AN ABSTRACT MODEL FOR MEASURE AND CATEGORY

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by

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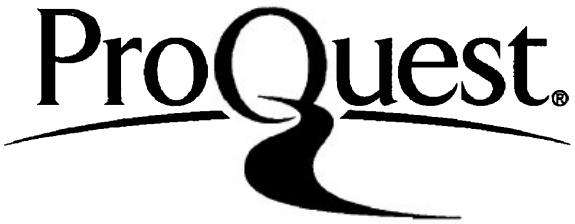

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ABSTRACT

This thesis is an attempt to establish an abstract model for Lebesgue measure and Baire category.

In the introduction we list several similarities between Lebesgue measurable sets and sets having the property of Baire. Then we abstract these similarities and use them as axioms.

In Chapter I, we introduce a generalized model and prove some results that are well-known both in measure and category.

In Chapter II, we define kernels and covers. After proving their existence for any set, we proceed to find some interesting results.

It is very natural to consider the quotient algebra if we have an algebra containing a proper ideal. Hence Chapter III inevitably comes into the scene.

In Chapter IV we introduce analytic sets through A-operations. This approach enables us to prove that every analytic set belongs to our model.

In Chapter V we consider the local properties of sets and prove some interesting results.

Chapter VI is taken from the work of J. C. Morgan II. We include his work here for the completeness of the thesis. Also, as we will see, it gives us a new insight into "negligible sets".

We conclude the thesis by setting up a list of questions which, we think, are rather challenging.

NOTATION

$\mathbb{R}$ denotes the real line.

$\mathbb{B}$ denotes the family of all Borel subsets of $\mathbb{R}$.

$\mathbb{N}$ denotes the set of positive integers.

N_0 denotes $\aleph_0$.

N_1 denotes $\aleph_1$.

ϕ denotes the empty set.

$|A|$ denotes the cardinality of A.

c denotes the cardinality of continuum.

$\mathbb{P}(A)$ denotes the power set of A.

$\overline{A}$ denotes the closure of A.

A' denotes the complement of A, i.e. $\mathbb{R}-A$.

A^k denotes the kernel of A in Propositions 44 to 52.

A^c denotes the cover of A in Propositions 44 to 52.

AB denotes $A \cap B$.

$A \Delta B$ denotes the symmetric difference of A and B.

$\cup F$ denotes $\bigcup_{F \in F} F$.

A class of sets that contains countable unions and arbitrary subsets of its members is called a σ-ideal.

A non-empty class S of subsets of $\mathbb{R}$ is called a σ-ring if it is closed under the operations of countable union and complementation. It is called a σ-algebra if $\mathbb{R}$ itself is a member of S.

A subset of $\mathbb{R}$ is said to have the property of Baire if it can be represented in the form $G \Delta F$ where G is open and F is of first category.

Throughout this thesis, we assume Γ as a σ-ideal that satisfies the following axioms:

(i) Γ does not contain any non-empty open set.

(ii) All singleton sets belong to Γ.

(iii) For each $A \in \Gamma$, there is a $K \in \mathbb{B} \cap \Gamma$ such that $A \subseteq K$.

(iv) Every non-empty perfect set has a non-empty perfect subset in Γ.

(v) There are at most N_0 disjoint Borel sets none of which is in Γ.

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INTRODUCTION

The purpose of this thesis is to put into an abstract setting results concerning Lebesgue measurable sets and sets having the property of Baire. Results of this paper are, of course, satisfied in both of the above cases. For the sake of simplicity, we restrict ourselves to the subsets of the real line $\mathbb{R}$. The continuum hypothesis "$c = N_1$" is assumed and used without further mention.

It is well known that any set can be represented as a disjoint union of a null set (set of measure zero) and a set of first category [12]. If a non-measurable set is given, we obtain a non-measurable set of first category. On the other hand, from a measurable set, we have a measurable set of first category. Hence, a set possessing the property of Baire can be measurable or non-measurable. Similarly, a measurable set may or may not have the property of Baire. Thus, it seems that measurable sets and sets having the property of Baire are totally different, but this is not the case; there are many similarities, some of which will be shown here. For example, a famous theorem proved by Sierpinski and refined by Erdös is as follows: 'assuming the continuum hypothesis, there exists a bijection f of the line to itself such that $f = f^{-1}$ and such that $f(E)$ is a null set if and only if E is of first category'. This theorem justifies the following principle of duality. Let P be any proposition involving solely the notion of null set and notions of pure set theory (for example,

cardinality, disjointness, or any property invariant under arbitrary one-to-one transformation). Let P* be the proposition obtained from P by replacing "null set" by "set of first category" throughout. Then each of the propositions P and P* implies the other, assuming the continuum hypothesis.

Some basic properties which are true in both models will be considered here. To begin with, a subset of a set of first category is again first category, and the countable union of sets of first category is also first category. A family of sets which is closed under countable union and hereditary is said to form a σ-ideal. Therefore sets of first category form a σ-ideal. Null sets also have the above properties and hence they too form a σ-ideal. Baire proved that an open interval is of second category and thus "every non-empty open set is not of first category." Also "every non-empty open set is not of measure zero." Singleton sets are nowhere dense, and so "all singleton sets are of first category." In measure, "all singleton sets are of measure zero." If a set A is of first category, then it is a countable union of nowhere dense sets. But the closure of a nowhere dense set is also nowhere dense. This implies that A is contained in a F_σ set of first category, and F_σ sets are Borel sets, therefore, "any set of first category is contained in a Borel set which is of first category." In measure, every null set is contained in a G_δ set of measure zero; and G_δ sets are Borel sets. Hence "any null set is contained in a Borel set which is of

measure zero." It is well known [18, p. 23] that any uncountable G_δ set contains an uncountable nowhere dense closed null set. From P. S. Alexandrov's result that every uncountable Borel set contains a non-empty perfect subset, it follows that "every non-empty perfect set contains a non-empty perfect subset of first category" and "every non-empty perfect set contains a non-empty perfect subset of measure zero." Kuratowski [10, p. 256] proved that every family of disjoint sets each with the property of Baire and none of which is of first category, is countable. This together with the fact that every Borel set has the property of Baire gives the following: "a family of disjoint Borel sets none of which is of first category, is countable." Due to the σ-finiteness of $\mathbb{R}$ for measure and the countable additivity of measure, we can easily prove that every family of disjoint measurable sets, none of which is of measure zero, is countable. This, together with the fact that every Borel set is measurable, implies that "a family of disjoint Borel sets, none of which is of measure zero is countable." The properties that are underlined do not exhaust all the similarities between Lebesgue measurable sets and sets having the property of Baire. In this thesis, the above properties are taken as axioms and other similarities are derived from these. So our model will be a common generalization of Lebesgue measurable sets and sets with the property of Baire. For similar work, see [9] and [13].

Although our results are proved in $\mathbb{R}$, many of them hold good if $\mathbb{R}$ is replaced by a complete separable metric

space of power N_1.

CHAPTER I

In this chapter we establish a model which is a generalization of the family of Lebesgue measurable sets and the family of sets with the property of Baire. Then we derive some well-known results including one of Sierpinski's and another concerning the Lusin set.

Proposition 1. *All countable sets belong to* Γ.

Proposition 2. $|\Gamma| = 2^{\mathfrak{c}}$.

Proof. Γ contains a non-empty perfect set (Axiom iv), and Γ is a σ-ideal, so $|\Gamma| \geq 2^{\mathfrak{c}}$. On the other hand, Γ is a subset of $\mathbb{P}(\mathbb{R})$, so $|\Gamma| \leq |\mathbb{P}(\mathbb{R})| = 2^{\mathfrak{c}}$. Hence $|\Gamma| = 2^{\mathfrak{c}}$.

Proposition 3. $|\mathbb{B}| = \mathfrak{c}$.

This result is well-known, for a proof, see [6, p. 26].

Proposition 4. $|\Gamma - \mathbb{B}| = 2^{\mathfrak{c}}$.

Proof. Since $|\Gamma| = 2^{\mathfrak{c}}$ and $|\mathbb{B}| = \mathfrak{c}$, therefore $|\Gamma - \mathbb{B}| = 2^{\mathfrak{c}}$.

Proposition 5. $|\mathbb{B} \cap \Gamma| = \mathfrak{c}$.

Proof. $|\mathbb{B} \cap \Gamma| \leq |\mathbb{B}| = \mathfrak{c}$, and Axiom ii implies that $|\mathbb{B} \cap \Gamma| \geq \mathfrak{c}$.

Definition 6. $\partial = \{A : A = B \Delta M$ for some $B \in \mathbb{B}$ and $M \in \Gamma\}$. Note that ∂ is a generalization of the family of Lebesgue measurable sets and the family of sets with the property of Baire.

Proposition 7. *$A \in \partial$ if and only if there is a Borel set B such that $A \Delta B \in \Gamma$.*

Proof. $A \in \partial$ implies $A = B_1 \Delta M_1$ for some $B_1 \in \mathcal{B}$ and $M_1 \in \Gamma$. The symmetric property of "Δ" gives $A \Delta B_1 = B_1 \Delta (B_1 \Delta M_1) = M_1$; let $B = B_1$. Now suppose that there is a Borel set B such that $A \Delta B \in \Gamma$. Since $A = (A \Delta B) \Delta B$, we have $A \in \partial$.

Theorem 8. *∂ is the σ-algebra generated by $\mathcal{B}$ and sets in Γ.*

Proof. $A \in \partial$ implies $A = B \Delta M$ for some $B \in \mathcal{B}$ and $M \in \Gamma$. This gives $A' = B' \Delta M$. But $B' \in \mathcal{B}$, therefore $A' \in \partial$. If $A_i = B_i \Delta M_i$ for $B_i \in \mathcal{B}$ and $M_i \in \Gamma$; let $A = \bigcup_{i=1}^{\infty} A_i$, $B = \bigcup_{i=1}^{\infty} B_i$, $M = \bigcup_{i=1}^{\infty} M_i$. Clearly $B - M \subseteq A \subseteq B \cup M$ which in turn gives $B - A \subseteq M$ and $A - B \subseteq M$. Therefore $A \Delta B \in \Gamma$. Hence $A \in \partial$ (Proposition 7).

Proposition 9. $|\partial - \Gamma| = 2^c$.

Proof. Let $I = [0,1]$. There exist 2^c Γ-sets (i.e. sets belonging to Γ) outside I. If A is a Γ-set outside I, then $A \cup I \in \partial - \Gamma$, so $|\partial - \Gamma| \geq 2^c$. Since $\partial - \Gamma \subseteq \mathcal{P}(\mathbb{R})$, the result follows.

Proposition 10. *If $A \in \partial - \Gamma$, then there exist $B \in \mathcal{B} - \Gamma$ and $M \in \Gamma$ such that $B \cap M = \phi$ and $A = B \cup M$.*

Proof. $A \in \partial$ means $A = B_1 \Delta M_1$ for some $B_1 \in \mathcal{B}$ and $M_1 \in \Gamma$. Axiom iii implies that there is a set $K \supseteq M_1$ with $K \in \mathcal{B} \cap \Gamma$. Since

$$\begin{aligned} A &= B_1 \Delta M_1 \\ &= [(B_1 - K) \cup (B_1 \cap K)] \Delta (M_1 \cap K) \\ &= [(B_1 - K) \Delta (B_1 \cap K)] \Delta (M_1 \cap K) \\ &= (B_1 - K) \Delta [(B_1 \Delta M_1) \cap K] \\ &= (B_1 - K) \cup [(B_1 \Delta M_1) \cap K] . \end{aligned}$$

Let $B = B_1 - K$, and $M = (B_1 \Delta M_1) \cap K$. Then the result follows.

Proposition 11. *If* $A \in \partial\text{-}\Gamma$ *then* A *contains a non-empty perfect subset belonging to* Γ.

Proof. $A \in \partial\text{-}\Gamma$ implies, by Proposition 10, that there exists $B \subseteq A$ with $B \in \mathfrak{B}\text{-}\Gamma$. P. S. Alexandrov proved that every uncountable Borel set contains a non-empty perfect subset [17]; this, together with Axiom iv implies the result.

Theorem 12. *If* $A \in \partial\text{-}\Gamma$, *then* A *contains* c *perfect sets.*

Proof. $A \in \partial\text{-}\Gamma$ implies, by Proposition 11, that it contains a non-empty perfect set in Γ. Let F be the family of all those perfect subsets of A which are in Γ. Now if $|F| \leq N_0$, then $\bigcup F \in \Gamma$. But $A - \bigcup F \in \partial\text{-}\Gamma$ implies that it contains a non-empty perfect subset P in Γ. Therefore $P \in F$, a contradiction. Hence $|F| \geq c$, and $|F| \leq |\mathfrak{B}| = c$, so $|F| = c$.

Now if $A \in \partial\text{-}\Gamma$, then there exists an $H \subseteq A$ such that H and $A\text{-}H$ intersect every non-empty perfect subset of A. This can be proved as follows: let F be the family of all non-empty

perfect subsets of A. Well order F, i.e. $F = \{F_\alpha : \alpha < \omega_c\}$ where ω_c is the first ordinal having c predecessors. Also, well-order each member in F. Let p_1, q_1 be the first two elements in F_1, let p_2, q_2 be the first two elements in F_2 that are different from p_1, q_1. If $1 < \alpha < \omega_c$ and if p_β and q_β have been defined for all $\beta < \alpha$, let p_α, q_α be the first two elements in $F_\alpha - \bigcup_{\beta<\alpha} \{p_\beta, q_\beta\}$; this can be done because $|F_\alpha| = c$ and $\bigcup_{\beta<\alpha} \{p_\beta, q_\beta\}$ is countable. Now let $H = \{p_\alpha : \alpha < \omega_c\}$. H intersects every perfect subset in A, and since A-H contains all q_α, A-H intersects each perfect subset in A.

H, constructed above, is a totally imperfect set, i.e. a set which contains no non-empty perfect subset.

Proposition 13. *If H is totally imperfect, then $K \subseteq H$ and $K \in \partial$ imply $K \in \Gamma$.*

Proof. Suppose $K \in \partial - \Gamma$; then K contains a non-empty perfect subset belonging to Γ (proposition 11), which is a contradiction.

Theorem 14. *If both H and H' are totally imperfect, then $H \notin \partial$.*

Proof. Suppose $H \in \partial$; then $H \in \Gamma$ (Proposition 13). This implies that $H' \in \partial$, and so $H' \in \Gamma$. Hence $\mathbb{R} = H \cup H' \in \Gamma$, which is a contradiction.

The results of Proposition 13 and Theorem 14 still hold

if H and A-H are totally imperfect provided $A \in \partial\text{-}\Gamma$ and $H \subseteq A$.

Theorem 15. *If* $A \notin \Gamma$, *then* A *contains a subset not in* ∂.

Proof. Suppose H and H' are totally imperfect, consider $A \cap H$ and $A \cap H'$. If they are both in ∂, then they are both in Γ (Proposition 13). Also $A = (A \cap H) \cup (A \cap H')$ implies that $A \in \Gamma$, which is a contradiction.

Corollary 16. *If* $\mathcal{P}(A) \subseteq \partial$, *then* $A \in \Gamma$.

Corollary 17. *If* $A \notin \Gamma$ *then either* $A \notin \partial$ *or* $A = H_1 \cup H_2$, *with* $H_1 \cap H_2 = \phi$ *and* $H_1, H_2 \notin \partial$.

The following is a generalization of Sierpinski's result. For similar proofs, see [18, p. 76] and [22, p. 77].

Theorem 18. *If* Γ, Γ^* *are two* σ*-ideals satisfying the Axioms, then there exists a bijection* f *of* $\mathbb{R}$ *to itself such that* $f(A) \in \Gamma^*$ *if and only if* $A \in \Gamma$.

Proof. By Proposition 5, $|\mathcal{B} \cap \Gamma| = c$. Well-order $\mathcal{B} \cap \Gamma$ by $\{B_t : t < \Omega\}$, where Ω is the first uncountable ordinal. Next define by transfinite induction a sequence $\{A_t : t < \Omega\}$ of disjoint sets in Γ such that $|A_t| = c$ for all t and for each

$A \in \Gamma$ there exists $t' < \Omega$ such that $A \subseteq \bigcup_{t<t'} A_t$.
This is done by first choosing A_0 to be any uncountable set in Γ. If sets A_k, $k < t < \Omega$ are chosen, then since $\bigcup_{k<t} A_k \in \Gamma$ and $(\bigcup_{k<t} A_k)' \in \partial\text{-}\Gamma$ it follows from Proposition 11 that $(\bigcup_{k<t} A_k)'$ contains a non-empty perfect subset P_t in Γ. Now define $A_t = P_t \cup (B_t - \bigcup_{k<t} A_k)$. Similarly, define a transfinite family $\{A_t^* : t < \Omega\}$ associated with Γ^*. Since $|A_t| = |A_t^*| = c$, there is a one to one function f that maps A_t onto A_t^* and since the A_t's are disjoint and $\mathbb{R} = \bigcup_{t<\Omega} A_t$, it follows that $f : \mathbb{R} \to \mathbb{R}$ is one to one and onto. Now $A \in \Gamma$ if and only if $A \subseteq \bigcup_{t<t'} A_t$ for $t' < \Omega$, if and only if $f(A) \subseteq f(\bigcup_{t<t'} A_t)$, if and only if $f(A) \subseteq \bigcup_{t<t'} A_t^*$, if and only if $f(A) \in \Gamma^*$.

If Γ is the family of sets of first category and Γ^* is the family of nullsets, then we obtain Sierpinski's result. Suppose further that there is a set $M \in \Gamma$ whose complement $M' \in \Gamma^*$. Then there exists a bijection f of $\mathbb{R}$ to itself such that $f = f^{-1}$ and such that $f(E) \in \Gamma^*$ if and only if $E \in \Gamma$. This is the generalization of Erdös refinement for Sierpinski's result. For a proof, see [18, p. 77].

The following (Corollary 19 to Theorem 23) are interesting results that are similar to those which Sierpinski collected in his masterpiece "Hypothése Du Continu".

Corollary 19. *If* $E \notin \Gamma$, *then* E *intersects uncountably many elements in* $\{A_t : t < \Omega\}$.

Proof. Let F be the family of sets in $\{A_t : t < \Omega\}$ which E intersects. If $|F| \leq N_0$, then $\bigcup F \in \Gamma$ and $K = E - \bigcup F \notin \Gamma$. Now $K \cap A_t = \phi$ for all $t < \Omega$ implies that $K \cap \mathbb{R} = \phi$, a contradiction.

Definition 20. *Any uncountable set E, such that no uncountable subset of E belongs to Γ, is called a Lusin set.*

Theorem 21. *$E \notin \Gamma$ if and only if it contains a Lusin set.*

Proof. If E contains a Lusin set, then obviously $E \notin \Gamma$. Suppose now $E \notin \Gamma$. Corollary 19 implies that $E \cap A_t \neq \phi$ for uncountably many t. Let A be formed by choosing one point from each of these. Then $|A| = c$. If $K \subseteq A$ is uncountable, then K is not covered by countably many A_t's and so $K \notin \Gamma$.

Corollary 22. *There exists a one to one mapping f of $\mathbb{R}$ onto a subset of itself such that $f(E) \notin \Gamma$ whenever E is uncountable.*

Proof. Since a Lusin set L has cardinality c, there is a one to one function f from R onto L. If $|E| = c$, then $f(E) \notin \Gamma$.

Theorem 23. *If $A \notin \Gamma$, then we can decompose A into c disjoint sets none of which belongs to Γ.*

Proof. Two methods of proof are exhibited, the first one is much shorter, the second one, though longer, has its own importance.

Method 1. $A \notin \Gamma$ implies it contains a Lusin set L (Theorem 21).

Since $|\mathbb{R}| = |L| = c$, there is a one to one function which maps $\mathbb{R}$ onto L. Also since there are c disjoint uncountable sets in $\mathbb{R}$ there are c disjoint subsets of A, and each one is not in Γ. Adjoin A minus their union to any one of them.

Method 2. This method is due to Ulam [18, p. 25] and the proof is as follows: $A \notin \Gamma$ implies $|A| = c$. Well-order A such that for each $y \in A$, the set $\{x : x < y\}$ is countable. Let $f(x,y)$ be a one-to-one mapping of this set onto a subset of positive integers. Then f is an integer-valued function defined for all pairs (x,y) of elements of A for which $x < y$. Since f is one-to-one, if $x < x' < y$, then $f(x,y) \neq f(x',y)$. For each $x \in A$ and each positive integer n, define $F^n_x = \{y : x < y, f(x,y) = n\}$ and establish an array as follows:

$$\begin{array}{llll} F^1_{x_1} & F^1_{x_2} & F^1_{x_3} & \cdots\cdots \\ F^2_{x_1} & F^2_{x_2} & F^2_{x_3} & \cdots\cdots \\ F^3_{x_1} & F^3_{x_2} & F^3_{x_3} & \cdots\cdots \end{array}$$

Note that there are only countable rows but c columns. We claim: (i) the sets in any row are mutually disjoint, (ii) the union of the sets in each column is cocountable. The proof of (i) is trivial: suppose $y \in F^n_x \cap F^n_{x'}$ for some n, y,x and x' with $x < x'$. And if $x' < y$ then $x < x' < y$ and $f(x,y) \neq f(x',y)$, which contradicts the definition of F^n_x. To prove (ii) observe that $x < y$ implies that y belongs to one of the sets F^n_x, i.e. to the one for which $n = f(x,y)$. Hence the union of the sets

$F_x^n(n = 1,2,\ldots)$ differs from A by the countable set $\{y : y \leq x\}$. Since there are c columns and N_0 rows, there must be a row such that there are c elements not in Γ. If not, then all rows have only countably many elements not in Γ, which implies that there are N_0 elements not in Γ in the whole array. Then there is a whole column such that each of its elements belongs to Γ. Hence $A \in \Gamma$, which contradicts the fact $A \notin \Gamma$. Therefore there are c disjoint elements none of which belong to Γ. In case the union of these sets so obtained is not equal to A, adjoin the complement of that union to any one of them.

Proposition 24. *There exist at most N_0 disjoint sets in ϑ-Γ.*

Proof. The result follows easily from Proposition 10 and Axiom v.

Corollary 25. *There exist N_0 disjoint sets in ϑ-Γ.*

Proof. Proposition 24 and Axiom i imply the result.

Proposition 26. *If $A \notin \Gamma$, then A can be decomposed into c disjoint sets none of which are in ϑ.*

Proof. Theorem 23 and Proposition 24 imply that there are c disjoint sets none of which are in ϑ. Adjoin A minus their union to any one of them.

Proposition 27. *If $A \in \Gamma$-$\mathcal{B}$, then A can be decomposed into c disjoint sets each of which belongs to Γ-$\mathcal{B}$.*

Proof. By Ulam's method as in the proof of Theorem 23.

From Axiom i, we get the following result.

Proposition 28. *If* $A \in \Gamma$ *then* A' *is dense in* $\mathbb{R}$.

Proposition 29. *There is no set* A *in* Γ *which intersects each uncountable set in* Γ.

Proof. Suppose there is such a set A. Since $A \in \Gamma$, $A' \in \partial\text{-}\Gamma$, and so A' contains a non-empty perfect set P in Γ (Proposition 11). But $|P| = c$ which means $A \cap P \neq \phi$, a contradiction.

Proposition 30. *If* $A,B \in \partial$, *then either* $A \cap B \in \Gamma$ *or* $A \cap B$ *contains a set* K *with* $K \in \mathbb{B}\text{-}\Gamma$.

Proof. Suppose $A \cap B \notin \Gamma$, then $A \cap B \in \partial\text{-}\Gamma$. The result therefore follows from Proposition 10.

Proposition 31. *If* $A \notin \Gamma$ *and* $B' \in \Gamma$, *then* $A \cap B \notin \Gamma$.

Proof. $A \cap B \in \Gamma$ and $A \cap B' \subseteq B'$ implies $A \in \Gamma$, a contradiction.

Proposition 32. *If* $E \in \Gamma$ *and it is* F_σ, *then* E *is of first category.*

Proof. If E is F_σ, then $E = \bigcup_{i=1}^{\infty} E_i$ where the E_i's are closed. By Axiom i, each E_i is nowhere dense. Hence E is of first category.

Proposition 33. *If* $A \in \partial\text{-}\Gamma$ *and* $K \in \partial$, *then* A *contains a*

subset $B \in \mathcal{B}\text{-}\Gamma$ *such that* $B \cap K \in \Gamma$ *or* $B \cap K' \in \Gamma$.

Proof. Suppose $A \cap K \in \Gamma$. Since $A \in \partial\text{-}\Gamma$, A contains a subset $B \in \mathcal{B}\text{-}\Gamma$ (Proposition 10). Hence $B \cap K \in \Gamma$. Next if $A \cap K \notin \Gamma$, then $A \cap K \in \partial\text{-}\Gamma$. This implies by Proposition 10, that $A \cap K \supseteq B$ with $B \in \mathcal{B}\text{-}\Gamma$. Therefore $B \cap K' = \phi \in \Gamma$.

CHAPTER II

Measurable kernels and covers play an important role in the investigation of inner and outer measure. For references, see [4], [6] and [15].

Some problems concerning the property of Baire can be solved by means of the Baire kernels and hulls, see [25] and [26].

In this chapter, we establish definitions of kernels and covers similar to those pertaining to measure and category. Then we prove their existence for any set, and then derive some interesting results.

Definition 34. *The kernel K of a set A is a maximal subset of A in ∂; in the sense that $K \subseteq A$, $K \in \partial$ and if $L \subseteq A$, $L \in \partial$, then $L-K \in \Gamma$.*

Definition 35. *The cover C of a set A is a minimal set in ∂ which contains A; i.e. if $A \subseteq L$ and $L \in \partial$, then $C-L \in \Gamma$.*

Theorem 36. *For any set A, there exists a kernel $K \subseteq A$.*

Proof. Zorn's Lemma implies the existence of a maximal set F of disjoint elements in $\mathbb{P}(A) \cap (\partial-\Gamma)$. Proposition 24 implies that $|F| \leq N_0$. Let $K = \bigcup F$. Obviously $K \in \partial$ and $K \subseteq A$. Suppose $L \in \partial$ and $L \subseteq A$; then $L-K \in \partial$. Since $L-K$ does not intersect any element in F, and F is maximal, it follows that $L-K \in \Gamma$.

Corollary 37. *For any set A, there exists a cover $C \supseteq A$.*

Proposition 38. *Every set has a kernel which is a Borel set.*

Proof. Let K be a kernel of A. By Proposition 10, $K = B \cup M$ where $B \in \mathcal{B}$ and $M \in \Gamma$. Consequently, the Borel set B is also a kernel of A.

Similarly, every set has a cover which is a Borel set.

Proposition 39. *If both* K_1 *and* K_2 *are kernels of* A, *then* $K_1 \Delta K_2 \in \Gamma$.

Proof. The result follows easily from the definition.

Proposition 40. *If both* C_1 *and* C_2 *are covers of* A, *then* $C_1 \Delta C_2 \in \Gamma$.

Proof. The result follows easily from the definition.

Proposition 41. *Let* $\{A_i\}$ *be a sequence of sets and* $\{K_i\}$ *be the corresponding sequence of kernels. Then* $\bigcap_{i=1}^{\infty} K_i$ *is a kernel of* $\bigcap_{i=1}^{\infty} A_i$.

Proof. Let $L \subseteq \bigcap_{i=1}^{\infty} A_i$, and $L \in \partial$. Then $L - (\bigcap_{i=1}^{\infty} K_i) = \bigcup_{i=1}^{\infty} (L - K_i)$ and the result follows easily from the definition.

Proposition 42. *Let* $\{A_i\}$ *be a sequence of sets and* $\{C_i\}$ *be the corresponding sequence of covers. Then* $\bigcup_{i=1}^{\infty} C_i$ *is a cover of* $\bigcup_{i=1}^{\infty} A_i$.

Proof. Let $E \in \partial$ and $E \supseteq \bigcup_{i=1}^{\infty} A_i$. Then $\bigcup_{i=1}^{\infty} C_i - E = \bigcup_{i=1}^{\infty} (C_i - E) \in \Gamma$.

Thus $\bigcup_{i=1}^{\infty} C_i$ is a cover of $\bigcup_{i=1}^{\infty} A_i$.

If H and H' are both totally imperfect, then by Proposition 13, their kernels are both in Γ. But $H \cup H' = \mathbb{R}$. Hence the following is proved.

Proposition 43. *If K_1, K_2 are kernels of A_1, A_2 respectively; it is not necessarily true that $K_1 \cup K_2$ is a kernel of $A_1 \cup A_2$.*

In the Propositions 44-52, we denote by A^c a cover of A and by A^k a kernel of A.

Proposition 44. *If $A \subseteq B$, then $A^c - B^c \in \Gamma$.*

Proof. Since $A \subseteq B \subseteq B^c$, the result follows easily from the definition.

Similarly, the following is proved.

Proposition 45. *If $A \subseteq B$, then $A^k - B^k \in \Gamma$.*

Proposition 46. *If $\{A_n\}$ is an increasing sequence of sets, then there exists an increasing sequence $\{A_n^c\}$ of covers corresponding to $\{A_n\}$ such that $(\lim A_n^c) \Delta (\lim A_n)^c \in \Gamma$.*

Proof. Let $\{C_n\}$ be a sequence of associated covers of $\{A_n\}$. Define $A_1^c = C_1$, and $A_n^c = (A_{n-1}^c - C_n) \cup C_n$ for $n \geq 2$. Obviously $\{A_n^c\}$ is an increasing sequence of covers corresponding to $\{A_n\}$. Since $A_n^c \supseteq A_n \supseteq A_m$ for $n \geq m$, it follows that $\lim A_n^c \supseteq A_m$. Also m is arbitrary implies $\lim A_n^c \supseteq \lim A_n$. But $\lim A_n^c \in \partial$

implies that $(\lim A_n)^c - \lim A_n^c \in \Gamma$. Moreover, from $\lim A_n \supseteq A_m$, Proposition 44 implies that $A_m^c - (\lim A_n)^c \in \Gamma$. Since m is arbitrary, $\lim A_n^c - (\lim A_n)^c \in \Gamma$. Hence $(\lim A_n^c)\Delta(\lim A_n)^c \in \Gamma$. Similarly, the following can be proved.

Proposition 47. *If* $\{A_n\}$ *is a decreasing sequence of sets, then there exists a decreasing sequence* $\{A_n^k\}$ *of kernels corresponding to* $\{A_n\}$ *such that* $(\lim A_n^k)\Delta(\lim A_n)^k \in \Gamma$.

Proposition 48. *If* $M \in \partial$, *then for any set* A,

$$M\Delta[(MA)^k \cup (M-A)^c] \in \Gamma.$$

Proof. Since $M-A \subseteq M$ and $M \in \partial$, implies that $(M-A)^c - M \in \Gamma$. This together with the fact that $(MA)^k \subseteq M$ implies $[(MA)^k \cup (M-A)^c] - M \in \Gamma$. Conversely, $M-(M-A)^c \subseteq M-(M-A)$, implies $M-(M-A)^c \subseteq MA$. But since $M-(M-A)^c \in \partial$, it follows from the definition that $[M-(M-A)^c] - (MA)^k \in \Gamma$. Therefore $M-[(MA)^k \cup (M-A)^c] \in \Gamma$. Hence $M\Delta[(MA)^k \cup (M-A)^c] \in \Gamma$.

One can compare this result with the following result in Measure Theory: If M is measurable, then for any set A, $\mu(M) = \mu_*(MA) + \mu^*(M-A)$ where μ_* and μ^* denote the inner and outer measure respectively.

Proposition 49. *If* $M \in \partial$, *then for any set* A, $(A^cM)\Delta(AM)^c \in \Gamma$.

Proof. Let $L = A^c-[(A^cM)-(AM)^c] = [A^c-(A^cM)] \cup [A^c \cap (AM)^c]$ $= (A^c-M) \cup [A^c \cap (AM)^c]$. Obviously $L \in \partial$. Since $A-M \subseteq A^c-M$

and $AM \subseteq A^c \cap (AM)^c$, then $L \supseteq (A-M) \cup (AM) = A$. It follows from the definition that $A^c - L \in \Gamma$, i.e. $A^c \cap [(A^cM)-(AM)^c] \in \Gamma$. Therefore $(A^cM)-(AM)^c \in \Gamma$. Conversely, $AM \subseteq A^cM$, and $A^cM \in \partial$. Then $(AM)^c-(A^cM) \in \Gamma$. Hence $(A^cM)\Delta(AM)^c \in \Gamma$.

Proposition 50. *If $M \in \partial$, then for any set A, $(A^kM)\Delta(AM)^k \in \Gamma$.*

Proof. Since $(AM)^k \in \partial$ and $(AM)^k \subseteq A$, then $(AM)^k-A^k \in \Gamma$. Therefore $(AM)^k-(A^kM) \in \Gamma$. Moreover, $A^kM \subseteq AM$ and $A^kM \in \partial$. Then $A^kM-(AM)^k \in \Gamma$. Hence $(A^kM)\Delta(AM)^k \in \Gamma$.

Proposition 51. *If $\{M_n\}$ is a sequence of sets and A is any set, then $(\bigcup_{n=1}^{\infty} A M_n)^c \Delta \bigcup_{n=1}^{\infty} (A M_n)^c \in \Gamma$.*

Proof. Since $(AM_n)^c$ is a cover of AM_n, Proposition 42 implies that $\bigcup_{n=1}^{\infty} (AM_n)^c$ is a cover of $\bigcup_{n=1}^{\infty} A M_n$. But $(\bigcup_{n=1}^{\infty} A M_n)^c$ is also a cover of $\bigcup_{n=1}^{\infty} A M_n$. Result then follows by Proposition 40.

Proposition 52. *If $\{M_n\}$ is a sequence of sets in ∂ and A is any set, then $(\bigcup_{n=1}^{\infty} A M_n)^k \Delta \bigcup_{n=1}^{\infty} (A M_n)^k \in \Gamma$.*

Proof. Let $M = \bigcup_{n=1}^{\infty} M_n$ and B be a kernel of AM. Define $E = BM$ and $E_n = E M_n$. It is obvious that E is a kernel of AM and E_n is a kernel of $A M_n$. Since $EM = E$, we have $E = \bigcup_{n=1}^{\infty} E_n$. Observe that $\bigcup_{n=1}^{\infty} A M_n = AM$; this together with Proposition 39 implies $E\Delta(\bigcup_{n=1}^{\infty} A M_n)^k \in \Gamma$. But $E\Delta \bigcup_{n=1}^{\infty} (A M_n)^k = (\bigcup_{n=1}^{\infty} E_n)\Delta \bigcup_{n=1}^{\infty} (A M_n)^k$ and E_n is a kernel of $A M_n$. Therefore $E\Delta \bigcup_{n=1}^{\infty} (A M_n)^k \in \Gamma$. Hence $(\bigcup_{n=1}^{\infty} A M_n)^k \Delta \bigcup_{n=1}^{\infty} (A M_n)^k \in \Gamma$.

CHAPTER III

In this chapter we define an equivalence relation on ∂ which enables us to obtain a quotient algebra. We prove that the quotient algebra is complete.

Theorem 53. *If* $C \subseteq \partial$, *then there exists a countable subset* K *of* C *such that for each* $C \in C$, $C - \bigcup K \in \Gamma$.

Proof. If L is a σ-ring generated by C, then obviously $C \subseteq L \subseteq \partial$. Let N be the family of sets each of which can be covered by a countable union of elements of C. Clearly N is a σ-ring containing C, and therefore $C \subseteq L \subseteq N$. Zorn's lemmma implies that there exists a maximal family F of disjoint elements of $L-\Gamma$. Since $F \subseteq L-\Gamma \subseteq \partial-\Gamma$, it follows from Proposition 24 that $|F| \leq N_0$ which, in turn, implies that $\bigcup F \in L$, for L is a σ-ring. This means that $\bigcup F \in N$ and hence there exists a countable subset K of C such that $\bigcup F \subseteq \bigcup K$. The fact that F is maximal implies $L - \bigcup F \in \Gamma$ for each $L \in L$. Now $C \in C$, implies $C \in L$. This means that $C - \bigcup K \subseteq C - \bigcup F \in \Gamma$. Hence $C - \bigcup K \in \Gamma$.

Definition 54. $A \sim B$ *if and only if* $A \Delta B \in \Gamma$.

Proposition 55. *"$\sim$" is an equivalence relation.*

Proof. $A \sim A$ since $\phi \in \Gamma$. Due to the symmetric property of "Δ", $A \sim B$ implies $B \sim A$. Now suppose $A \sim B$ and $B \sim C$; then $A-C \subseteq (A-B) \cup (B-C)$ and $C-A \subseteq (C-B) \cup (B-A)$ and so $A-C$ and $C-A$ are both

in Γ. Therefore $A \sim C$.

Since "$\sim$" is an equivalence relation, it decomposes ∂ into disjoint classes, i.e. the equivalence classes of the relation "$\sim$": A_1, A_2 are in the same class if and only if $A_1 \sim A_2$. The class containing an element $A \in \partial$ will be denoted by $[A]$. By definition, the following conditions are equivalent: $A \sim B$, $A \in [B]$, $[A] = [B]$.

Let ∂/Γ denote the set of all equivalence classes $[A]$, $(A \in \partial)$. ∂/Γ is an algebra and it is called the quotient algebra of ∂ modulo Γ.

Theorem 56. $|\partial/\Gamma| = c$.

Proof. Suppose $A \in \partial$; then A contains a Borel set B such that $A \sim B$ and so $|\partial/\Gamma| \leq |\mathfrak{B}| = c$. Now suppose A, B are two open intervals, and suppose $A \nsubseteq B$. Then there exists $x \in A-B$ and an open neighbourhood N_x of x such that $N_x \subseteq A-B$. Hence $A-B \notin \Gamma$, and so $[A] \neq [B]$. Since there are c distinct open intervals, there are at least c distinct equivalence classes. Hence $|\partial/\Gamma| = c$.

R. Sikorski proved in [22, p. 74] that ∂/Γ is a σ-algebra and, for every N_0-indexed set $A_t \in \partial (t \in T)$, $\bigvee_{t \in T}[A_t] = [\bigcup_{t \in T} A_t]$ and $\bigwedge_{t \in T}[A_t] = [\bigcap_{t \in T} A_t]$ where "$\bigvee$", "$\bigwedge$" denote the join and the meet respectively. Sikorski also established the following: an element $[A]$ $(A \in \partial)$ is the join of an indexed set of elements $[A_t]$ $(t \in T)$ in ∂/Γ if and only if (i) $A_t - A \in \Gamma$ for every

$t \in T$, (ii) If $A_t - A_0 \in \Gamma$ $(A_0 \in \partial)$ for every $t \in T$, then $A - A_0 \in \Gamma$.

Definition 57. *An algebra U is said to be complete if and only if both the join and the meet exist for all subsets of U.*

There is a theorem in algebra [5, p. 26] which states that if every subset of an algebra has a join, then that algebra is complete. Hence there is no need to check the existence of the meet.

Theorem 58. *∂/Γ is a complete algebra.*

Proof. Let $\{[A_t]\}_{t\in T}$ be a subset of ∂/Γ, and $C = \{A_t\}_{t\in T}$. Since $C \subseteq \partial$, by Theorem 53 there exists a countable subset $K = \{A_{\alpha_t}\}_{\alpha_t \in \mathbb{N}}$ of C, such that $A_t - \bigcup K \in \Gamma$ for every $t \in T$. Suppose $A_t - A_0 \in \Gamma$ $(A_0 \in \partial)$ for every $t \in T$, then $\bigcup K - A_0 = \bigcup_{\alpha_t \in \mathbb{N}} (A_{\alpha_t} - A_0)$. But $A_{\alpha_t} - A_0 \in \Gamma$ for every $\alpha_t \in \mathbb{N}$, implies $\bigcup K - A_0 \in \Gamma$. Hence $[\bigcup K]$ is the join of $\{[A_t]\}_{t\in T}$.

CHAPTER IV

There are many approaches to analytic sets (see [8]). We define analytic sets through the A-operations. This approach has advantages over the others as we will see in this chapter.

Let F be a class of sets. Assign to each finite collection $(k_1,k_2,\ldots,k_n)$ of natural numbers, a definite set $A_{k_1\ldots k_n}$ in F. The set

$$F = \bigcup_{k_1\ldots k_n\ldots} \bigcap_{n=1}^{\infty} A_{k_1\ldots k_n},$$

where the union is over all infinite sequences of natural numbers, is called the result of A-operations on the sets of the class F. If F is a class of closed sets, then F is called an analytic set, briefly A-set. Let $A(F)$ denote the family of all sets which are the results of the A-operations on the sets of F. Also, let $S(F)$ and $I(F)$ be the families of all sets which are the unions and intersections, respectively, of a countable aggregate of sets which belong to F. The following results are proved in [20, p. 212].

(i) $A(F) = A(A(F))$.

(ii) $S(F) \subseteq A(F)$.

(iii) $I(F) \subseteq A(F)$.

In particular, $F \subseteq A(F)$. Now if C denotes the family of all closed sets, then $A(C)$ is the family of all analytic sets. Since $A(A(C)) = A(C)$, it follows that $S(A(C)) \subseteq A(A(C)) = A(C)$ and

$I(A(\mathcal{C})) \subseteq A(\mathcal{C})$; hence analytic sets are closed under countable unions and countable intersections. Also $\mathcal{C} \subseteq A(\mathcal{C})$, i.e. every closed set is analytic. From these observations, it can be seen easily that every Borel set is an A-set. The fact that analytic sets form a strictly larger class than the class of Borel sets was proved by Kuratowski in [10], §38, VI.

∂ is closed under countable unions and complementation. Proposition 24 implies that each class of disjoint sets in ∂-Γ is at most countable. These conditions coincide with conditions 1°,2° and 3' in E. Szpilrajn's paper [23], therefore ∂ is invariant under A-operations.

Proposition 59. $\partial = A(\partial)$.

Proof. ∂ is invariant under the A-operation, so that $A(\partial) \subseteq \partial$. This together with $\partial \subseteq A(\partial)$ implies that $\partial = A(\partial)$.

Theorem 60. *All analytic sets belong to* ∂.

Proof. Let $\mathcal{C}$ be the family of all closed sets. It is obvious that $A(\mathcal{C}) \subseteq A(\partial)$ for $\mathcal{C} \subseteq \partial$. Therefore $A(\mathcal{C}) \subseteq \partial$ by Proposition 59.

Sierpinski [20, p. 224] proved that every uncountable analytic set contains a non-empty perfect subset. Now suppose both H and H' are totally imperfect. If A is any uncountable set in Γ, then either $A \cap H$ or $A \cap H'$ is an uncountable set in ∂ which contains no non-empty perfect subset. Hence the following result

is proved.

Proposition 61. *Every uncountable set in* Γ *contains a non-analytic subset.*

Kuratowski in [10] defined the projective sets of class 0 as the Borel sets. The projective sets of class $2n+1$ are continuous images of the projective sets of class $2n$ (lying in the same space); the projective sets of class $2n$ are the complements of the projective sets of class $2n-1$. In particular, the projective sets of class 1, i.e. the continuous images of Borel sets, are called analytic sets; their complements, i.e. the projective sets of class 2 are called analytic complements. Kuratowski also proved that the definition he used for analytic sets and the definition of analytic sets as given here are equivalent. If L_n denotes the projective sets of class n, then L_0, L_1, L_2 belong to ∂. K. Gödel [3] proved that the hypothesis of existence of non-measurable sets of L_3 does not contradict the axioms of set theory. However no actual example of a non-measurable set of L_3 is known (see [24]). Thus it is impossible to prove that $L_3 \subseteq \partial$ by the usual system of axioms of set theory.

CHAPTER V

This chapter is devoted to the discussion of the local properties of sets. We use the definitions introduced by Kuratowski in [10]. Then we prove some interesting results.

Definition 62. *A is said to be of Γ at the point p if there exists a neighbourhood E of p such that $(A \cap E) \in \Gamma$. Let A^* denote the set of points at which A is not of Γ.*

Note that the neighbourhood E of p in the above definition may be required to be open since each neighbourhood E of p contains an open neighbourhood G of p. The following properties are given by Kuratowski in [10], §7, IV.

Proposition 63. *(i) A^* is closed.*

(ii) $A \subseteq B$ implies $A^* \subseteq B^*$.

(iii) $A^{**} \subseteq A^* = \overline{A^*} \subseteq \overline{A}$.

(iv) If G is open, then $G \cap A^* = G \cap (G \cap A)^*$.

(v) $(\bigcap_{\alpha \in T} A_\alpha)^* \subseteq \bigcap_{\alpha \in T} A_\alpha^*$ and $\bigcup_{\alpha \in T} A_\alpha^* \subseteq (\bigcup_{\alpha \in T} A_\alpha)^*$, where T is any set of indices.

(vi) $(A \cup B)^* = A^* \cup B^*$.

(vii) $A^* - B^* \subseteq (A - B)^*$.

Theorem 64. *$A \in \Gamma$ if and only if A is of Γ at each of its points.*

Proof. If $A \in \Gamma$, it is trivial that A is of Γ at each of its points. Now suppose A is of Γ at each of its points, $\mathbb{R}$

has a countable open base $R_1, R_2, \ldots$. For $p \in A$, let $R_{n(p)}$ be any basic open neighbourhood of p. Since $(A \cap R_{n(p)}) \in \Gamma$, this implies that $\bigcup_{p \in A} (A \cap R_{n(p)}) = A \in \Gamma$.

Proposition 65. *The following are equivalent:*

(i) $A \in \Gamma$.

(ii) $A \cap A^* = \phi$.

(iii) $A^* = \phi$.

Proof. Result follows easily from Theorem 64.

Proposition 66. $(A-A^*)^* = \phi$.

Proof. Proposition 63 (ii) implies $(A-A^*)^* \subseteq A^*$. Since $(A-A^*) \cap (A-A^*)^* \subseteq (A-A^*) \cap A^* = \phi$, then by Proposition 65, $(A-A^*)^* = \phi$.

Proposition 67. $A^*-A^{**} = \phi$.

Proof. By Proposition 63 (vii), $A^*-A^{**} \subseteq (A-A^*)^*$ and the result follows from Proposition 66.

Proposition 68. $A^* = A^{***}$.

Proof. Proposition 63 (iii) and Proposition 67 implies the result.

Proposition 69. $(A \cap A^*)^* = A^*$.

Proof. Since $A = (A-A^*) \cup (A \cap A^*)$, by Proposition 63 (vi), $A^* = (A-A^*)^* \cup (A \cap A^*)^*$. Therefore $A^* = (A \cap A^*)^*$.

Proposition 70. *If* $A \in \partial$, *then* A *contains a Borel set* B *such that* $A^* = B^* \subseteq \overline{B}$.

Proof. If $A \in \Gamma$, then $B = \phi$. Now suppose $A \in \partial\text{-}\Gamma$. Proposition 10 implies that $A = B \cup M$ for some $B \in \mathcal{B}\text{-}\Gamma$ and $M \in \Gamma$. This gives $A^* = (B \cup M)^* = B^* \cup M^*$. But $M^* = \phi$, hence $A^* = B^* \subseteq \overline{B}$.

Proposition 71. *If* $A_1 = \{x \in A : A$ *is of* Γ *at* $x\}$, *then* $A_1 \in \Gamma$.

Proof. Obviously $A_1 = A\text{-}A^*$, the result follows from Proposition 65 and Proposition 66.

Proposition 72. *If* $A^*\text{-}A \notin \Gamma$, *then* A *contains a set* K *such that* $K^* = \overline{K} = A^*$.

Proof. $A^*\text{-}A \notin \Gamma$ implies that $A \notin \Gamma$. Let $K = A \cap A^*$, obviously $K \neq \phi$ and $A\text{-}K \in \Gamma$. Also $\overline{K} \subseteq \overline{A}^* = A^*$ because A^* is closed. $A = K \cup (A\text{-}K)$ implies $A^* = K^* \cup (A\text{-}K)^* = K^*$. But $K^* \subseteq \overline{K}$ implies that $A^* \subseteq \overline{K}$, hence $A^* = K^* = \overline{K}$.

Proposition 73. *For any set* A, *there exists a Borel set* B *containing* A *such that for all* $C \supseteq A$ *with* $C^*\text{-}C \in \Gamma$, *then* $B\text{-}C \in \Gamma$.

Proof. Since $A\text{-}A^* \in \Gamma$ (Proposition 71), Axiom iii implies that there is a set K containing $A\text{-}A^*$ with $K \in \mathcal{B} \cap \Gamma$. Let $B = K \cup A^*$, then $B \in \mathcal{B}$ and $B \supseteq A$. Suppose $C \supseteq A$ and $C^*\text{-}C \in \Gamma$. Then $B\text{-}C = (K \cup A^*)\text{-}C = (K\text{-}C) \cup (A^*\text{-}C)$. But $K\text{-}C \subseteq K \in \Gamma$. Proposition 63 (ii) implies that $A^* \subseteq C^*$. Then $A^*\text{-}C \subseteq C^*\text{-}C \in \Gamma$. Therefore $B\text{-}C \subseteq (K\text{-}C) \cup (C^*\text{-}C)$ which means that $B\text{-}C \in \Gamma$.

CHAPTER VI

We have seen in Chapter II how elements in Γ behave as "negligible sets". In the present chapter we will exhibit the "negligibility" of sets in Γ from another approach. The work is taken from [14], a masterpiece written by J. C. Morgan II, who has generalized in that paper, the Banach-Mazur game. We enter his work here for completeness of the thesis.

Definition 74. *A family C of subsets of a non-empty set X is called an* M*-family if C satisfies the following axioms:*

1. The intersection of any descending sequence of C-sets is non-empty.

2. Suppose x is a point in X. Then (a) there is a C-set containing x, i.e. $X = \bigcup C$; and (b) for each C-set A, there is a C-set $B \subseteq A$ such that $x \notin B$.

3. Let A be a C-set and let D be a non-empty family of disjoint C-sets which has power less than the power of C. (a) If $A \cap (\bigcup D)$ contains a C-set, then there is a D-set D such that $A \cap D$ contains a C-set. (b) If $A \cap (\bigcup D)$ contains no C-set, then there is a C-set $B \subseteq A$ which is disjoint from all sets in D.

Definition 75. *Let C be a family of subsets of X. A subset S of X is singular with respect to C, or more briefly, C-singular, if each C-set A contains a C-set B disjoint from S. A countable union of C-singular sets is called a C_1-set.*

Game-theoretical definitions 76. θ *will denote the empty sequence. A play is a descending sequence* $\langle A_n \rangle_{n=1}^{\infty}$ *of C-sets, and the result of the play is the set* $\bigcap_{n=1}^{\infty} A_n$. *A strategy for player I is a function* σ *defined for all finite sequences of even length in C and the empty sequence such that* $\sigma(\theta)$ *is a C-set and, for* $n \geq 1$, $\sigma(A_1,\ldots,A_{2n})$ *is a C-set contained in* A_{2n}. *A strategy for player II is a function* τ *defined for all finite sequences of odd length in C such that, for each* $n \geq 1$, $\tau(A_1,\ldots,A_{2n-1})$ *is a C-set contained in* A_{2n-1}. *To each strategy* σ *for player I and each strategy* τ *for player II there is associated a play* $\langle \sigma,\tau \rangle$ *defined inductively as follows:* $A_1 = \sigma(\theta)$, *and* $A_{2n} = \tau(A_1,\ldots A_{2n-1})$ *and* $A_{2n+1} = \sigma(A_1,\ldots A_{2n})$ *for each* $n \geq 1$. *If* σ *and* τ *are strategies for players I and II, respectively, then a play* $\langle A_n \rangle_{n=1}^{\infty}$ *is consistent with* σ *if* $A_1 = \sigma(\theta)$ *and* $A_{2n+1} = \sigma(A_1,\ldots A_{2n})$ *for* $n \geq 1$; *consistent with* τ *if* $A_{2n} = \tau(A_1,\ldots A_{2n-1})$ *for* $n \geq 1$; *and consistent with* σ *and* τ *if it is consistent with both* σ *and* τ. *Let S be a subset of X. A strategy* σ *for player I is winning for player I in the game* $\gamma(S,C)$ *if, for every strategy* τ *of player II, the result of the play* $\langle \sigma,\tau \rangle$ *intersects S. A strategy* τ *for player II is winning for player II in the game* $\gamma(S,C)$ *if, for every strategy* σ *of player I, the result of the play* $\langle \sigma,\tau \rangle$ *does not intersect S. The game* $\gamma(S,C)$ *is determined if either player I or player II has a winning strategy.*

Morgan proved the following theorem.

Theorem 77. *Let C be an* M*-family of subsets of X and let S be a subset of X. Player II has a winning strategy in the game $\gamma(S,C)$ if and only if S is a C_1-set.*

Proposition 78. *The family $C = \{A : A' \in \Gamma\}$ is an* M*-family.*

Proof. (1) If $A_1 \supseteq A_2 \supseteq \ldots$ is a descending sequence of C-sets, then $(\bigcap_{i=1}^{\infty} A_i)' = \bigcup_{i=1}^{\infty} A_i'$ belongs to Γ. Therefore $\bigcap_{i=1}^{\infty} A_i \neq \phi$.

(2) (a) Since $\phi \in \Gamma$ implies $\mathbb{R} \in C$, $\mathbb{R} = \cup C$ trivially.

(b) Suppose $x \in \mathbb{R}$ and $A \in C$, then $A-\{x\} \in C$.

(3) It is sufficient to prove that if A and D are C-sets, then $A \cap D$ is also a C-set. This is because $(A \cap D)' = A' \cup D'$ which belongs to Γ. Hence $(A \cap D) \in C$.

Proposition 79. *S is C-singular if and only if $S \in \Gamma$.*

Proof. Suppose $S \in \Gamma$ and $A \in C$, then $(A-S)' = A' \cup S$ which belongs to Γ. Therefore $A-S \in C$. Now suppose S is C-singular then there exists a C-set A such that $S \cap A = \phi$. This implies $(S \cap A)' = S' \cup A' = \mathbb{R}$ which means that $\mathbb{R}-S' \subseteq A'$. But $A' \in \Gamma$, hence $\mathbb{R}-S' = S$ belongs to Γ.

Corollary 80. *S is C_1 if and only if $S \in \Gamma$.*

By Theorem 77 the following is true.

Theorem 81. *Let $C = \{A : A' \in \Gamma\}$ and let S be a subset of $\mathbb{R}$. Player II has a winning strategy in the game $\gamma(S,C)$ if and only if $S \in \Gamma$.*

Corollary 82. *$S \notin \Gamma$ if and only if there exists a C-set A such that for every C-set $B \subseteq A$, $S \cap B \notin \Gamma$.*

Proof. Since there exists a set B such that $S \cap B \notin \Gamma$, obviously $S \notin \Gamma$. Suppose for every C-set A, there is a C-set $B \subseteq A$ such that $S \cap B \in \Gamma$. Then player II has a winning strategy in the game $\gamma(S,C)$ and consequently $S \in \Gamma$.

Corollary 83. *If $S \in C \cup \Gamma$, then the game $\gamma(S,C)$ is determined.*

Proof. Suppose player II has no winning strategy, then $S \notin \Gamma$. Let $S = C \cup M$ where $C \in C$ and $M \in \Gamma$. Since $S' = C' \cap M' \in \Gamma$, then for any C-set A, player I has a winning strategy by choosing the C-set $A-(A \cap S')$.

Remark. Due to Theorem 81, Γ may be defined as the family of "negligible sets".

CONCLUSIONS

With slight modifications, this thesis can be generalized to the setting of a complete separable metric space with power N_1. Here certain questions are listed.

1. The family of nullsets and the family of all sets of first category satisfy the Axioms. But does there exist another non-trivial σ-ideal which satisfies the Axioms also?

2. Can Axiom v be derived from Axioms i to iv?

3. In Theorem 18, there is a bijection f of $\mathbb{R}$ to itself such that $f(A) \in \Gamma^*$ if and only if $A \in \Gamma$. Now under what conditions is it true that there exists a bijection g of $\mathbb{R}$ to itself such that $g(A) \in \partial^*$ if and only if $A \in \partial$, where $\partial^* = \{A \subseteq \mathbb{R} : A = B \Delta M^* \text{ for some } B \in \mathbb{B} \text{ and } M^* \in \Gamma^*\}$, and ∂ is defined as before?

4. Can the hypothesis "If $B \in \mathbb{B} \cap \Gamma$, then $x+B = \{x+b : b \in B\}$ belongs to Γ" be proved without adding more axioms?

5. A Vitali set (for definition, see [18]) is non-measurable and it does not possess the property of Baire. Is it possible also to prove that it does not belong to ∂, even if we assume the hypothesis "If $B \in \mathbb{B} \cap \Gamma$, then $x+B = \{x+b : b \in B\}$ belongs to Γ"?

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