{1/2}(a,b)) = 2 \bullet B_{1/2}(f(a), f(b))$$

$= f(a) - f(b)$. Thus f is a linear map from M into N .

Further by lemma 3.6.1. and 3.6.2., $v(a) = d(O, a) = d(O + O', a + O')$
 $= d(O', f(a)) = \mu(f(a))$. Since d is translation invariant, $d(a, b) = d(a + O', b + O') =$
 $d(f(a), f(b))$. Since f is one-one, $a = b$ implies $f(a) = f(b)$, for $f(b) = b + O' = a +$
 $O' = f(a)$. Finally f is onto, since for every $b \in N$ there exists $b \in M$.

Now $O' \in M$ implies there exists $-O' \in M$ (the additive inverse of O in M).

Thus $b - O' \in M$. Let $a = b - O'$, therefore $f(a) = (b - O') + O' = b$.

Chapter IV

Convexity in Metric Spaces.

4.1. Introduction: There are different ways in which one can introduce a notion of convexity in a metric space.

In section 2, Menger [17] convexity (M-convexity) is defined and it is shown that it is not a good generalisation of convexity in a normed linear space.

In section 3, we discuss the notion of convexity introduced by W. Takahashi [19].

In section 4, using Doss's definition of a mean in metric space, we introduce B-Convexity and obtain some pleasant properties. We also observe that the results concerning Takahashi's convexity in Machado [15] and Tallman [20] can be derived from B-convexity.

4.2. Menger Convexity: First we give a definition of Menger Convexity which is a modification of Busemann's definition (1.1). However, our definition yields a convexity space (2.1).

4.2.1. Definition: Let (X,d) be a metric space. Given $x,y \in X$, z is between x and y in the Menger sense if $d(x,z) + d(z,y) = d(x,y)$ and we define $M(x,y) = \{z: d(x,z) + d(z,y) = d(x,y)\}$.

The following example shows M-convexity is not a good generalisation of convexity in normed linear space since balls are not necessarily M-convex.

4.2.2. Example: Let $(x_1, y_1), (x_2, y_2), \dots$ be points in \mathbb{R}^2 . Then $\| (x_1, y_1) \| = |x_1| + |y_1|$

is a norm and with the corresponding metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

By definition $z = (x_3, y_3)$ is between $x = (x_1, y_1)$ and $y = (x_2, y_2)$ in the Menger sense if $d(x, y) = d(x, z) + d(z, y)$. (i)

Case(a). If the line joining x and y is parallel to the x -axis, then $M(x, y)$ is the line segment between x and y .

Case(b). If the line joining x and y is parallel to the y -axis, then $M(x, y)$ is the line segment between x and y .

Case(c). Otherwise $M(x, y)$ is the rectangle (square) with diagonal the line segment joining x and y .

To see this let $z = (x_3, y_3)$ be any point in the rectangle with diagonal the line joining $x = (x_1, y_1)$ and $y = (x_2, y_2)$ then $x_1 \leq x_3 \leq x_2$ and $y_1 \leq y_3 \leq y_2$. Now

$$d(x, z) = |x_1 - x_3| + |y_1 - y_3| = x_3 - x_1 + y_3 - y_1 \quad \text{and} \quad d(y, z) = |x_2 - x_3| + |y_2 - y_3| = x_2 - x_3 + y_2 - y_3.$$

Therefore $d(x, z) + d(z, y) = x_2 - x_1 + y_2 - y_1 = d(x, y)$.

Further we claim z cannot lie outside the rectangle, since the equality (i) does not hold.

Now consider the closed ball $S(0, 1)$, that is, $S(0, 1) = \{(x_1, y_1) : |x_1| + |y_1| \leq 1\}$. Now, if we take two points on the boundary of the ball say, $a = (a_1, a_2)$ and $b = (b_1, b_2)$, then $M(a, b)$ is the rectangle with a diagonal as the line segment joining a and b . Evidently $M(a, b) \not\subset S(0, 1)$.

4.2.3. Example: Let $X = S_3 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ and define a metric on X as $d(x,y)$ is the length of the shorter arc along the great circle through x and y . Evidently $d(x,y) \leq \pi$. Now it can be shown that (X,d) is a metric space.

For any $x,y \in X$, $M(x,y)$ is the shorter arc along the great circle joining x and y .

Let us consider the closed ball $\bar{S}(p,r)$, where p is the pole and $r = \pi/2 + \varepsilon$.

$0 < \varepsilon < \pi/2$ and $E = \{x : d(x,p) = \pi/2\}$ is the equator. Now if we take two points x_1 and x_2 below the equator on $\bar{S}(p,r)$, then $M(x_1, x_2)$, the shorter arc, lies outside the ball $\bar{S}(p,r)$. Hence $\bar{S}(p,r)$ is not M -convex.

4.2.4. Example: Consider $X = \ell_1^2$ with the metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|^{1/2} + |x_2 - y_2|^{1/2}$ as in example 3.3.4. Now consider the closed ball

$S(0,1)$, that is, $S(0,1) = \{x = (x_1, x_2) : |x_1|^{1/2} + |x_2|^{1/2} \leq 1\}$. Now if we consider $(-1,0)$ and $(1,0)$, then $M((-1,0), (1,0))$ is the square with the diagonal as the line segment joining $(-1,0)$ and $(1,0)$, by example 4.2.2.

Therefore $M((-1,0), (1,0)) \not\subseteq S(0,1)$.

4.3. Takahashi Convexity structure :

A convexity structure was introduced by W. Takahashi [19] and has been studied by H. Machado [15]

4.3.1. Definition: Let (X,d) be a metric space and $\alpha \in [0,1]$. A Takahashi convexity structure (TCS) on X is a function $W: X \times X \times [0,1] \longrightarrow X$

which has the property that for every $x, y \in X$ and $\alpha \in [0, 1]$,

$d(z, W(x, y, \alpha)) \leq (1-\alpha) d(z, x) + \alpha d(z, y)$ for every $z \in X$. If (X, d) has a

TCS, we say X is a W -convexity space.

A set $A \subseteq X$ is a W -convex set provided $W(x, y, \alpha) \in A$ for each $x, y \in A$ and $\alpha \in [0, 1]$. It is clear from the definition of mean that $W(x, y, \alpha) \in B_\alpha(x, y)$.

Therefore when means are singletons, there exists one and only one TCS.

In a normed linear space the means are singletons and $B_\alpha(x, y) = (1-\alpha)x + \alpha y$

by theorem 3.3.9. Thus we have the following :

The only TCS in normed linear space is the usual convexity and

$$B_\alpha(x, y) = W(x, y, \alpha) = (1-\alpha)x + \alpha y.$$

The main properties of TCS can be derived from the corresponding properties of means. We have the following theorem:

4.3.2. Theorem: There exists a TCS on X if and only if $B_\alpha(x, y) \neq \phi$, for any

$x, y \in X$ and $\alpha \in [0, 1]$

Proof: Suppose $B_\alpha(x, y) \neq \phi$, then $\{ B_\alpha(x, y) : x, y \in X, \alpha \in [0, 1] \}$ is family of

nonempty sets. Therefore there exists a choice function $\mathfrak{S} : X \times X \times [0, 1] \rightarrow$

$\cup B_\alpha(x, y)$ such that $\mathfrak{S}(x, y, \alpha) \in B_\alpha(x, y)$. Then $\mathfrak{S} = W$, gives a TCS.

The converse part of the theorem follows from the definition of mean.

In general it is clear that the function W , as defined in the above proof, need not be continuous. However, the following theorem indicates that W is continuous on some points. Further if means are singletons (W is uniquely defined) and X is finitely compact, then W is continuous as the Theorem 4.3.4. shows.

4.3.3. Theorem: If the means are singletons on a metric space (X,d) , then

$B_{\alpha}(x,x)$ is continuous at each point $(x,x,\alpha) \in X \times X \times [0,1]$.

Proof: Let $\{x_n\}$, $\{y_n\}$, and $\{\alpha_n\}$ be sequences in X and $[0,1]$ which converge

to x , x and α , respectively. We also note that $B_{\alpha}(x,x) = x$. It will be

sufficient to prove that $\{B_{\alpha_n}(x_n, y_n)\}_{n=1}^{\infty}$ converges to x which follows since, for

each n , $d(x, B_{\alpha_n}(x_n, y_n)) \leq (1-\alpha_n)d(x, x_n) + \alpha_n d(x, y_n)$.

4.3.4. Theorem: If the means are singletons on a finitely compact metric space (X,d) , then the mean is a continuous function from $X \times X \times I$ to X .

Proof: Let $\{(x_n, y_n, t_n)\}$ be a sequence in $\{X \times X \times [0,1]\}$ which converges to

(x, y, t) and let w be a limit point of the sequence $\{B_{t_n}(x_n, y_n)\}_{n=1}^{\infty}$. Select a

subsequence $\{B_{t_{nk}}(x_{nk}, y_{nk})\}_{n=1}^{\infty}$ which converges to w . Then for any $z \in X$,

$d(z, B_{t_{nk}}(x_{nk}, y_{nk})) \leq (1-t_{nk})d(z, x_{nk}) + t_{nk}d(z, y_{nk})$ for $k = 1, 2, 3, \dots$. By continuity

of the metric, $d(z, w) \leq (1-t)d(z, x) + td(z, y)$. Since the means are singletons,

therefore $w = B_t(x,y)$ is the only limit point of the sequence $\{B_{t_n}(x_n, y_n)\}_{n=1}^{\infty}$.

Since X is compact $\{B_t(x_n, y_n)\}$ must converge to $B_t(x,y)$ and hence the theorem is proved.

4.3.5. Remarks: L.Tallman [20] proved the above two theorems with different but equivalent hypotheses.

4.4. B-convexity: Let (X,d) be an arbitrary metric space. In this section we begin studying a convexity structure in X which is given by using the mean studied in Chapter III.

4.4.1. Definition: The set of all points in the segment in the sense of Doss's [8]

mean, between x and y is defined to be $\cup B_{\alpha}(x,y)$, $\alpha \in [0,1]$ and is denoted by

$\langle x, y \rangle$. A set $C \subseteq X$ is B-convex iff $\langle x, y \rangle \subseteq C$, for all $x, y \in C$.

Let \mathcal{C}_B denote the family of B-convex subsets of X . We observe that (X, \mathcal{C}_B) is an axiomatic convexity space. The following lemma shows that the segments defined by the resulting hull operator coincides with the segments defined by means.

4.4.2. Lemma: For all $x, y \in X$, $\langle x, y \rangle = \mathcal{C}_B(x,y)$.

Proof: Let $x, y \in X$. If $C \in \mathcal{C}_B$ and $x, y \in C$ then $\langle x, y \rangle \subseteq C$.

So, $\langle x, y \rangle \subseteq \{ C \in \mathcal{C}_B : C \supseteq \{x, y\} \} = \mathcal{C}_B(x,y)$.

On the other hand, $\langle x, y \rangle$ is B-convex, for if $a, b \in \langle x, y \rangle$, then there

exists $\rho, \rho' \in [0,1]$ such that $a \in B_\rho(x,y)$ and $b \in B_{\rho'}(x,y)$. So for any $\rho^* \in [0,1]$

and, by lemma 3.2.11, we get $B_{\rho^*}(a,b) \subseteq B_{\rho+\rho'\rho^*-\rho\rho^*}(x,y)$.

Thus $\langle a, b \rangle \subseteq \langle x, y \rangle$. Therefore $C_B(x, y) \subseteq \langle x, y \rangle$.

The following lemmas are the direct consequence of the definition of mean.

4.4.3. Lemma: If $x, y \in X$ and $B_\alpha(x, y) \neq \emptyset$, for all $\alpha \in [0,1]$, $\alpha \longmapsto B_\alpha(x, y)$

is an injection.

Proof: Suppose $B_\alpha(x, y) = B_\beta(x, y)$, then by lemma 3.2.10., for some

$p \in B_\alpha(x, y) \cap B_\beta(x, y)$, $d(p, x) = \alpha d(x, y) = \beta d(x, y)$. This implies $\alpha = \beta$.

4.4.4. Lemma: The open and closed balls $S(x, r), \bar{S}(x, r)$, with center x and radius r , in X are B -convex subsets of X .

Proof: Let $y, z \in S(x, r)$, then $d(x, y) < r$ and $d(x, z) < r$. Again, let $w \in \langle y, z \rangle$

$= \cup \{B_\alpha(x, y) : \alpha \in [0,1]\}$. Then there exists $\alpha \in [0,1]$ such that $w \in B_\alpha(y, z)$.

Therefore, for all $x \in X, d(w, x) \leq (1-\alpha)d(x, y) + \alpha d(x, z) < (1-\alpha)r + \alpha r = r$.

Therefore $w \in S(x, r)$. Hence $S(x, r)$ is a B -convex set in X . Similar proof follows for $\bar{S}(x, r)$.

4.5. Properties of the segment :

The following lemmas list some properties of segments.

4.5.1. Lemma: For all $a \in X, \langle a, a \rangle = a$.

Proof: By definition $\langle a, a \rangle = \bigcup_{\alpha} B_{\alpha}(a, a) = \bigcup_{\alpha} \{a\} = a$.

4.5.2. Lemma: If $c \in \langle a, b \rangle$ then $\langle a, c \rangle \subseteq \langle a, b \rangle$.

Proof: By lemma 4.4.2, $c \in \langle a, b \rangle = \mathcal{C}_B(a, b)$ which implies $\mathcal{C}_B(a, c) = \langle a, c \rangle \subseteq \mathcal{C}_B(a, b) = \langle a, b \rangle$.

4.5.3. Lemma: If $c \in \langle a, b \rangle$ then $\langle a, c \rangle \cup \langle c, b \rangle \subseteq \langle a, b \rangle$.

Proof: By lemma 4.5.2, $\langle a, c \rangle \subseteq \langle a, b \rangle$ and $\langle c, b \rangle \subseteq \langle a, b \rangle$

Thus $\langle a, c \rangle \cup \langle c, b \rangle \subseteq \langle a, b \rangle$.

The following example shows that equality does not hold in general.

4.5.4. Example: Let $X = \{a, b, c, d\}$ be a discrete metric space with metric ∂ , except $\partial(a, b) = 2$. From example 3.2.3, we have, $\mathcal{C}_B(a, b) = \{a, b, c, d\}$.

Then $\mathcal{C}_B(a, c) = \{a, c\}$ and $\mathcal{C}_B(c, b) = \{c, b\}$. Thus $\langle a, c \rangle \cup \langle c, b \rangle$ is a proper subset of $\langle a, b \rangle$.

In general Pasch's axiom does not imply JHC (Chapter II). However in a B-convexity space when means are singletons Pasch's axiom implies JHC as the following lemma shows.

4.5.5. Lemma: Let (X, \mathcal{C}_B) be a B-convexity space satisfying :

(i) $B_{\rho}(x, y)$ is singleton for all $x, y, z \in X$, and $\rho \in [0, 1]$

(ii) Pasch's axiom, i.e., for all $\alpha, \beta \in [0, 1]$, there exists $\gamma, \partial \in [0, 1]$ such that

$B_\beta(B_\alpha(x,y), z) = B_\beta(B_\gamma(x,z), y)$. Then, if $H \subseteq X$ is B-convex and $x_0 \in X \setminus H$,

$$C_B(H \cup x_0) = U \{ C_B(x_0, y) : y \in H \}$$

Proof: By lemma 2.4.2, $U \{ C_B(x_0, y) : y \in H \} \subseteq C_B(H \cup x_0)$. Further to complete

the proof, it is shown that $U \{ C_B(x_0, y) : y \in H \}$ is B-convex.

Let $a, b \in U \{ C_B(x_0, y) : y \in H \} = U \{ U \{ B_\alpha(x_0, y) : \alpha \in [0, 1] \} : y \in H \}$. Therefore,

for some suitable $\rho_1, \rho_2 \in [0, 1]$ and $y_1, y_2 \in H$, $a = B_{\rho_1}(x_0, y_1)$ and

$b \in B_{\rho_2}(x_0, y_2)$. Let, for some $\rho \in [0, 1]$, $z = B_\rho(a, b) = B_\rho(B_{\rho_1}(x_0, y_1), B_{\rho_2}(x_0, y_2))$.

By using Pasch's axiom and lemma 3.2.8. repeatedly we get the following steps:

$$z = B_\gamma(B_\beta(x_0, B_{\rho_2}(x_0, y_2)), y_1) = B_\gamma(B_{1-\beta}(B_{\rho_2}(x_0, y_2), x_0), y_1)$$

$$= B_\gamma(B_\zeta(B_\xi(x_0, x_0), y_2), y_1) = B_\gamma(B_\zeta(x_0, y_2), y_1) = B_\gamma(B_{1-\gamma}(y_2, x_0), y_1)$$

$$= B_\mu(B_\lambda(y_2, y_1), x_0) = B_{1-\mu}(x_0, B_\lambda(y_2, y_1)).$$
 Since H is B-convex

$B_\lambda(y_2, y_1) \in H$. So $z \in U C_B(x_0, y)$ and $\langle a, b \rangle \subseteq C_B(x_0, y)$.

4.5.6. Remarks: L. A. Tallman [20] obtained a similar result for the so called "strict TCS". This notion holds when means are singletons.

The following lemma gives a relation between M-convexity and B-

convexity.

4.5.7. Lemma: Every M-convex set is a B-convex set.

Proof: $M(x,y) = \{ z : d(x,z) + d(z,y) = d(x,y) , z \text{ is between } x \text{ and } y \}$

Let $a, b \in M(x,y)$, then by definition $\langle a, b \rangle = \cup \{ B_\rho(a,b) : \rho \in [0,1] \}$. Let

$p \in \langle a, b \rangle$, then there exists some ρ such that $p \in B_\rho(a, b)$. Hence by

lemma 3.2.10. we have $d(a, p) + d(p, b) = d(a, b)$, i.e., p is between a and b

so that $p \in M(x, y)$. Thus $\langle a, b \rangle \subseteq M(x, y)$.

However the converse of the theorem is not true in general as the following example shows.

4.5.8. Example: Consider example 3.3.4., where $X = \ell^2_{1/2}$ and $d((x_1, x_2),$

$(y_1, y_2)) = |x_1 - y_1|^{1/2} + |x_2 - y_2|^{1/2}$. By example 4.2.4., $M((0,1), (1,0)) =$ rectangle

ABCD. Further, by definition, $\langle (0,1), (1,0) \rangle = \cup \{ B_\rho((0,1), (1,0)) : 0 \leq \rho \leq 1 \}$.

Now, by example 3.3.4, $B_\rho((0,1), (1,0)) = \emptyset$. This implies $\langle (0,1), (1,0) \rangle$ is not

M-convex set.

4.6. Linearization of B-convex space : For the remainder of this chapter X will

denote a metric space with the properties defined below :

PROPERTIES : For any $x, y, z \in X$, $\alpha, \beta, \gamma \in [0,1]$:

- I. $B_\alpha(x,y)$ is singleton.

II. Distance is homothetic i.e. $d(B_\alpha(z,x), B_\alpha(z,y)) = \alpha d(x,y)$.

III. A specialised Pasch's axiom is satisfied, i.e., $B_\beta(z, B_\beta(y,x))$.

$$= B_{\beta(1-\alpha)}(B_{\alpha\beta/(1+\alpha\beta-\beta)}(z, x), y) = B_{\alpha\beta}(B_{\beta(1-\beta)/(1-\alpha\beta)}(z,y), x).$$

In this section we will show, following a construction given by Machado[15], that if X is a B-convex metric space satisfying properties I–III, then X is isometrically isomorphic to a convex subset of a normed linear space E . The construction of the normed linear space is lengthy and involved, so we will divide it into a sequence of steps.

STEP 1: Construction of a cone $G = \mathbb{R}^+ X$:

In order to define positive scalar multiples of X , we select an element O as the origin in X and for $\alpha \in [0,1]$ and $x \in X$, define the scalar multiplication by $\alpha x = B_\alpha(O,x)$ as defined in section 3.5.

The following properties are immediate :

(1) $0x = \alpha O = O$ follows from lemmas 3.2.5. and 3.2.4.

(2) $\alpha(\beta x) = (\alpha\beta)x$, where $\alpha, \beta \in I$, follows from lemma 3.5.9.

(3) $d(\alpha x, \alpha y) = \alpha d(x,y)$, follows from property II, when $z=O$.

(4) $\beta B_\alpha(y,x) = B_{1-\beta+\alpha\beta}(y, \beta/(1-\beta+\alpha\beta))x$.

Proof of (4): Substituting $z = O$ in property III, we have $B_\beta(O, B_\alpha(y, x))$

$$= B_{\beta(1-\alpha)}(B_{\alpha\beta/(1+\alpha\beta-\beta)}(O, x), y) \text{ which implies } \beta B_\alpha(y, x) =$$

$$B_{\beta(1-\alpha)}(\{\alpha\beta/(1+\alpha\beta-\beta)\}x, y) \text{ and } \beta B_\alpha(y, x) = B_{1-\beta(1-\alpha)}(y, \{\alpha\beta/(1+\alpha\beta-\beta)\}x).$$

$$\text{Therefore } \beta B_\alpha(y, x) = B_{1+\alpha\beta-\beta}(y, \{\alpha\beta/(1+\alpha\beta-\beta)\}x).$$

Next we consider the Cartesian product $\mathbf{R}^+ \times X$, where \mathbf{R}^+ is the set of all positive real numbers. Now we define a relation " \wedge " on the set $\mathbf{R}^+ \times X$ such that $(\lambda, x) \wedge (\partial, y)$ holds if and only if $\lambda x / (\lambda + \partial) = \partial y / (\lambda + \partial)$.

We claim the relation defined on the set $\hat{\mathbf{A}}^+ \times X$ is an equivalence relation. Evidently the relation is reflexive and symmetric. To prove transitivity consider $(\lambda, x) \wedge (\partial, y)$ and $(\partial, y) \wedge (\gamma, z)$ i.e. we have $\lambda x / (\lambda + \partial) = \partial y / (\lambda + \partial)$ and $\partial y / (\partial + \gamma) = \gamma z / (\partial + \gamma)$. Thus $\lambda x / (\gamma + \lambda) = \gamma z / (\gamma + \lambda)$ i.e. $(\lambda, x) \wedge (\gamma, z)$.

Let $G = \mathbf{R}^+ \times X / \wedge$. For sake of simplicity the equivalence class for the pair (λ, x) and the pair itself will be denoted by (λ, x) . The zero class is written $(1, O) = \{(\lambda, O), \lambda > 0, O \in X\}$.

STEP 2: Scalar Product of non negative numbers and vectors in G:

For $\partial \in \mathbf{R}^+$ and $(\lambda, x) \in G$ define (5) $\partial(\lambda, x) = (\partial\lambda, x)$, and $0(\lambda, x) = (1, O)$.

To see that scalar multiplication, by non negative scalars and vectors in G is well defined, consider the pairs $(\lambda, x), (\partial, y)$ and $\gamma \in \mathbf{R}^+$ where $(\lambda, x) \wedge (\partial, y)$.

Now $\lambda x / (\partial + \lambda) = \partial y / (\partial + \lambda)$, so $\gamma \lambda x / (\gamma \partial + \gamma \lambda) = \gamma \partial y / (\gamma \partial + \gamma \lambda)$. Thus $(\gamma \lambda, x) \wedge (\gamma \partial, y)$ and $\gamma(\lambda, x) \wedge \gamma(\partial, y)$ as desired.

We note that we are dealing with classes and furthermore $\gamma(\xi(\lambda, x)) = \gamma\xi(\lambda, x)$,

$1(\lambda, x) = (\lambda, x)$ and $\lambda(1, 0) = (1, 0)$. We note

STEP 3: Addition of vectors in G:

For $(\lambda, x), (\partial, y) \in G$, $\partial, \lambda \in \mathbf{R}^+$ define

$$(6) (\lambda, x) + (\partial, y) = (\lambda + \partial, B_{\partial / (\lambda + \partial)}(x, y)).$$

To see that addition is well defined, consider the pairs $(\lambda, x), (\partial, y)$

and (γ, z) where $(\lambda, x) \wedge (\partial, y)$. As $\lambda x / (\lambda + \partial) = \partial y / (\lambda + \partial)$, we have $\lambda x / (\lambda + \partial + \gamma) =$

$\partial y / (\lambda + \partial + \gamma)$. By taking the mean with z we have :

$$B_{(\partial + \lambda + \gamma) / (\partial + \lambda + 2\gamma)}(z, \lambda x / (\lambda + \partial + \gamma)) = B_{(\partial + \lambda + \gamma) / (\partial + \lambda + 2\gamma)}(z, \partial y / (\lambda + \partial + \gamma)).$$

By property III,

$$(\gamma + \partial) / (\lambda + \partial + 2\gamma) B_{\lambda / (\lambda + \gamma)}(z, x) = (\partial + \gamma) / (\lambda + \partial + 2\gamma) B_{\partial / (\partial + \gamma)}(z, y). \text{ That is,}$$

$$(\lambda + \gamma) B_{\gamma / (\lambda + \gamma)}(x, z) = (\partial + \gamma) B_{\partial / (\partial + \gamma)}(y, z) \text{ which}$$

implies $(\lambda, x) + (\gamma, z) = (\partial, y) + (\gamma, y)$ as desired.

Addition is commutative, for, $(\lambda, x) + (\partial, y) = (\lambda + \partial, B_{\partial / (\lambda + \partial)}(x, y))$

$$= (\partial + \lambda, B_{\lambda / (\lambda + \partial)}(y, x)) = (\partial + \lambda) + (\lambda, x).$$

The zero is $(1, O) = \{(\lambda, O) : \lambda > 0, O \in X\}$, since, $(\lambda, O) + (\partial, y) =$

$(\lambda + \partial, B_{\partial/(\lambda+\partial)}(O, y)) = (\lambda + \partial, \partial y/(\lambda + \partial))$ and $(\partial, y) \wedge (\lambda + \partial, \partial y/(\lambda + \partial))$ and the proof is complete.

In order to prove associativity, consider the identity,

$$B_{\gamma/\Omega}(B_{\lambda/\Omega}/(1-(\gamma/\Omega)) (y, x), z) = B_{\lambda/\Omega}(B_{\partial/\Omega}/(1-\lambda/\Omega) (z, y), x) \text{ where } \Omega = \lambda + \partial + \gamma.$$

$$\text{Therefore } B_{\gamma/\Omega} (B_{\lambda/(\lambda+\partial)} (y, x), z) = B_{\lambda/\Omega} (B_{\partial/(\partial+\gamma)} (z, y), x).$$

$$\text{Now } ((\lambda, x) + (\partial + y)) + (\gamma + z) = (\lambda + \partial, B_{\partial/(\lambda+\partial)}(x, y)) + (\gamma + z)$$

$$= (\lambda + \partial + \gamma, B_{\gamma/(\lambda+\partial+\gamma)} (B_{\partial/(\lambda+\partial)}(x, y), z))$$

$$= (\lambda + \partial + \gamma, B_{\gamma/(\lambda+\partial+\gamma)} (B_{\lambda/(\lambda+\partial)}(y, x), z))$$

$$= (\lambda + \partial + \gamma, B_{\lambda/(\lambda+\gamma+\partial)} (B_{\partial/(\partial+\gamma)}(z, y), x))$$

$$= (\lambda + \partial + \gamma, B_{(\partial+\gamma)/(\lambda+\partial+\gamma)} (x, B_{\partial/(\partial+\gamma)}(z, y)))$$

$$= (\lambda, x) + (\partial + \gamma, B_{\partial/(\partial+\gamma)}(z, y)) = (\lambda, x) + (\partial + \gamma, B_{\gamma/(\partial+\gamma)}(y, z))$$

$$= (\lambda, x) + ((\partial + y) + (\gamma + z)).$$

The two distributive laws are evident:

$$(\gamma + \xi)(\lambda, x) = \gamma(\lambda, x) + \xi(\lambda, x) \text{ and } \gamma((\lambda, x) + (\partial, y)) = \gamma(\lambda, x) + \gamma(\partial, y)$$

$$\text{Firstly, } (\gamma + \xi)(\lambda, x) = ((\gamma + \xi)\lambda, x) = (\gamma\lambda + \xi\lambda, x).$$

Also, $\gamma(\lambda, x) + \xi(\lambda, x) = (\gamma\lambda, x) + (\xi\lambda, x) = (\gamma\lambda + \xi\lambda, B_{\xi\lambda/(\gamma\lambda + \xi\lambda)}(x, x)) = (\gamma\lambda + \xi\lambda, x)$.

Thus $(\gamma + \xi)(\lambda, x) = \gamma(\lambda, x) + \xi(\lambda, x)$.

Again $\gamma((\lambda, x) + (\partial, y)) = \gamma(\lambda + \partial, B_{\partial/(\lambda + \partial)}(x, y)) = (\gamma(\lambda + \partial), B_{\partial\gamma/(\lambda\gamma + \partial\gamma)}(x, y))$.

Therefore $\gamma((\lambda, x) + (\partial, y)) = (\lambda\gamma, x) + (\partial\gamma, x)$.

STEP 4. A metric d' on G:

(7) $d'[(\lambda, x), (\partial, y)] = (\lambda + \partial) d[\lambda x/(\lambda + \partial), \partial y/(\lambda + \partial)]$.

To prove d' is well defined let us consider the pairs (λ, x) , (∂, y) and (γ, z)

where $(\lambda, x) \wedge (\partial, y)$. Now $(\lambda, x) \wedge (\partial, y)$ implies $\lambda x/(\lambda + \partial) = \partial y/(\lambda + \partial)$

and we have $\lambda x/(\lambda + \partial + \gamma) = \partial y/(\lambda + \partial + \gamma)$.

Thus $d(\lambda x/(\lambda + \partial + \gamma), \gamma z/(\lambda + \partial + \gamma)) = d(\partial y/(\lambda + \partial + \gamma), \gamma z/(\lambda + \partial + \gamma))$.

By property III., $\{(\lambda + \gamma)/(\lambda + \partial + \gamma)\} d(\lambda x/(\lambda + \partial), \gamma z/(\lambda + \gamma))$

$= \{(\partial + \gamma)/(\lambda + \partial + \gamma)\} d(\partial y/(\partial + \gamma), \gamma z/(\partial + \gamma))$, i.e., $d'((\lambda, x), (\gamma, z)) = d'((\partial, y), (\gamma, z))$ as desired.

The function d' satisfies all the properties of a metric. First, $d'((\lambda, x), (\partial, y)) = (\lambda + \partial) d(\lambda x/(\lambda + \partial), \partial y/(\lambda + \partial)) = 0$ iff $\lambda x/(\lambda + \partial) = \partial y/(\lambda + \partial)$ iff $(\lambda, x) \wedge (\partial, y)$. If (λ, x) and (∂, y) do not belong to the same class then $\lambda x/(\lambda + \partial) \neq \partial y/(\lambda + \partial)$ and $(\lambda + \partial) d(\lambda x/(\lambda + \partial), \partial y/(\lambda + \partial)) > 0$, i.e., $d'((\lambda, x), (\partial, y)) > 0$. Further d' is symmetric.

To prove the triangle inequality let us consider:

$d(\lambda x/\Omega, \partial y/\Omega) \leq d(\lambda x/\Omega, \gamma z/\Omega) + d(\gamma y/\Omega, \partial z/\Omega)$. By property II.,

$$\{(\lambda+\partial)/\Omega\} d(\lambda x/(\lambda+\partial), \partial y/(\lambda+\partial)) \leq \{(\lambda+\gamma)/\Omega\} d(\lambda x/(\lambda+\gamma), \gamma z/(\lambda+\gamma)) +$$

$$\{(\gamma+\partial)/\Omega\} d(\gamma z/(\gamma+\partial), \partial y/(\gamma+\partial)) \text{ which implies } d'((\lambda, x), (\partial, y)) \leq$$

$$d'((\lambda, x), (\gamma, z)) + d'((\gamma, z), (\partial, y)).$$

(8) The metric d' is homothetic i.e. if g and $g' \in G$ and $\lambda \geq 0$ then $d'(\lambda g, \lambda g') =$

$\lambda d'(g, g')$. This can be seen from the definition.

(9) The metric is translation invariant, i.e., for all $g, g', g'' \in G$,

$d'(g+g''+g'+g'') = d(g, g')$, where g is of the form of (λ, x) . To see this, let $g=(\lambda, x)$,

$g'=(\partial, y)$, $g''=(\gamma, z)$. Then $d(B_{\Omega'/\Omega'}(z, \lambda x/\Omega), B_{\Omega'/\Omega'}(z, \partial y/\Omega)) = \{\Omega/\Omega'\} d(\lambda x/\Omega,$

$\partial y/\Omega)$ where $\Omega'=\lambda+\partial+2\gamma$ and $\Omega=\lambda+\partial+\gamma$. By using property II., and relation (4)

we have, $\Omega' d(\{(\lambda+\gamma)/\Omega'\} B_{\lambda/(\lambda+\gamma)}, \{(\partial+\gamma)/\Omega'\} B_{\partial/(\partial+\gamma)}(y, z)) = \{\partial(\lambda+\partial)/\Omega\}$

$d(\lambda x/(\lambda+\partial), \partial y/(\lambda+\partial))$ which implies

$$\Omega' d(\{(\lambda+\gamma)/\Omega'\} B_{\lambda/(\lambda+\gamma)}(x, z), \{(\partial+\gamma)/\Omega'\} B_{\partial/(\partial+\gamma)}(y, z)) = (\lambda+\partial) d(\lambda x/(\lambda+\partial), \lambda y/(\lambda+\gamma)).$$

Therefore $d'((\lambda, x) + (\gamma, z), (\partial, y) + (\gamma, z)) = d'((\lambda, x), (\partial, y))$.

The result so far can be summarized as follows :

4.6.1. Summary:

(i) Addition on G is commutative, associative and there is a zero element

$(1, 0)$ which we will denote from now on by g_0 .

(ii) Multiplication by nonnegative scalars is distributive and associative .

Also, if $g \in G$ and $\lambda > 0$, then $1g = g$, $\lambda g_0 = g_0$ and $0g = g_0$.

(iii) (G, d') is a metric space which satisfies property II, i.e., is homothetic and d' is translation invariant.

STEP 5: An isometry of X into G which preserves convexity:

(10) The mapping $f: X \longrightarrow G$ is given by $f(x) = (1, x)$ has the following properties :

(i) $f(0) = (1, 0)$ which is trivial.

(ii) Preserves convex combination, i.e., for all $x, y \in X$, $f(B_\alpha(x, y)) = (1-\alpha)f(x) + \alpha f(y)$. This holds since $(1, B_\alpha(x, y)) = (1-\alpha, x) + (\alpha, y) = (1-\alpha)(1, x) + \alpha(1, y)$.

(iii) $d'((1, x), (1, y)) = 2 d(x/2, y/2) = d(x, y)$ by definition and property II.

As a direct consequence of results (8),(9),(10) we have

(11) $g+g'' = g'+g''$ implies $g = g'$, since d' is translation invariant, i.e., $d'(g, g') = d'(g+g'', g'+g'') = 0$.

(12) If $\lambda g = \lambda g'$ and $\lambda > 0$ then $g = g'$

since d' is homothetic, i.e., $d(\lambda g, \lambda g') = \lambda d(g, g') = 0$.

(13) If $\lambda g = \lambda' g$ and $g \neq g_0$ then $\lambda = \lambda'$, for, $\lambda g = \lambda(\partial, x) = (\lambda \partial, x)$ where $g = (\partial, x)$ and $\partial > 0$ and $\lambda' g = (\lambda' \partial, x)$. So $\lambda g = \lambda' g$ implies $\lambda \partial x / (\lambda \partial + \lambda' \partial) = \lambda' \partial x / (\lambda \partial + \lambda' \partial)$ and

$\lambda/(\lambda+\lambda') = \lambda'/(\lambda+\lambda')$ since $x \neq 0$ and $\partial > 0$. Thus $\lambda = \lambda'$.

Now since $(\lambda, x) = \lambda(1, x)$, for every $\lambda > 0$ and $x \in X$, we identify X and $f(X)$.

We can write $G = \mathbb{R}^+X$, that is the cone spanned by X .

STEP 6. Equivalence class in $(G \times G)$:

We define an equivalence relation \sim in $(G \times G)$ as follows.

If $e, f, g, h \in G$, we say $(g, h) \sim (e, f)$ if and only if $g+f = e+h$.

Define $E = G \times G / \sim$. We define on the quotient space E , addition and scalar multiplication as follows.

Using the same symbol for the pair (c, d) and the corresponding equivalent class and we define the addition as follows.

$$(14) \quad (e, f) + (g, h) = (e+g, f+h).$$

To see that addition is well defined consider the pairs $(c, d), (e, f), (g, h)$ where $(e, f) \sim (g, h)$. Now $(c, d) + (e, f) = (c+e, d+f)$ and $(c, d) + (g, h) = (c+g, d+h)$. Also $(e, f) \sim (g, h)$ iff $g+h = e+f$. Therefore $c+d+g+f = c+d+e+f$. This implies $(c+e, d+f) \sim (c+g, d+h)$.

Further it can be easily seen that addition is associative and commutative. The equivalence class of (g_0, g_0) , is the set $\{(g, g), g \in G\}$, is the zero element for addition and every element (e, f) has an additive inverse (f, e) .

The product by the real scalars is defined below.

$$(15) \quad \lambda(e, f) = (\lambda e, \lambda f) \text{ if } \lambda \geq 0 \quad \text{and} \quad \lambda(e, f) = |\lambda| (e, f) \text{ if } \lambda < 0$$

It can be easily seen that the above product is well defined and satisfies the

following identities.

$$(16) \lambda(\partial(e,f)) = \lambda\partial(e,f).$$

$$(17) \lambda\{(e,f)+(g,h)\} = \lambda(e,f) + \lambda(g,h).$$

$$(18) (\lambda+\partial)(e,f) = \lambda(e,f) + \partial(e,f).$$

$$(19) 1(c,d) = (c,d).$$

STEP 7. A norm on E:

First we define a metric d'' on E as follows :

$$(20) d''((c,d), (e,f)) = d'(c+f, e+d)$$

In order to show d'' is well defined, let us consider the pairs (e,f) , (g,h) and (c,d) where $(e,f) \wedge (g,h)$. Then $(e,f) \wedge (g,h)$ implies $e+h = g+f$.

Therefore $d''((e,f), (c,d)) = d'(e+d, f+c) = d'(e+d+g+h, f+c+g+h) = d'(d+g, c+h) = d''((g,h), (c,d))$.

d'' is homothetic and translation invariant, for,

$$d''((e,f)+(g,h), (j,k)+(g,h)) = d''((e+g, f+h), (j+g, k+h)) = d'(e+g+k+h, f+h+j+g)$$

$$= d'(e+k, j+f) = d''((e,f), (j,k)) \text{ and } d''(\lambda(e,f), \lambda(g,h)) = d''((\lambda e, \lambda f), (\lambda g, \lambda h))$$

$$= d'(\lambda e + \lambda h, \lambda f + \lambda g) = d'(\lambda(e+h), \lambda(f+g)) = \lambda d''((e+h), (f+g)) = \lambda d''((e,f), (g,h)).$$

Now all the properties put together imply that E is a normed linear space with norm given by the formula.

(21) $\|v\| = d''(v, 0)$ where $v = (e,f)$ is a typical element of E and $O = (g_0, g_0)$ is the zero element.

STEP 8. Isometry of X into E :

We consider the mapping ψ from the cone G to the space $E = G \times G / \sim$ defined by $\psi(g) = (g, g_0)$. Evidently ψ is one-one. Further $d''(\psi(g), \psi(g')) = d''((g, g_0), (g', g_0)) = d'(g + g_0, g' + g_0) = d'(g, g')$ and $\psi(g_0) = (g_0, g_0)$. Further ψ preserves convex combinations, for, $\psi((1-\lambda)g_1 + \lambda g_2) = ((1-\lambda)g_1 + \lambda g_2, g_0) = ((1-\lambda)g_1 + \lambda g_2, (1-\lambda)g_0 + \lambda g_0) = ((1-\lambda)g_1, (1-\lambda)g_0) + (\lambda g_2, \lambda g_0) = (1-\lambda)(g_1, g_0) + \lambda(g_2, g_0) = (1-\lambda)\psi(g_1) + \lambda\psi(g_2)$.

The conclusion is that E contains a copy of G viz, $\psi(G)$; in fact with this identification in mind we can write $E = G - G$ in the sense that any vector $v = (e, f) = (e, g_0) - (f, g_0) = \psi(e) - \psi(f)$.

Finally consider the composition $\Phi = \psi \circ f$ of two mappings f and ψ . Then $\Phi(x) = (\psi \circ f)(x) = \psi(f(x)) = \psi(1, x) = ((1, x), (1, 0))$ for every $x \in X$. As both ψ and f preserve convex combinations, the same is true for Φ . Thus we have the following theorem:

4.6.2. Theorem: If (X, d) is a B-convexity space which satisfies properties I, II, and III, then there exists an isometry \tilde{f} from X onto a convex subset of some normed linear space which preserves convex combinations, that

$$\text{is, } \Phi(B_\alpha(x, y)) = (1-\alpha)x + \alpha y.$$

4.6.3. Remarks: The image $\Phi(X)$ is a B-convex subset of a normed space E

which cannot be distinguished from a B-convexity space so far as the metric and convex structure is concerned. The point O , selected as a reference (origin) in X , gets identified with the zero vector in the normed space E . After making the identification $X \longrightarrow \Phi(x)$ we can write $E = \mathbf{R}^+X - \mathbf{R}^+X$, or $E = \text{Span } X$.

Indeed if, $v = (e, f) \in E$ and, say, $e = (\lambda, x)$, $f = (\partial, y)$ then we have $v = (e, f) = \psi(e) - \psi(f) = \psi(\lambda(1, x)) - \psi(\partial(1, y)) = \lambda\Phi(x) - \partial\Phi(y)$. So that we can think of the normed space E as the minimal extension of X .

CHAPTER VCONCLUSION5.1. Summary and Remarks:

In this thesis we have developed B-convexity and compared it to other notions of convexity defined on metric spaces. Several results concerning B-convexity have been derived and, although the notion of B-convexity is quite general, it has several expected and desirable properties. Following the construction given by Machado [15] we show (Theorem 4.6.2) that if X is a B-convex metric space satisfying properties I - III, then X is essentially a convex subset of a normed space and the space is unique.

Further, as shown in Theorem 3.7.1, if a B-convex metric space satisfies property GM and the consistent midpoint property then B-convexity space becomes a normed space which is unique up to an isometric isomorphism.

Many of the fixed point theorems that have been proved in the convex metric space (in the sense of Takahashi) setting can be obtained for B-convex metric spaces. Other results along these lines and other possible areas of application are topics for further investigation. Also the structural properties of B-convexity spaces could be further studied .

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