BEYOND RECURSIVE PATTERNING: VISUAL REPRESENTATIONS TO PROMOTE ALGEBRAIC THINKING WITH PRIMARY STUDENTS

by

Kate Cooper

A Thesis

submitted in partial fulfillment of the requirements

for the degree of

Master of Education

FACULTY OF EDUCATION

LAKEHEAD UNIVERSITY

THUNDER BAY, ONTARIO

November 2015

Abstract

This case study examined the impact of using designed visual representations of simple linear functions on Grade 2/3 students' development of algebraic thinking. I also looked for the connections between the use of visual representations of linear functions and young students' understanding of multiplication. A pre-assessment interview, five-lesson intervention, and post-assessment interview sequence was used over the span of one and a half weeks with a retention task that followed approximately two weeks later. The five-lesson intervention was developed to prominently feature designed visual representations, along with other representations (e.g. table of values, pattern rules, narrative contexts), of simple linear functions to encourage students' development of explicit reasoning skills. All students developed some level of explicit reasoning and were able to generalize about simple linear functions by the end of the study. Most students moved beyond recursive thinking and were able to generate and apply explicit pattern rules for simple linear functions in order to generalize about any term within that function. Students' development of explicit reasoning in order to work with simple linear functions often sparked a need for a new operation: the invention of repeated addition and, or, multiplication. Activities involving explorations of simple linear functions that prominently feature designed visual representations led most of the students in the study to develop explicit reasoning skills and an early understanding of multiplication.

Acknowledgements

There are many people without whom the completion of my thesis would not have been possible and I would like to take the time to express my gratitude. First, I would like to thank Dr. Alex Lawson, my supervisor, who encouraged me to begin my journey as a Master of Education student and who was also there to support me every step of the way. Your patience, expertise and thoughtful advice have been instrumental in the completion of this thesis and have helped me to grow in countless ways as both a learner and an educator.

I am also very grateful to the amazing teacher who participated in this study. You were essential to both the development and implementation of this research project. Thank you for providing me with the wonderful opportunity to learn from you and your students.

In addition, I would like to thank my committee member Dr. Joan Chambers, my internal examiner Dr. Ruth Beatty and my external examiner Dr. Ann Anderson for their valuable guidance and thoughtful feedback throughout this process.

Finally, I must thank Harley Cooper, my continuously supportive husband, for always helping me to see the bigger picture and for pushing me to consider alternative perspectives. I am also grateful to my parents and first teachers, Karen and Neil Gillies, for their continual encouragement and for teaching me to ask questions.

Table of Contents

List of Tables

List of Figures

List of Appendices

Chapter 1: Introduction

Algebra has historically been an agent of inequality; it is a gateway to higher mathematics that is often only accessible to the elite segments of a population (Kaput, 2008; Stephens et al., 2013). Many researchers have asserted that it is poor instruction, rather than limited ability that has caused this situation (Carpenter, Loef Franke, & Levi, 2003). Algebra is usually introduced in the middle grades or high school years as an isolated area of mathematics limited to the memorization of meaningless procedures (Carraher, Schliemann, & Schwartz, 2008; Kaput, 2008). The memorization of procedures has long been a central tenet of school mathematics (Van de Walle, Folk, Karp, & Bay-Williams, 2011). However, the National Council of Teachers of Mathematics (NCTM), along with numerous educators and researchers, have been calling for curricular and instructional shifts towards *reform* oriented mathematics instruction for two decades (Battista, 1994).

More recently, researchers have called for a restructuring of algebra and its introduction into mathematics curricula in the primary grades through meaningful activities connected to arithmetic (Carpenter, et al., 2003). We have some limited evidence that early algebra is not only possible, but also lays the foundation for a greater understanding of many areas of mathematics for many students in the later grades (Carpenter, et al., 2003; Stephens et al., 2013). It has been suggested that this early algebra would revolve around helping students to develop a meaningful understanding of symbol systems, equation structure, number patterns and constructing generalizations that apply to all numbers (Carpenter, et al., 2003; Carraher, et al., 2008; Ferrini-Mundy, Lappan, & Phillips, 1997; Schifter, Russell, & Bastable, 2009).

The study of number patterns and relationships, or functions, from the primary grades through to the intermediate grades, has been identified as one area of mathematics that can provide a rich context for learning algebraic concepts and developing the ability to reason algebraically (Beatty & Bruce, 2012; Beatty, Day-Mauro, & Morris, 2013; Ferrini-Mundy, et al., 1997; Moss & London McNab, 2011). Functions have been one of the reoccurring themes that make up high-level studies of algebra yet they are often a source of difficulty for students who have been taught algebra following methods of traditional instruction (Beatty & Bruce, 2012). Therefore, researchers have suggested that it is important for educators to approach studies of functions in a fundamentally different way; functions need to be taught in such a way that it encourages the development of mathematical power among young people (*mathematical power* refers to the notion that students should be encouraged to look for connections and become flexible thinkers who are capable of working through complex problems). Beatty and Bruce further contend that the use of multiple representations can make the study of functions accessible to students of all ages; moreover, experience working with multiple representations can support the development of algebraic thinking.

1.1 The Context: Reform Mathematics Education

These calls for change in algebra instruction come out of a more global call for change or reform in mathematics instruction. The reform movement of mathematics instruction and curricula development was born out of an evaluation of mathematics education by the NCTM in 1989 (Battista, 1994). The NCTM is an organization comprised of teachers, mathematics educators and researchers from both Canada and the United States (Van de Walle, et al., 2011). In 1989 the NCTM released the document

Curriculum and Evaluation Standards for School Mathematics, which called for a major rethinking of both the content of mathematics curricula and the ways in which teachers viewed teaching and learning (Battista, 1994; Van de Walle, et al., 2011). This curricular and instructional rethinking of mathematics education was referred to as the reform movement or reform mathematics (Van de Walle, et al., 2011).

The reform movement represents a shift from a curricular focus on mathematical procedures to a focus on mathematical understanding. The movement calls for teachers to abandon mathematics curricula that focus solely on pen-and-paper computation. Instead teachers are asked to focus the mathematical learning of students on problem solving and developing conceptual understandings of mathematical ideas (Battista, 1994; Hiebert et al., 1996; Van de Walle, et al., 2011). One of the main goals of the reform movement is "to replace the current obsolete, mathematics-as-computation curriculum with a mathematics curriculum that genuinely embraces conceptual understanding, reasoning, and problem solving as the fundamental goals of instruction" (Battista, 1994, p. 463). The reform movement was born out of a realization that a curricular focus on computational skills alone did not ensure that students understood the concepts behind the required procedures, and that conceptual understanding was necessary as these students pursued higher-level studies of mathematics (Battista, 1994; Hiebert et al., 1996; Van de Walle, et al., 2011). While early calls for change focused largely on number sense, as the movement gained strength other areas such as algebra were included in the discussion.

1.1.1 The role of constructivist and sociocultural theories of learning. The reform movement is underpinned by constructivism and the sociocultural theory of learning. Within a constructivist lens, learning is believed to occur through active construction rather than passive reception of knowledge. Students cannot simply absorb the teacher's knowledge through repetition of the teacher's mathematical thinking. Reform mathematics is therefore underpinned by the belief that mathematics can be made more accessible to students if, among other things, it builds on the student's prior knowledge and involves a relevant context. Reformers assert that students will construct stronger mathematical understandings when teachers expect them to act as mathematicians and engage in worthwhile mathematical tasks (Geist, 2000; Hiebert et al., 1996). They also believe that students acquire knowledge through interactions with others; social interactions are the medium through which mathematical knowledge is explored, refined and then integrated with that individual's existing knowledge (Smith & Stein, 2011). The movement towards reform mathematics education is based on two central beliefs: teaching through problem solving will support students to construct their own understandings; and, productive discussions will help children to refine, adjust and precisely communicate their mathematical understandings (Chapin, O'Connor, & Anderson, 2009; Hiebert et al., 1996; Hufferd-Ackles, Fuson, & Gamoran Sherin, 2004; Smith & Stein, 2011).

1.2 Purpose of the Study

The purpose of this study was to explore the ways in which patterning activities that involve simple *linear functions* (a directly proportional relationship between two sets of data which are connected by a pattern rule) can be used to promote the algebraic thinking in primary students by capitalizing on the basic tenets of reform mathematics instruction and newer ideas presented in recent research on the role of representations in early algebra, patterning and generalization. This study comprised a Grade 2/3 class and the impact of a patterning unit on the students' abilities to reason and communicate ideas algebraically. The patterning unit was taught with an emphasis on the development of *explicit reasoning* (focusing on the functional relationship between multiple sets of data) and determining patterning rules rather than *recursive thinking* (finding the next term in a pattern or attending to patterns in only one set of data).

1.3 Research Questions

In what ways do primary students use or develop explicit reasoning skills when examining linear functions using designed visual representations?

How does working with these visual representations of linear growing patterns affect the algebraic thinking of young students?

- In what ways can students move beyond additive or recursive thinking when working with linear functions?

How do the various representations of linear growing patterns help or encourage students to invent multiplication?

1.4 Significance of the Study

There is some evidence that primary students are capable of developing and employing algebraic reasoning when working with linear functions (Beatty, et al., 2013; Blanton & Kaput, 2011; Moss & London McNab, 2011). Some researchers also suggest that early and continuous exposure to patterning activities that promote student fluency with multiple representations of linear functions are beneficial to the mathematical development of children (Beatty & Bruce, 2012; Blanton & Kaput, 2011). It is accepted that early algebra must be approached very differently than the traditional and rote algebra that has long been common to high school mathematics classrooms (Carpenter, et al., 2003; Carraher, et al., 2008; Kaput, 2008; Stephens et al., 2013). There is however, still much to be learned about the kinds of patterning activities that are mathematically worthwhile when teaching young children; there is only limited research that has been done on the types of activities that support the development of algebraic reasoning in young children. Searches of education databases, such as ERIC or CBCA, return few related articles with the search terms “linear function” or “linear growing pattern” and “early algebra” or “early algebra education.” This study will contribute to the knowledge base on primary students’ capacities for algebraic reasoning. This study will also add to the research on the role and effectiveness of patterning activities that employ multiple representations of simple linear functions in early algebra.

1.5 Limitations of the Study

The main limitations of this research project include the design of the project and the interview tasks that were used. The study employed an embedded case study design and as such, it provided rich and context-based descriptions of six case studies bound by one overall study and context (Yin, 2009). As a result I did not attempt to generalize findings to other groups of students. Instead, the data were used to try and better understand the issue at hand (Baxter & Jack, 2008), that is: how can designed visuals of simple linear functions be used to support young students’ development of explicit reasoning skills? The class of Grade 2/3 students who participated in the study were not considered to be representative of all students at the Grade 2 or 3 level. This case study was also not designed to compare the effectiveness of various types of instruction methods; instead the intent of the study was to examine six representative children from one class as they worked with simple linear functions to determine whether or not multiple representations of such patterns encouraged the students to reason algebraically about the relationships underlying the patterns.

The pre and post-assessment interviews differed from the questions students worked on during the five-lesson intervention. Although all were designed to address the same content at a comparable difficulty level, the differences in wording and numbers used may have impacted student achievement on each task. The students were also video recorded for the pre and post-assessment interviews; all attempts were made to reduce the intrusiveness of the video recorder on student-researcher interactions. However, the presence of the video recorder may have influenced student responses on the interview tasks. The students may have also become more comfortable with the presence of the video recorder by the end of the unit, leading to possible inconsistencies across the data.

Chapter 2: Literature Review

2.1 Algebra: How Important is it in the Study of Mathematics?

Often, a student's mastery of algebra is viewed as an indicator of their potential for success in advanced mathematics (Kaput, 2008; Stephens et al., 2013). Many students are prevented from pursuing a high school or university level mathematics education due to a lack of understanding of algebraic concepts (Stephens et al., 2013). Secondary students commonly view algebra as a disconnected area of mathematics and yet it "pervades all of mathematics and is essential for making mathematics useful in daily life" (Van de Walle, et al., 2011, p. 262). Algebra is involved in nearly all of the mathematics that an individual will encounter in later life and it has a central role in the majority of university, college and high school level mathematics courses (Kaput, 2008). Therefore, students of all ages need to be provided with opportunities to form a strong foundation of the algebraic understandings upon which they can build the capacity for complex mathematical thought (Carpenter, et al., 2003). Considering the importance of algebra and the impact it can have on one's mathematical future, it is necessary to first determine: *what is algebra?*

2.2 Defining Algebra and its Centrality in Mathematics

The way that teachers, administrators and policymakers understand and define algebra or algebraic reasoning has a huge impact on the ways in which they will approach algebra instruction (Kaput, 2008). Kaput (2008) argued that it is important that individuals involved in mathematics education come to realize "algebra's breadth, richness, and organic relation to naturally occurring human cognitive and communicative powers" (p.8). A more comprehensive understanding of what it means to think algebraically will make it possible for teachers to use algebra to strengthen and add depth to the existing curriculum (Blanton & Kaput, 2003; 2011). It is therefore important for teachers, researchers and administrators to also recognize the diversity within the field of algebra.

While there are different ways to organize the field of algebra, many mathematics educators and researchers delineate a number of common areas:

- *symbolization* (working with and manipulating variables);
- *equality and conjectures* (understanding that the equal sign represents a mathematical relationship where two expressions have the same value and using the property of equality to argue that some statements may be true for all numbers);
- *patterns and rules* (repeating and growing patterns as well as mathematical functions); and,
- *representations* (representing mathematical relationships in many ways such as through the use of graphs, tables, equations, narratives and images) (Chapin & Johnson, 2000).

These four areas of algebra are interconnected yet different.

2.2.1 Symbolization. Many researchers assert that symbolization is strongly linked to all other areas of algebra (Chapin & Johnson, 2000; Kaput, Blanton, & Moreno, 2008). It is difficult for students to develop an understanding of equality, pattern rules, and representations if they do not have a strong grasp of the symbol systems that are the cornerstones of algebra (Chapin & Johnson, 2000; Kaput, et al., 2008). Kaput (2008) suggests that some form of symbolization or generalization is required in order for a reasoning process to be termed *algebraic*. He contends "that the heart of algebraic reasoning is comprised of complex symbolization processes that serve purposeful generalization and reasoning with generalizations" (Kaput, 2008, p. 9). However, symbolization is not limited to the manipulation of variables or of typically written symbols (e.g. x, $f(x)$, $=$, $\sqrt{\ }$). Instead symbolization is embodied in any reasoning process where the individual looks *through* symbols or uses some type of symbolization or generalization to express, manipulate, or analyze a mathematical idea (Kaput, 2008, Kaput et al., 2008). This ability to use symbols facilitates an individual's ability to work within the other areas of algebra.

While symbolization is important because it provides the foundation for learning in other areas of algebra, the ability to express and analyze mathematical ideas symbolically is itself mathematically powerful. Algebra and symbolization are inextricably linked because it is symbolization that makes it possible for individuals to communicate mathematical generalizations, which are central to the study of algebra (Kaput, et al., 2008). Mathematical generalizations are dependent upon and are the inspiration for symbolization; it is through generalization and the symbolization of these generalizations that one may begin to reason algebraically (Kaput, et al., 2008; Radford, 2011). However, some researchers argue it is possible to reason algebraically without the use of formal symbol systems because "[thinking is characterized] as algebraic [when] it deals with *indeterminate* quantities conceived of in *analytic* ways" (Radford, 2011, p. 310). It is important to realize that it is possible for students to engage in mathematical symbolization and generalization (e.g. through the use of a detailed diagram or the use of a simple shape to represent an unknown variable) without knowledge of the traditional or formal systems of algebraic notation or with an incomplete understanding of these systems.

2.2.2 Equality and conjectures. As previously mentioned, many researchers in the field of algebra education agree that symbolization is important and there is one symbol – the equal sign – that has been the focus of a great deal of research (Carpenter, et al., 2003; Falkner, Levi, & Carpenter, 1999; Kieran, 1981). Many researchers suggest that a comprehensive understanding of the meaning of the equal sign, and the property of equality that it represents, will help students to efficiently manipulate, extend and understand equations (Carpenter, et al., 2003; Falkner, et al., 1999; Fosnot & Jacob, 2010). Developing a strong understanding of equality, and what it means for expressions to be equal, is fundamental to understanding the structure of equations (Carpenter, et al., 2003; Fosnot & Jacob, 2010).

A strong understanding of the equal sign is also essential to the process of recognizing and precisely communicating conjectures about numbers and operations (Carpenter, et al., 2003). A conjecture is a mathematical hypothesis such as, *addends can be reversed and still result in the same sum*, or more precisely: $a + b = b + a$. Conjectures can help students make their mathematical knowledge explicit through the use of precise language, which is central to the communication of conjectures and the study of mathematics more generally (Carpenter, et al., 2003; Schifter, et al., 2009). Furthermore, “in the process of carefully articulating, refining, and editing conjectures, students confront important mathematical ideas and engage in basic forms of mathematical argument” (Carpenter, et al., 2003, p. 48). Through discussions that surround the process of proving a conjecture, students come to value the importance of their word selection when explaining their ideas and consequently develop a more thorough understanding of the mathematical principle in question. This need for precise language can also lead to the student's realization that the use of variables and symbols can aid in the clear communication of mathematical ideas (Carpenter, et al., 2003; Russell, Schifter, & Bastable, 2011).

2.2.2.1 Equality and relational thinking. Although equality is the foundational knowledge required to make sense of the structure of equations, a strong understanding of this property will also encourage students to develop the ability to recognize and efficiently capitalize on the relationships that exist within our number system (Carpenter, et al., 2003). A comprehensive understanding of equality and the meaning of the equal sign encourages the development of relational thinking; an understanding of equality allows individuals to view mathematics as a field to explore interconnected number relations (Carpenter, et al., 2003; Fosnot & Jacob, 2010). Relational thinking is based on constructing an understanding of the relationships that exist between real numbers and the operations. For example, a student who recognizes the relationships between numbers and operations may recognize: $4 \times 7 = 7 + 7 + 7 + 7$ (Carpenter, et al., 2003). A relational approach to mathematics encourages the comparison of expressions and helps individuals realize that they do not always need to immediately carry out a calculation when working with the equal sign (Carpenter, et al., 2003). For example, students can solve $7 + 8 = 6 + \square$ by thinking relationally about what number is 1 more than 8 rather than completing a string of calculations: $7 + 8 = 15$ and then $15 – 6 = 9$. Students who have strong relational thinking skills will then be able to better understand how to manipulate and solve traditional algebraic equations in later schooling because they will have a strong understanding as to how the relationships between real numbers and how the operations apply to the variables and symbols traditionally used in algebra (Carpenter, et al., 2003; Fosnot & Jacob, 2010; Stephens et al., 2013).

2.2.3 Functions and representations. Although symbolization and equality are important in the study of algebra, patterning activities often comprise the first encounters that students will have with algebra (Chapin & Johnson, 2000). Certain types of patterns can be represented as mathematical functions and these representations can be communicated through the use of various algebraic symbols, which require an understanding of the property of equality (Chapin & Johnson, 2000). Functions are a type of mathematical relationship "in which two sets are linked by a rule that pairs each element of the first set with exactly one element of the second set" (Chapin & Johnson, 2000, p. 131). In the early elementary grades, students may encounter growing patterns, which are an earlier version of linear functions. Functions are a central aspect of high school and post-secondary algebra and they are a part of our daily lives. For example, the relationship between the amount of tax and the cost of an item is a functional relationship that we encounter on a daily basis (Chapin & Johnson, 2000).

The ability to recognize, manipulate and work with functions is central to the study of algebra and mathematics in general (Beatty & Bruce, 2012; Beatty, et al., 2013; Blanton & Kaput, 2011; Ferrini-Mundy, et al., 1997). Beatty and Bruce (2012) suggest that the study of linear functions, or growing patterns, "helps students develop a deep understanding of relationships among quantities that underlie mathematical relationships. It also helps students develop the capacity for generalizing" (p.7). A strong understanding of linear functions provides students with mathematical knowledge that is the foundation for many areas of high-level mathematics, algebra included (Beatty & Bruce, 2012; Blanton & Kaput, 2011).

Many researchers have suggested that early experiences with patterning are valuable because they encourage students to attend to the relationships between sets of data; the study of patterns focuses student attention on the dependent relationships that underlie linear functions (Beatty & Bruce, 2012; Blanton & Kaput, 2011; Moss & Beatty, 2006). Patterns and linear functions are also powerful because they provide a context from which students can begin to generalize and symbolize their mathematical ideas in a meaningful way (Beatty & Bruce, 2012; Blanton & Kaput, 2011; Coulombe & Berenson, 2001; Moss & Beatty, 2006). Working with linear functions helps students to develop the ability to engage in functional thinking where they are "building and generalizing patterns and relationships using diverse linguistic and representational tools and treating generalized relationships, or functions, that result as mathematical objects useful in their own right" (Blanton & Kaput, 2011, p. 8). Patterning is important in the early elementary grades because it encourages students to make generalizations by *looking through* the different symbols that may be used to represent a pattern (Beatty & Bruce, 2012; Blanton & Kaput, 2011; Kaput, et al., 2008).

Different representations of these functions can be used to analyze and understand various characteristics of the function (Carraher, Schliemann, & Brizuela, 2006; Chapin & Johnson, 2000). Representation refers to the communication of functional relationships through graphs, symbols, pictures, verbal explanations, tables and pattern rules or equations (Beatty & Bruce, 2012; Chapin & Johnson, 2000). In order to create and analyze the many different representations of functions, a strong understanding of equality and symbol systems is required. Algebra curricula that encourage the study of functions and representations, patterns, equality and symbolization have been proposed as a solution to the problems surrounding traditional algebra instruction (Carraher, et al., 2006; Chapin & Johnson, 2000).

2.3 The Problem of the Traditional Instruction of Algebra

Researchers focusing on algebra instruction have identified what they characterize as *the algebra problem,* which refers to ineffective algebra instruction that is causing students to be ill prepared for advanced mathematics (Kaput, 2008). *The algebra problem* refers to the fact that students often have a fragile understanding of algebraic concepts due to the procedural ways in which algebra is typically taught (Kaput, 2008; Stephens et al., 2013). According to Kaput (2008), algebra education, like any other form of education, has been shaped by both historical and societal contexts. Algebra has traditionally followed arithmetic and often it is limited to the manipulation of symbols based on specific rules. This notion that algebra must follow arithmetic dates back to the start of the 20th century. At that time, students were required to complete only elementary school, and the majority of the population was expected to know only basic arithmetic. Therefore, algebra was accessible to only the elite segments of the population who were able to attend secondary school (Kaput, 2008). When high school was made publicly accessible, the historical context in which algebra was initially integrated into school curricula continued to negatively affect the ways in which algebra was taught and presented. As a result, algebra largely remains an isolated, procedure-based area of the mathematics curriculum that is inaccessible to many students, especially those who are socially or economically disadvantaged (Kaput, 2008; Stephens et al., 2013).

2.3.1 Early algebra: Addressing the algebra problem. Researchers focusing on algebra education have suggested that algebra could be made accessible to the majority of students if it was introduced in the early elementary years and focused on symbolization, equality, patterns and representations, as mentioned earlier (Carpenter, et al., 2003; Carraher, et al., 2008; Chapin, et al., 2009). Many researchers believe that algebra should be treated "as a K-12 strand, as opposed to an isolated eighth- or ninth-grade course, as a way to…ensure that more students have access to algebra and the opportunity to be academically and economically successful" (Stephens et al., 2013, p. 173). We have evidence from current research that algebra in the elementary grades, or early algebra, would be beneficial to students' understanding of algebraic concepts and could help them to develop algebraic reasoning skills (Carpenter, et al., 2003; Kaput, 2008). Researchers suggest that the early introduction of algebra is part, but not all, of the solution. This early algebra instruction must also be inherently different from traditional algebra courses and it should make use of symbolization, equality and conjectures, patterns and rules, and representations (Chapin & Johnson, 2000). There is extensive existing research on the ways in which instruction can be designed to develop students' understandings of symbolization, equality and conjecture (e.g. Carpenter, et al., 2003; Carraher, et al., 2008; Chapin, et al., 2009; Falkner, et al., 1999; Fosnot & Jacob, 2010; Schifter, Monk, Russell, & Bastable, 2008; Schifter, et al., 2009; Stephens et al., 2013). However, gaps in the research are present concerning the ways in which early algebra instructions should be designed to address the areas of functions and representations.

2.4 Developing an Early Algebra Curriculum in Functions and Representations

Since the NCTM has suggested that algebra should have a place in mathematics curricula from Kindergarten to Grade 8[1], researchers and mathematics educators have been trying to determine the best ways to approach algebra in the elementary years (Carraher, et al., 2008; Chapin & Johnson, 2000). There are many problems with existing algebra curricula because curriculum developers and those who created mathematics textbooks did not traditionally consider the developmental needs and abilities of children when designing materials (Fosnot & Jacob, 2010). Instead, mathematics and algebra were viewed as a process of accumulating sets of skills so that one would be able to perform algebraic procedures (Fosnot & Jacob, 2010; Kaput, 2008). However, this view of what it means to understand algebra is too simplistic as algebra involves a great deal more than the carrying out of multiple procedures (Fosnot & Jacob, 2010; Kaput, et al., 2008; Kaput, 2008). Instead of a focus on the memorization of procedures, early algebra instruction must encourage students to explore patterns, work with symbols and make generalizations leading to a strong understanding of functions and representations (Carraher, et al., 2008; Chapin & Johnson, 2000). Early algebra curricula suggested by various researchers tend to follow two paths: one path involves generalized arithmetic (e.g. Carpenter, et al., 2003), and the other focuses on patterns and functions (e.g. Beatty & Bruce, 2012; Beatty, et al., 2013). A great deal of research has been conducted to explore the path involving generalized arithmetic, however less is known about the path toward algebra in the early grades that involves the study of patterns and functions.

---

[1] See the recent call in 2011 for submissions for a focus issue on beginning algebra in the NCTM's journal *Mathematics Teacher* http://www.nctm.org/publications/article.aspx?id=30915

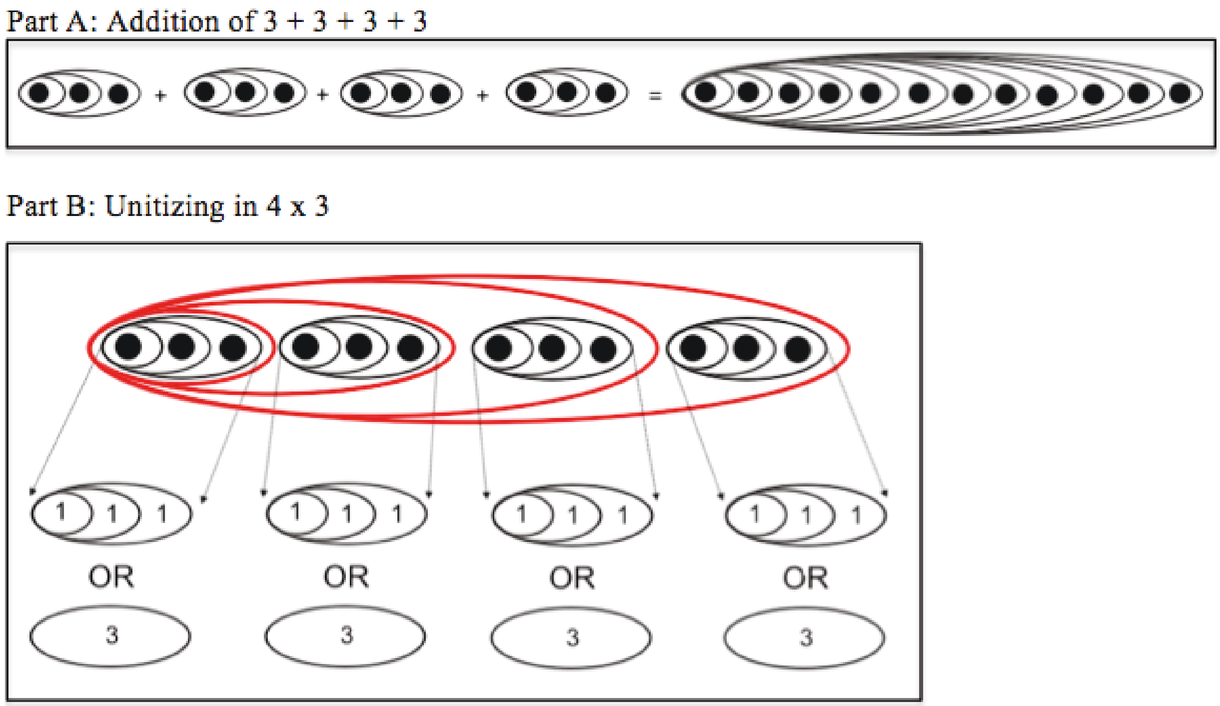
2.4.1 Common student difficulties with patterning activities. One entry point into early algebra is the study of early patterning; however, researchers and educators have found that many elementary students currently struggle with patterning concepts (Beatty & Bruce, 2012; Beatty, et al., 2013; Moss & Beatty, 2006; Moss & London McNab, 2011; Noss, Healy, & Hoyles, 1997). This is often due to the ways in which patterning exercises are presented to students (Beatty & Bruce, 2012; Beatty, et al., 2013; Moss & Beatty, 2006; Noss, et al., 1997); patterning in the elementary grades is often limited in scope and attention is "focused on the numeric attributes of the output" (Noss, et al., 1997, p. 205). When teachers reduce a patterning activity to an arithmetic task, where students are simply required to find a few values based on a pattern, the students are not encouraged to generalize or think mathematically (Beatty & Bruce, 2012).

2.4.1.1 Reliance on recursive thinking. The first problem that students may encounter when working with patterning activities is that they may rely too heavily on recursive or additive thinking when trying to determine the rule for a pattern (Beatty & Bruce, 2012; Beatty, et al., 2013; Moss & London McNab, 2011). Recursive thinking allows students to determine the next terms in a pattern using the previous term but it limits the student's ability to create a generalized rule for the pattern. For example, a child looking at the following table of values (see Figure 1) could determine the next output simply by looking at the output column and adding on another 2, rather than thinking multiplicatively and looking for relationships between the inputs and outputs to determine that any input number multiplied by 2 plus 1 will give the output. Researchers working in Canadian and American primary classrooms have found that patterning activities often only require recursive or additive thinking; patterning activities in textbooks commonly use a table like that in Figure 1 and ask things like *what is the next output?* rather than asking *what is the output for an input of 25?* or *what is the output for any input?* (Beatty & Bruce, 2012; Blanton & Kaput, 2011; Moss & London McNab, 2011). If the student is simply required to look over the table of values and determine which number would come next, they will develop a narrow understanding of functions and patterns in general. The prevalence of tasks that involve ordered tables of values and only require students to find the next term of a pattern are in part responsible for students' reliance on recursive thinking and their limited understandings of functions. Instead students need to be encouraged to explore in depth the relationships between the two sets of data that are a part of the function (Beatty & Bruce, 2012; Beatty, et al., 2013). They need to learn how to work *across* the table rather than only being able to work *down* the output column.

Input	Output
1	3
2	5
3	7
4	9
5	

Figure 1. Table of values for the linear function *Output* = *Input* x 2 + 1

2.4.1.1.1 Challenges with multiplication. Linear functions involve two sets of data that are connected through a pattern rule that involves multiplication and sometimes the addition or subtraction of a constant; however, many students struggle to move beyond recursive thinking because the operation of multiplication is difficult for young students to conceptualize when it is first introduced in the primary years (Van de Walle et al., 2015). In the early primary grades, students spend a great deal of time developing an understanding of addition and subtraction which is why they feel comfortable using recursive thinking to work with linear functions. They explore the part-whole relationships involved in these operations and they come to conceptualize addition as the repeated adding of units of one. At the same time, these units of one are also involved in inclusion (or hierarchical) relationships: 1 is included within 2 and 2 is contained within 3 and so on (see Figure 2, Part A). When adding 3 + 3 + 3 + 3 (as shown in Figure 2 Part A), "the groups are combined *successively*, on one level, as 3 + (3 more ones)" and so on until the correct number of ones have been added (Clark & Kamii, 1996, p. 42). When students are expected to then develop multiplicative reasoning, they often build on what they know about addition (Young-Loveridge, 2005). However, the structures involved in multiplication are much more complex than those present in addition because there are more inclusion relationships occurring (Clark & Kamii, 1996). For example, when thinking about 4 x 3, within a group of 3 there are 3 units of 1 where the 1 is included in 2 and the 2 is included in 3 (see Figure 2 Part B). At the same time, the groups of 3 can also be conceptualized as one unit: 1 group of three is included in 2 groups of three, 2 groups of threes are included in 3 groups threes, and finally 3 groups of threes are included in 4 groups of threes (the red ovals in Figure 2 Part B). Researchers agree that the transition from additive reasoning to multiplicative reasoning is difficult for students because "a major conceptual hurdle [that students face] in working with multiplicative structures is understanding groups of things as single entities while also understanding that a group contains a given number of objects" (Van de Walle et al., 2015, p.157). Researchers such as Clark and Kamii (1996) use the term *many-to-one correspondence* to refer to the idea that one can have a unit that is both one group and three individual objects, where as other researchers such as Lamon (1996) and Fosnot and Jacob (2010) use the term *unitizing* (see Figure 2, Part B). Due to its prevalence in more recent research, for the purpose of this project, the term *unitizing* will be used whenever referring to this concept. It is essential for students to develop a strong conceptual understanding of unitizing and the complex part-whole and inclusion relationships involved in multiplication before they will be able to effectively use or apply multiplicative reasoning within the context of linear functions (Chapin & Johnson, 2000; Clark & Kamii, 1996; Fosnot & Jacob, 2010; Young-Loveridge, 2005). It is also valuable for students to have time to explore working with groupings in contextual situations before they are expected to generalize the idea that repeated addition is one way of conceptualizing multiplication (i.e. it is hard for students to make the jump that 3 + 3 + 3 + 3 is the *same* as 4 x 3) (Clark & Kamii, 1996; Van de Walle, et al., 2011).



(Adapted from Clark & Kamii, 1996, p. 42)

Figure 2. Comparing structures of addition and multiplication

2.4.1.2 Whole-object reasoning. The second problem that students may experience when working with patterns is that they "may use what is referred to as the whole-object strategy" (Beatty & Bruce, 2012, p. 5). This means that some elementary students may use proportional reasoning in ways that may not be correct in the context of the function— particularly if the function involves a constant (Beatty & Bruce, 2012; Lannin, 2005; Moss & London McNab, 2011). For example, when working with the pattern shown in Figure 1, if asked, *Given an input of 8, what would the output be?* some primary students may use a whole-object strategy to double the output for an input of 4 (which is 9) and state that the output for an input of 8 would be 18 (Moss & London McNab, 2011). A student who makes this type of error is incorrectly using proportional reasoning and failing to attend to the relationship between the input and output numbers.

2.4.1.3 Difficulty justifying pattern rules. The third problem that students face when working with patterns is that they may not feel confident enough to explain and justify the rules they have identified for a pattern (Beatty & Bruce, 2012). Students are often hesitant to discuss or refine the rules that they formulate for patterns and this can limit their understanding of patterning in general. Researchers have suggested alternative curricula to address the three problems.

2.4.2 A more effective curriculum for linear functions. Some researchers suggest that when students are beginning to work with algebra, it is valuable for them to have a familiar context from which to construct representations of mathematical ideas (Coulombe & Berenson, 2001). Mathematicians commonly place high value on the use of visualization and they mentally employ many different representations and symbolizations of mathematical ideas before moving to transcribe these representations using algebraic notation (Noss, et al., 1997). However, traditional algebra instruction limits opportunities for visualization and context-based representations while placing high value on mathematical products or completing a procedure to find the answer (Coulombe & Berenson, 2001; Noss, et al., 1997). Some argue that students need to be provided with opportunities to explore symbolic representations of their mathematical thinking so that they will have the “cognitive room to explore more complex ideas in later elementary grades” (Blanton & Kaput, 2011, p. 12). There is some existing research that suggests primary students are capable of using forms of symbolic notation and algebraic reasoning (Beatty, et al., 2013; Blanton & Kaput, 2011; Moss & London McNab, 2011).

Many researchers suggest that even in the early elementary years, teachers can support their students' algebraic development and nurture their capacities to represent and generalize mathematical ideas (Blanton & Kaput, 2003; Blanton & Kaput, 2011; Carpenter, et al., 2003; Kaput, et al., 2008; Moss & London McNab, 2011). Blanton and Kaput (2011) suggest "instruction should begin to scaffold students' thinking toward symbolic notation from the start of formal schooling so that students can transition from an opaque to transparent use of symbols as they progress through the elementary grades" (p.14). In the early elementary years, students need to be provided with opportunities to formulate mathematical generalizations and employ symbolization throughout the process of communicating, proving and justifying generalizations. How can teachers scaffold children as they make algebraic generalizations and use symbolization in the process of developing an understanding of linear functions?

2.4.2.1 Representations or mathematical models? Many researchers have suggested that mathematical models (Fosnot, 2007; Fosnot & Jacob, 2010; Gravemeijer, 2002) and representations (Beatty, 2010; Beatty & Bruce, 2012; Chapin & Johnson, 2000; Moss & London McNab, 2011) can be used to help children construct an understanding of, and proficiency with, algebra. Although mathematical models and mathematical representations overlap and are interrelated, there are some differences when applied to the area of early algebra and the study of linear functions.

Mathematical models introduced in the elementary grades are often called emergent models, meaning that they are built upon a specific problem context that can eventually be applied to multiple situations or problems (Fosnot, 2007; Fosnot & Jacob, 2010; Gravemeijer, 2002; van Oers, 2002). These models allow students to flexibly and efficiently reason with mathematical problems (Fosnot & Jacob, 2010). For the purpose of this research project, models used by elementary students are conceptualized as a class of mathematical representations that contain regularity and aid in the carrying out of a calculation. In the area of early algebra the open double numberline would be one model that can be developed from a specific context, such as comparing the jumps a of frog and toad who travel the same distance but take different sized jumps (Fosnot & Jacob, 2010). Children can use the double numberline that emerges to make relational calculations across the equal sign in many new situations, including any problems that deal with common multiples, least common multiples and greatest common factors (Fosnot & Jacob, 2010).

While it is accepted that mathematical models such as the double numberline are powerful tools that can help students efficiently solve a variety of algebraic problems, they are not well suited to the study of functions. Models (as defined by Fosnot and Jacob) are not as well suited to the study of linear functions because the goal of working with a function is to analyze aspects of a mathematical relationship rather than to complete a calculation. Some researchers contend that instead, *representations* are well suited to the study of functions and that they can be used to help students develop a comprehensive understanding of functional relationships (Beatty & Bruce, 2012; Carraher, et al., 2008; Moss & London McNab, 2011).

Mathematical representations refer to “the display of mathematical relationships graphically, symbolically, pictorially, or verbally” (Chapin & Johnson, 2000, p. 140). Unlike mathematical models, representations are not a tool (like a calculator) that students can use to complete calculations. Studies involving students in the primary to intermediate grades have identified the following common mathematical representations that support some students' developing understandings of functions: a table of values, pattern rules, visuals, graphs and oral or written descriptions (Beatty & Bruce, 2012; Chapin & Johnson, 2000; Moss & London McNab, 2011).

They contend that mathematical representations should be central to the study of functions because they allow students to explore many aspects of one relationship (Beatty & Bruce, 2012). The use of multiple representations to study functions should ensure that one does not come to view the representation *as* the function (Carraher, et al., 2008). It is important that individuals do not come to equate a representation with a function because this will provide them with a very limited understanding of that mathematical relationship. For example, "equating numbers with their written forms can lead to serious problems such as the mistaken view that ¾ and 0.75 are different numbers" (Carraher, et al., 2008, p. 265). The same can be said about the representation of functions; each method of representing a function communicates only certain aspects of that mathematical relationship. For example, "the table tends to be a poor representation for conveying the continuity of a function. The graph conveys continuity, but can be ill-suited for displaying precise values of the function" (Carraher, et al., 2008, p. 266). Although it is difficult to see the continuity of a linear function in a table of values, that continuity is still a part of the function (which may go unnoticed if the individual does not study other representations of the function). Therefore, the use of multiple representations of a function ensures that the individual develops a comprehensive understanding of the multiple aspects of that functional relationship (Beatty & Bruce, 2012; Carraher, et al., 2008).

2.4.2.2 The purposeful use of supporting visual representations. Even with these multiple representations of linear functions, adults and students of all ages continue to struggle with their study of linear functions. Beatty and Bruce (2012) state that many students “have difficulty making connections among patterns, pattern rules, and other representations of linear relationships” (p.7). Therefore, some researchers have developed ways to make aspects of linear relationships more clearly visible through the purposeful use of a particular *supporting visual representations* of patterns that include: position numbers (corresponding to the input number in a table of values), and different coloured tiles (Beatty & Bruce, 2012; Beatty, et al., 2013; Moss & London McNab, 2011). A supporting visual representation of a function, as depicted in Figure 3, is meant to scaffold student construction of the linear function by drawing their attention to the constant (black tiles) and growing aspects (gray lined tiles) of the pattern, while encouraging connections between two sets of data (connecting the total number of tiles with the position number through the use of the term or position numbers below each pattern term) (Beatty & Bruce, 2012; Beatty, et al., 2013).

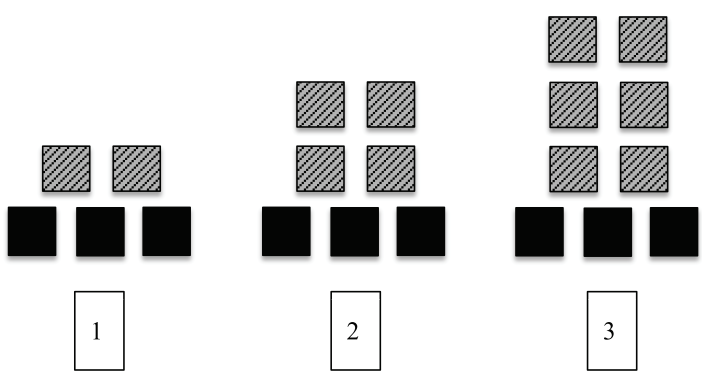


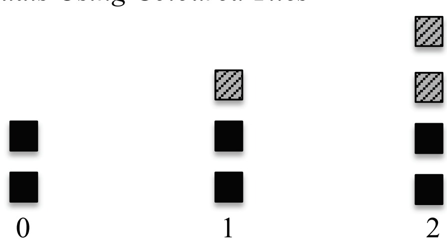
Figure 3. A supporting visual representation

Beatty and Bruce (2012) suggest, “students need to explore the interactions among representations to learn how changes to one or more representations can affect other representations" (p.9). Creating and identifying multiple representations of functions is valuable because these processes encourage the development of a comprehensive understanding of the properties of that relationship (Beatty & Bruce, 2012; Chapin & Johnson, 2000; Moss & London McNab, 2011). The various representations shown in Figure 4 represent the same function but they highlight different aspects of that relationship.

1. Written Description or Narrative Context

A construction company is planning an apartment building and would like the building to have one floor or story for each family that will live in the building. They also need one floor for the swimming pool and one floor for the building management offices. The height of the building is a function of the number of families for which the building is built.

2. Visuals Using Coloured Tiles



3. A Table of Values

Number of Families	Number of Stories
1	3
2	4
3	5

4. Pattern Rule

Number of Stories = Number of Families + 2 or $s = f \times 1 + 2$

Figure 4. Examples of multiple representations of a linear function

The role of purposely scaffolded visual representations of a linear function developed from a contextual problem and linked to other representations of a linear function (a table of values and a pattern rule) has been studied by a handful of researchers (Beatty, 2010; Beatty & Bruce, 2012; Beatty, et al., 2013; Moss & London McNab, 2011). Given the positive results of their work, more research is needed to explore the effectiveness of this aspect of an early algebra curriculum.

2.5 Summary

Algebra is a cornerstone of mathematics; algebraic thinking often determines success in many areas of mathematics (Kaput, 2008) and yet it is often poorly taught. Therefore, it is very important for teachers to adopt reform methods of mathematics instruction that will promote student understanding of essential algebraic concepts (Beatty & Bruce, 2012; Blanton & Kaput, 2003; Carpenter, et al., 2003; Carraher, et al., 2008). Many researchers studying algebra instruction look to the study of functions as a valuable and worthwhile endeavor (Beatty & Bruce, 2012; Beatty, et al., 2013; Blanton & Kaput, 2011; Ferrini-Mundy, et al., 1997; Moss & Beatty, 2006; Moss & London McNab, 2011). However, there is only limited evidence suggesting primary students are capable of algebraic reasoning when working with linear functions (Beatty, et al., 2013; Blanton & Kaput, 2011; Moss & London McNab, 2011). More research is therefore needed to explore early elementary students' capacities for algebraic thinking and the ways in which the study of functions can be made accessible to these young students. Some researchers suggest that the use of multiple representations of linear functions can make the study of functions accessible to young students and support the building of algebraic understanding in students of all ages (Beatty & Bruce, 2012; Beatty, et al., 2013; Moss & London McNab, 2011). More research surrounding the role of multiple representations of linear functions (in particular a contextual description, supporting visual representations of the pattern, table of values and a pattern rule) could help to inform the practice of mathematics educators and give them the necessary information to plan and implement mathematics programs that would build strong algebraic thinkers.

Chapter 3: Methodology

3.1 Research Design

This research study employed a qualitative case study design that examined primary students' capacities for explicit reasoning when working with visual representations of simple linear functions (along with other representations of these functions). Reform oriented mathematics instruction was the central context of the case study and the instructional interventions focused on developing a conceptual understanding of linear functions. I used an embedded case study design where six students were studied as embedded cases within the overall case of the five-lesson intervention (a five-day early algebra unit) and in the context of one grade 2/3 classroom. Within the embedded cases, I studied six children's mathematical development over the course of the intervention. The intention was to understand how the children's thinking changed over the course of the five-lesson intervention. This overall case was used to explore the ways in which patterning activities and visual representations of patterns can be used to promote algebraic thinking in young students. The study sequence included: pre-assessment interviews, a five-lesson intervention, post-assessment interviews, and a retention task.

3.2 Research Sample

A convenience sample was used for this study based on the availability of participants (Creswell, 2014). A Grade 2/3 class and their teacher from a school board in Northwestern Ontario participated in the research project. This class was purposely chosen as a convenience sample because I had spent many hours in the classroom as a volunteer and substitute teacher – which is believed to reduce the impact of an observer effect on the data collection process (Creswell, 2014). The classroom teacher and I identified two high level students, two average students and two low achieving students to be the focus of the case studies and therefore the data collection efforts.

3.3 Procedure

Ethics approval was required from Lakehead University, the school board, and the school's administration. Since data was collected about the students within the class, letters explaining the research project as well as permission forms were sent to the students' parents or guardians (Appendices A and B). The research project was also verbally explained to the students and they were provided with the opportunity to refuse participation in the study (Appendix C). The teacher was also given a letter and consent form before the study began (Appendices D and E). The principal of the school also received a letter and consent form to be completed before the commencement of data collection (Appendices F and G). Any names or terms that could indicate the identity of research participants have been altered in the appendices, and following sections, in order to maintain the anonymity of the participants.

The research project spanned a two-week period and the retention test was administered approximately two weeks after the post-assessment interviews. Each math lesson was approximately 1 hour and 20 minutes in length, occurring in the mid-morning. The tasks that were used in the pre and post-assessment interviews were developed and had been previously field tested by researcher Ruth Beatty (Appendix H). The five-lesson intervention was designed by modifying lessons from Beatty and Bruce's (2012) resource *From Patterns to Algebra: Lessons for Exploring Linear Relations.* Modifications were required in order to make the material more accessible and appropriate for the younger participants in this project (grade 2/3 students compared to grade 4-10 students in Beatty and Bruce’s resource); the modifications drew on the findings of Moss and London McNab’s (2011) study with Grade 2 students. The lesson sequence is outlined in Table 1 (with more information to follow in the Results Chapter) and it was designed to push students beyond recursive thinking through the use of many representations of linear growing patterns (Beatty & Bruce, 2012; Moss & London McNab, 2011). Beatty and Bruce along with Moss and London McNab have suggested that patterning activities can push students towards algebraic thinking in the early elementary years when linear growing patterns are represented in multiple ways and students are provided with opportunities to develop the skills to explore the explicit pattern rule for any given linear function. The lesson sequence outlined in Table 1 was designed to provide students with many opportunities to work with a variety of representations of linear growing patterns in order to challenge them to develop algebraic reasoning skills.

Table 1

Five-Lesson Intervention

Sequence	Topic	Representations of linear functions
1	Exploring linear functions with multiplicative rules. Developing narrative contexts for the study of linear functions and pattern rules.	- Narrative context to explain connection between pattern and pattern rule - Written pattern rules - Visual representation - Table of values
2	Exploring linear functions with multiplicative rules. Developing narrative contexts for the study of linear functions and pattern rules. Building linear functions from a pattern rule.	- Narrative context to explain connection between pattern and pattern rule - Written pattern rule - Visual representation - Tables of values
3	Determining near terms, far terms and pattern rules. Exploring and building linear functions with multiplicative rules.	- Written pattern rule - Visual representations - Table of values
4	Creating a pattern rule and building a visual to match that rule.	- Written pattern rule, with use of symbols - Multiple visual representations - Table of values
5	Comparing multiplicative rules to rules that involve multiplicative growth and a constant.	- Narrative context - Students can create any other representation independently

The research project began with structured pre-assessment interviews following the interview guide created by Beatty (Appendix H). I video recorded the interviews in order to review the interviews and fully analyze the students' understandings of linear growing patterns. We explained to the students that they were not being marked on their answers and that the questions would be used to guide the work we would be doing with patterning in the weeks to come. Interviews were selected as the main method of data collection at this stage of the project due to the age of the students and their limited written communication skills. Interviews provided me with rich data surrounding the conceptual understandings that the students held about linear growing patterns.

The pre-assessment interviews were followed by the five-lesson intervention on linear functions. The lessons were co-taught by the classroom teacher and me. The lesson sequence in Table 1 was taught using reform methods of mathematics instruction that had previously been employed throughout the school year. The lesson plans were developed based on the existing research and through consultation with the classroom teacher.

The five-lesson intervention ended with post-assessment interviews following the interview guide outlined in Appendix H. The students were once again interviewed individually and asked the same interview questions as in their pre-assessment interview to measure any changes to their understandings of linear functions. The retention task was then administered approximately two weeks following the end of the unit on linear growing patterns. The retention task was based on components of the five-lesson intervention using different numbers and a slightly different context. The items included in the retention task were selected after examining student work from the lesson sequence and the results of the pre and post-assessment interviews. The retention task was used only for the purposes of the study; that is, the goal of determining the effectiveness of the five-lesson intervention.

3.4 Data Collection

As Baxter and Jack (2008) suggest, an important aspect of “case study research is the use of multiple data sources, a strategy which also enhances data credibility” (p. 554).

Therefore, I used many separate sources of data, and attempted to converge that data during the analysis. As outlined in Table 2, the main sources of data were videotaped whole-group lessons, videotaped activities where the selected high, average and low students were working with their partner, as well as their responses to the pre and post-assessment interviews. I also recorded observation notes, which were brief written notes I recorded directly on a student's work sample or in a separate notebook while working with a student or following each lesson. These observation notes were used to clarify students' strategies or methods based on conversations I had with the student or what I had observed while they were working. Copies and photos of work samples were also included in the data set.

Table 2

Timing and Type of Data Sources

Day	Activity	Data sources
1	Pre-assessment	Video recorded pre-intervention interviews
2	Lesson 1	Video record group lesson Student work samples Observation notes
3	Lesson 2	Video record group lesson Video record students working with partner Student work samples Observation notes
4	Lesson 3	Video record group lesson Video record students working with partner Student work samples Observation notes
5	Lesson 4	Video record group lesson Video record students working with partner Student work samples Observation notes
6	Lesson 5	Video record students working with partner Student work samples Observation notes
7	Post-assessment	Video-recorded post-intervention interviews
8	Retention task	Student work samples

Before the start of the study, the classroom teacher and I identified six students to be the focus of the data collection efforts. The focus students were selected based on their general mathematics achievement level: two high achieving students, two average students (at grade level), and two low achieving students. With parental and student consent, additional students were occasionally selected to be video recorded during the five-lesson intervention based on the mathematical thinking they were developing. I video recorded the six focus students working with their partners and during whole group lessons. The teacher and I selected student partnerships for the six focus students to ensure each student was matched with a peer of a similar ability; this partner was usually not one of the other six students selected to be a focus for data collection. I also collected work samples from these six students and focused my observation notes on the students' developing algebraic reasoning processes.

Additionally, the collected work samples from the six selected students were used to supplement the video recorded data. Upon collecting a work sample, I recorded notes pertaining to the students' explanations or their thinking surrounding the work sample (i.e. comments students made either independently or following a prompt from myself or the teacher). The work sample was then photographed or photocopied and subsequently added to the database. These samples were used to support the analysis of video recorded data of the students working on various problems.

I also recorded observation notes during and following each lesson. These observations aided in the planning of subsequent lessons. I documented any important mathematical thinking, conjectures, or ideas presented by the students throughout the course of the lesson sequence and used these notes to help clarify the students' strategy development.

3.5 Data Analysis

I undertook three main stages in my data analysis procedure:

Stage 1: general viewing of the data to develop overall themes about children's algebraic and multiplicative development

Stage 2: coding of the data and refining of the themes

Stage 3: interpretation.

Stage 1 of my data analysis process involved mainly inductive analysis, whereas Stages 2 and 3 involved both inductive and deductive analyses procedures because as Creswell (2014) suggests, a researcher should “build their patterns, categories, and themes from the bottom up by organizing the data into increasingly more abstract units of information…Then deductively, the [researcher will] look back at their data from the themes to determine if more evidence can support each theme” (p.186). Initially in Stage 1, I focused on an inductive data analysis process as I attempted to pull themes from my data about how the students approached linear functions. These themes resulted in more generalized ideas about how explicit reasoning developed, how the students’ strategies progressed and the ways in which they developed an understanding of multiplication. Then the addition of deductive processes in Stages 2 and 3, allowed me to review specific segments of my database, as well as the existing research, in order to verify my theories about the development of explicit reasoning, students’ strategies and multiplicative reasoning in primary students.

3.5.1 Stage 1: General viewing of the data. In the first stage of my data analysis I watched the video recorded pre- and post-assessment interviews in order to determine the six students’ general progressions throughout the study. I examined the pre- and post-assessment interviews of the six selected students focusing on the strategies they used and looking for ways in which their methods changed from the pre- to the post-assessment interview. I then looked at the student work samples, as well as video clips of whole-class lessons and students working on problems with their partners, in order to identify major themes in the data. The themes identified centered around: the students’ strategy development, their level of explicit reasoning, and their evolving understandings of multiplication. I then began to code the data.

3.5.2 Stage 2: Coding procedures. All data gathered for each of the students who had parental permissions to participate in the study were assigned a unique identifying number. However, only the data pertaining to the six students who were selected as the focus of this study were analyzed. This identifying number was attached to any data collected that pertained to that particular student. The six students were subsequently given an alias. Video clips of each student responding to interview tasks for both the pre and post-assessment interviews were labeled with the participant's identification number. Any work samples collected were also labeled with the participant's identification number and names or information that could indicate the author of the work were cut or blacked out. No participants were made aware of their identifying number or the identifying numbers of other participants.

All video recorded data of the pre- and post-assessment interviews, whole class lessons and students working with their partners were input into Atlas.ti, qualitative data analysis software. Student work samples for all students throughout the five-lesson intervention and the retention test were also input into Atlas.ti. A unique *Primary Document* (PD) number was generated for each piece of data input into Atlas.ti resulting in a total of 200 PDs.

All data for the six selected students were coded based on five main categories (the code category was indicated by a prefix attached to each code name) and following the order outlined in Table 3. Data were first coded for correctness ("A" codes) and, where applicable, the type of pattern term a student was working to determine ("T" codes). Then the data were coded for the student's strategy ("Stgy" codes) based on Fosnot and Jacob's (2010) *Landscape of Learning for Algebra* (see Appendix I) as well as additional strategy codes (see Appendix J for a full list of codes). Next, I coded for the student's level of explicit reasoning ("EorR" codes). Then the data were coded focusing on the ways in which the student used or did not use the visual representations of linear functions ("V" codes). Based on the data and the ways in which the students thought about or approached the patterning activities, I generated the following additional categories of codes to accurately label and analyze the students' thinking:

1. "PofG" codes – were a subset of the "Stgy" codes and they identified different ways that students proved their generalizations or pattern rules
2. "SorR" codes – were a subset of the "V" codes and they indicated the type of symbolization or representation processes that a student used
3. "IwP" codes – indicated various ways that students interacted with a pattern
4. "O" codes – indicated instances when students made observations about the relationships between numbers and operations

Codes were generated based on the existing research and the specific goals of this research project (see Appendix J for a full list of codes). Once all of the data for the six selected students were coded, the student work samples and video tapes were re-analyzed to check for consistency in data coding processes and to refine themes or theories I had about the data. As I coded, I also used Atlas.ti to attach memos to the data that contained theories and ideas I could later pursue in the third stage of the analysis.

Table 3

Main Code Categories and Coding Sequence

Sequence	Main code category	Purpose of code category
1	"A" codes	Identify information about a student's answer and the correctness of student responses
2	"T" codes	Indicate the type of pattern term that a student was determining (i.e. were they finding the next, near or far term?)
3	"Stgy" codes	Identify the type of strategy that a student used to determine a pattern term or to work with a linear function
4	"EorR" codes	Indicate the student's level of explicit reasoning
5	"V" codes	Identify the ways in which a student used or did not use the visual representation of a linear function

3.5.3 Stage 3: Interpretation. In the third stage of the data analysis procedure I revisited segments of the database in order to further refine my developing theories and to confirm or invalidate my various interpretations. I used Atlas.ti to filter the database in order to reexamine specific segments of the data. I also used previous and new memos within Atlas.ti to record my analysis of segments of data and revisited many of these memos multiple times. I reviewed the existing literature at this stage in an attempt to find additional evidence to either substantiate or contradict my theories.

It was intended that the multiple sources of data be "converged in the analysis process rather than handled individually" (Baxter & Jack, 2008, p. 554). These sources of data were expected to collectively show that students in Grades 2 and 3 are capable of some level of explicit reasoning. It was also expected that the various representations of linear functions would encourage the participants to progress towards the use of explicit reasoning and away from recursive thinking when working with linear functions.

Chapter 4: Results

4.1 Overview

The research project began with the pre-assessment interviews on the first day. The six students, who ranged in ability, and had been selected to be the focus of data collection efforts, were interviewed on the same day, just prior to the five-lesson intervention. I interviewed each student individually using the interview guide in Appendix H. The interview questions had the students solve seven problems, five of which involved examining different patterns and building the next term using tiles, then mentally determining the number of tiles needed to build position 10 of the pattern (a near term) and position 100 of the pattern (a far term). The students then participated in a five-lesson intervention over the course of five consecutive days. The lessons in the five-lesson intervention were video recorded and samples of the students' work were collected. Following the five-lesson intervention, the same six students who participated in the pre-assessment were interviewed again in a post-assessment that followed the same structure and involved the same questions as the pre-assessment. All of the students then participated in a retention activity approximately two weeks later that presented students with a problem similar to those covered in the five-lesson intervention.

The data from the pre-assessment and the post-assessment interviews were reviewed and analyzed to determine how the whole group preformed. These results are reported in the next section. For each of the six selected students, the analysis of data from the five-lesson intervention and retention test, along with their pre- and post-assessment interviews were reported as individual case studies to determine how each student progressed during the research project.

4.2 Pre- and Post-Assessment Interviews

In both the pre- and post-assessment, the students were asked to solve 7 problems, however Problems 3 to 7 only will be discussed in detail while Problems 1 and 2 will not be discussed as they asked the students to describe the general nature of patterns and the data collected was not informative. In the pre-assessment, over the four items where the students were asked to find the next term of a pattern, all students were able to build the next term (see Table 4). For each problem that will be discussed in both the pre-and post-assessment interviews the students were shown the first three terms of a pattern (see Part A of Figure 5) and all students were able to independently build the next term of that pattern (see Part B of Figure 5). However, when asked to determine the near terms, the six students interviewed were able to accurately do so 21 of 30 times (see Table 4). The results involving determining the far terms were similar; when asked to determine the far terms, the students were able to do so 20 of 30 times (see Table 4).

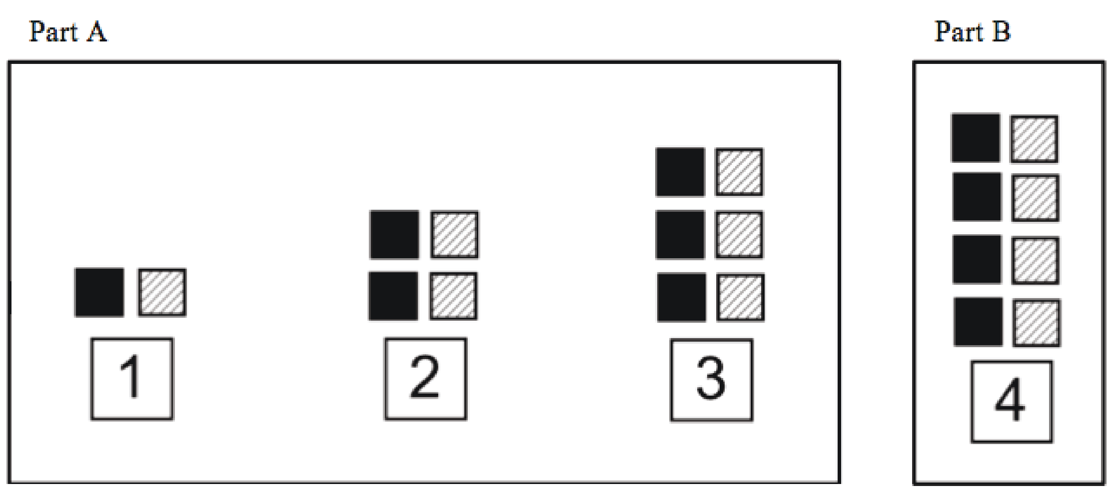


Figure 5. Building next terms in the pre- and post-assessments

Table 4:

Pre- and Post-Assessment Results

Student	Next terms n=24		Near terms n=30		Far terms n=30		Total n=84	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Amy	4	4	0	1	0	1	4	6
Nicole	4	4	3	5	3	4	10	13
Brandon	4	4	4	4	3	4	11	12
Corey	4	4	5	4	4	5	13	13
Alison	4	4	4	4	5	5	13	13
Eric	4	4	5	5	5	5	14	14
Total correct	24	24	21	23	20	24	65	71

In the post-assessment, all students were again able to build the next term for each pattern. With respect to determining the near and far terms of the patterns, there was some improvement from the pre to the post-assessment. In the post-assessment students were able to determine the near terms 23 of 30 times; in the post-assessment, the students identified two more correct near terms compared to the pre-assessment (see Table 4). Students were able to accurately determine the far terms 24 of 30 times in the post-assessment, or 4 more correct far terms compared to the pre-assessment (see Table 4).

4.3 Looking at the Problems in the Pre-Assessment Interviews

The problems that the students worked with in the pre-assessment interview will be discussed in order of difficulty, with Problems 4, 5 and 6 being the easiest problems, Problem 7 being slightly more difficult and Problem 3 being the most difficult. Problems 4, 5 and 6 build upon each other and are easier because they only involve multiplicative growth (i.e. input times one, input times two and input times three respectively) with no constant.

4.3.1 Problems 4, 5, 6: Multiplicative growth (no constant) and full visual. In the pre-assessment interviews, Amy was the only student who could not identify the near terms in Problems 4 to 6, and Alison was the only other student who made an error when determining the near term for the pattern in Problem 6 (see Table 5).

Table 5

Student Success Determining Near Terms in Pre- and Post-Assessment Interviews

Student	Problem 4: O = I x 1[a] n=6		Problem 5: O = I x 2 n=6		Problem 6: O = I x 3 n=6		Problem 7: O = I x 4 n=6		Problem 3: O = I x 2 +1 n=6	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Amy				✓						
Nicole	✓	✓	✓	✓	✓	✓		✓		✓
Brandon	✓	✓	✓	✓	✓	✓	✓	✓		
Corey	✓	✓	✓	✓	✓	✓	✓	✓	✓	
Alison	✓	✓	✓	✓			✓	✓	✓	✓
Eric	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Total correct	5	5	5	6	4	4	4	5	3	3

[a] O = I x 1 refers to the pattern rule where the I indicates the Input or the position number and the O indicates the Output or number of tiles at a given position.

In the pre-assessment interviews, Amy was the only student who could not determine the far terms in Problems 4 to 6, and Brandon could not figure out the far term in Problem 4 and yet correctly found the far term in Problem 5 and 6 (see Table 6). Even though Brandon struggled with Problem 4 and could not find the far term, it appears that working with the pattern in Problem 4 may have given him some time to think about how these patterns worked. He was then able to figure out the far terms with the next patterns (Problems 5, 6 and 7). Working with Problem 4 where the multiplier was one, may have helped Nicole, Brandon, Corey, Alison and Eric accurately determine the far terms in Problems 5 and 6. Since the multiplier in the linear function was one, Problem 4 allowed most students to identify a clear connection between the term number and the number of tiles (i.e. at term one there is one tile, at term two there are two tiles and the term number tells you how many tiles there will be), which may have helped them attend to the term number for all subsequent patterns. Amy who was using less sophisticated strategies was unable to figure out the far terms for any of the patterns.

Table 6

Student Success Determining Far Terms in Pre- and Post-Assessment Interviews

Student	Problem 4: O = I x 1 n=6		Problem 5: O = I x 2 n=6		Problem 6: O = I x 3 n=6		Problem 7: O = I x 4 n=6		Problem 3: O = I x 2 +1 n=6	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Amy				✓						
Nicole	✓	✓	✓	✓	✓	✓		✓		
Brandon		✓	✓	✓	✓	✓	✓	✓		
Corey	✓	✓	✓	✓	✓	✓	✓	✓		✓
Alison	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Eric	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Total correct	4	5	5	6	5	5	4	5	2	3

4.3.2 Problem 7: Multiplicative growth (no constant) and partial visual. Problem 7 was more challenging because the students were not given a complete visual of the first 3 terms of the pattern. With Problem 7, the students were shown the pattern in Figure 6 and asked to imagine that an additional column of green tiles was added beside the last yellow column of tiles, then they had to determine the near and far term of that new pattern (now with 4 columns of tiles). Although Problem 7 also built upon the previous patterns, some of the students struggled with this pattern with two of the six students in the pre-assessment (Amy and Nicole) unable to determine the near or far terms (see Table 6 and Table 5).

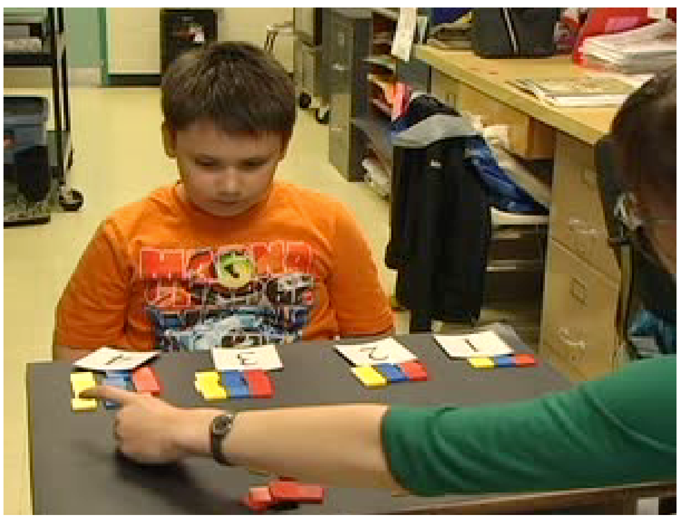


Figure 6. Introducing problem 7: Students have to imagine an additional green column

4.3.3 Problem 3: Multiplicative growth with constant and full visual. The first pattern that the students analyzed (Problem 3) was the most difficult because it involved a constant as well as multiplicative growth (see Table 6, Table 5 and the interview guide in Appendix H). Only three of the students were able to determine the near term for Problem 3 (see Table 5). The students struggled with finding the far terms of Problem 3 and only Alison and Eric were able to accurately determine the far terms in the pre-assessment (see Table 6).

4.4 Looking at the Problems in the Post-Assessment Interviews

The problems in the post-assessment interviews will also be discussed according to the difficulty of the problem, starting with the least difficult problems (Problems 4, 5, and 6), then moving on to Problem 7 and finally discussing Problem 3 which was the most difficult.

4.4.1 Problems 4, 5, 6: Multiplicative growth (no constant) and full visual. In the post-assessment, for Problems 4, 5 and 6 Amy was able to determine only the near term for Problem 5 (see Table 5). Alison accurately determined the near terms for Problems 4 and 5 but made a minor computational error when determining the near term for Problem 6, and all other students were able to accurately determine the near terms for these three problems.

In the post-assessment, Amy was the only student who could not determine all the far terms for Problems 4 and 6 (see Table 6). She was able to accurately determine the far term for the pattern in Problem 5 and all other students accurately found the far terms for all three problems.

4.4.2 Problem 7: Multiplicative growth (no constant) and partial visual. In the post-assessment, Amy was the only student who was unable to determine the near and far terms for Problem 7; all other students were able to determine both the near and far terms (see Table 6 and Table 5).

4.4.3 Problem 3: Multiplicative growth with constant and full visual. Problem 3 was the most difficult for the students as it involved multiplicative growth and a constant. Nicole, Alison and Eric were the only students able to accurately determine the near term for the pattern in Problem 3 (see Table 5). Alison and Eric were also able to determine the far term for Problem 3, but Nicole was not able to accurately do so as the calculation seemed too difficult for her and instead she made an estimate (See Table 6). When trying to determine the near term (term 10) for Problem 3, Corey looked at the number of tiles in term 5 (11 tiles) and then doubled the number of tiles at term 5 to find the number of tiles at term 10. However, this strategy resulted in an error because the pattern involved a constant and only part of the pattern grew from term to term. Corey did not make the same mistake when determining the far term for Problem 3 and he used a strategy that employed more explicit reasoning when working with the far term and was able to accurately find the far term.

4.5 Going Deeper: The Focus Students as Embedded Case Studies

Although the data collected through the pre- and post-assessment interviews provided some indication of the students' progress, tabulating this data (as was done in Table 4, Table 5 and Table 6) only provided information surrounding the *correctness* of student responses to each interview task. Within each task on the pre- and post-assessment interviews, students could answer one of the interview tasks correctly but the types of strategies that students used, and the efficiency of their reasoning processes, varied greatly across students and individually from the start of the project to the end of the project. Due to the focus on the students' levels of correctness in the data from the pre- and post-assessment interviews, this general overview of the data belies some of the real progress that many of the students made throughout the study. Therefore, I now turn to each of the six focus students to explore in more depth the many ways in which their strategies and reasoning processes evolved throughout the research project.

The following individual case studies will discuss each student's progression through the research project. After the pre-assessment interviews, the six focus students and their classmates participated in a five-lesson intervention where each lesson was designed to encourage the students to develop explicit reasoning skills. The five-lesson intervention followed the sequence outlined in Table 1 and Figure 7. Each lesson exposed the students to various representations of linear functions in order to help the students focus on the multiplicative growth present in linear functions (most of the lessons prominently featured a visual of a growing pattern along with other representations such as a narrative context, table of values and/or a pattern rule). All lessons were designed to encourage the students to reason explicitly and move toward beginning multiplicative reasoning, rather than focusing solely on recursive or additive reasoning. In each case study that follows, and for every day of the project (some days may be missing for particular students due to absences), wherever possible I will discuss:

1. the student's level of explicit reasoning,
2. the strategies they used to determine near or far terms, and
3. the ways in which the student used the visual images of linear functions.

Monday	Tuesday	Wednesday	Thursday	Friday
Day 1: Interviews 2 **-Pre-assessment interviews with:** Amy, Nicole, Brandon, Corey, Alison and Eric	**Day 2**: Lesson 1 3 -**Function Machine Activity**: As a class, selected random position number to put in *machine*, analyzed the visual of the position that came out. Recorded findings in table of values and discussed pattern rules. -**Math Journal:** look at image of one pattern term and discuss what the pattern rule could be.	**Day 3**:Lesson 2 4 - **Function Machine Activity**: Same as Day 2, with more challenging patterns -**Guess my Rule Game in Pairs:** Partner A makes up a pattern rule and builds pattern terms of that pattern so Partner B can guess the pattern rule	**Day 4**: Lesson 3 5 - **Worms in the Garden Problem in Pairs:** students given visual of 5 worms in garden on day 1, 10 worms on day 2, 15 worms on day 3, 20 worms on day 4 and empty table of values. Asked to solve: *How many worms would there be on the 10^{th} day, 20^{th} day and 100^{th} day?* - **Guess my Rule Game in Pairs:** Same as Day 3	**Day 5**: Lesson 4 6 -**Pattern Building Activity:** Students created pattern rules, then built the first three terms of their pattern. Students did a gallery walk to see their peers' patterns and selected 2 patterns to study. For each peer's pattern the student recorded the next term, the near term of 10 and some form of pattern rule.
9	**Day 6**: Lesson 5 10 -**Paying a Sitter Problem:** Students compared two linear functions: *Is it better to be paid a $10 flat fee and $2 per day or $3 per day? Which is the better option if they work 5 days, 10 days or 20 days?*	**Day 7**: Interviews 11 - **Post-assessment interviews with:** Amy, Nicole, Brandon, Corey, Alison and Eric	12	13
16	17	18	19	20
23	**Day 8**: Retention Task 24 - **Growing Cucumbers Problem:** Students given visual of 3 cucumbers on day 1, 6 cucumbers on day 2, 9 cucumbers on day 3, 12 cucumbers on day 4 and empty table of values. Asked to solve: *How many cucumbers would there be on the 10 day, 25^{th} day and 100^{th} day?*	25	26	27

Figure 7. Sequence of the research project

4.5.1 Identifying a student's level of explicit reasoning. In order to better understand and analyze students' reasoning processes, I drew upon existing research (e.g. Beatty & Bruce, 2012; Beatty, et al., 2013; Lannin, 2005) to create Figure 8. For the purpose of this project, the students' progression from recursive toward explicit reasoning was identified based on their degree of proficiency in determining a pattern core and generalizing about linear functions (this progression is outlined in Figure 8). Before a student can reason with a growing pattern, they must be able to determine the core of the pattern or "the string of one or more elements that repeats" (Van de Walle et al., 2015, p.

268) and furthermore, be able to extend that core. Once students can identify and extend the pattern core, they must begin to "look for a generalization or an algebraic relationship that will tell them what the pattern will be at any point along the way" (p. 270). Students who were relying mainly on recursive reasoning showed that they understood patterns to be made of iterated units (a predictable repetition of the pattern core) and they thought about linear functions as "adding" one unit of the pattern core at each successive pattern term. Some students then developed a whole-object strategy indicating that they had some understanding of a linear function's multiplicative growth but were unable to generalize about any term of the pattern. Since the problems used in this study focused on multiplicative growth alone, the presence of whole-object reasoning was not problematic for students but this type of reasoning could be difficult to accurately apply to linear functions involving multiplicative growth and a constant. Then as students moved closer to explicit reasoning, they were able to predict the visual structure of a linear function beyond the next term and they had some idea of what the pattern would *look like* at any given term. Students at this moderate level of explicit reasoning were able to predict the structure of the pattern beyond the next term due to some kind of understanding that there existed a relationship between the term number and the characteristics of that specific term. As students became more proficient in their explicit reasoning and their abilities to generalize about growing patterns, they were able to provide a general description of how the pattern was growing or a general pattern rule using everyday language. This general pattern rule also included some kind of description of the way in which the term number could be used to determine information about the growing pattern at any term. Finally, as students became more confident in their explicit reasoning abilities, they were able to

generate and use an explicit pattern rule that would allow them to determine the characteristics of any term of the pattern (these explicit pattern rules were often structured based on the following notation that was developed during the Function Machine Activity on Days 2 and 3: Output = Input x ___).

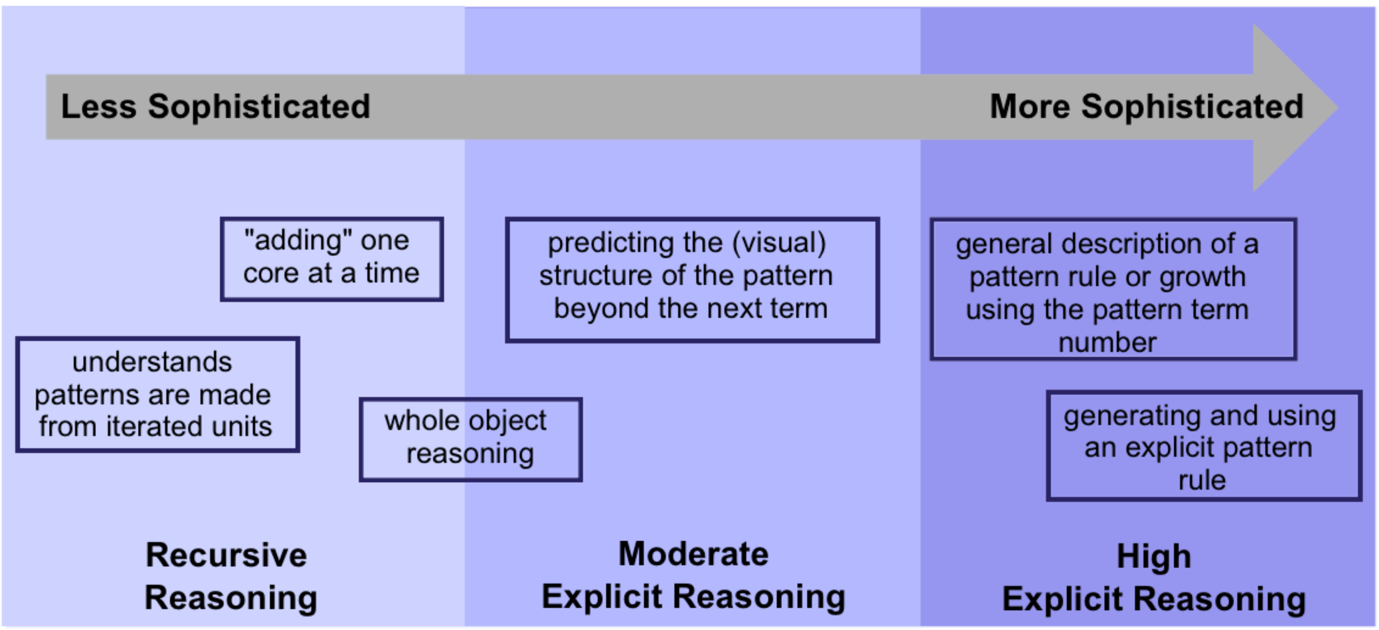


Figure 8. Students' shifting algebraic thinking: From recursive to explicit reasoning

4.5.2 Identifying the strategies used to determine near and far terms. The strategies that students used to determine near and far terms also provided a lot of information about how a student was thinking about a problem and what they understood about multiplication. Therefore, drawing upon existing research (e.g. Beatty & Bruce, 2012; Beatty, et al., 2013; Carpenter, Fennema, Loef Franke, Levi, & Empson, 1999; Chapin & Johnson, 2000; Fosnot & Jacob, 2010; Lamon, 1996; Young-Loveridge, 2005), I created Figure 9 in order to consistently label each student's strategies and to be able to better analyze their strategy development. In the primary years, students begin to build on what they know about addition in order to work with multiplicative problems. As students gain experience working within multiplicative contexts they can develop strategies increasing in complexity. Figure 9 depicts a progression from strategies founded on addition that form the basis for the early foundations of multiplicative reasoning toward strategies that require multiplicative thought and lead to the beginning of multiplicative reasoning. For the purpose of this project, and due to the young age of the study's participants, a focus has been placed on the early foundations and beginning of multiplicative reasoning. However this progression will continue beyond the progression depicted in Figure 9 and toward more complex multiplicative reasoning as the students move into the junior grades and beyond. In sections 4.5.2.1 and 4.5.2.2 that follow, the strategies the students used to determine the near and far terms of simple linear functions as well as how these strategies fit within the stages of the early foundations of multiplicative reasoning and the stages of beginning multiplicative reasoning are discussed (refer to Figure 9).

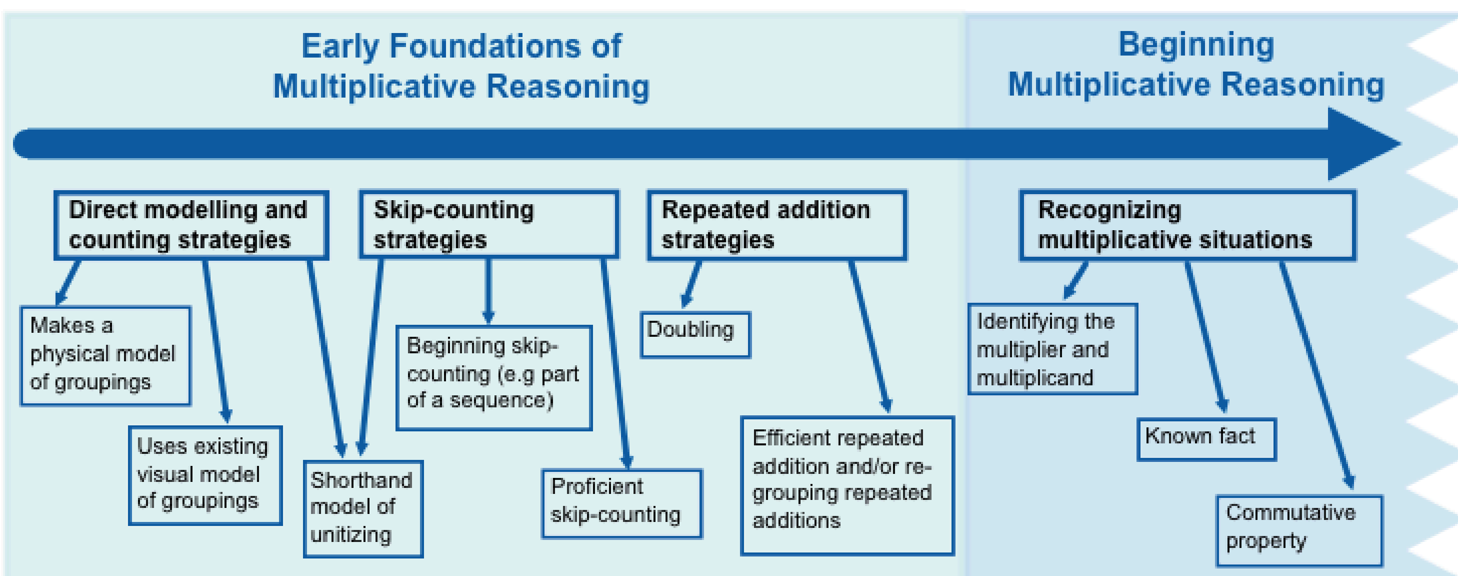


Figure 9. Progression of strategies used to determine pattern terms with linear functions

4.5.2.1 Strategies in the early foundations of multiplicative reasoning. Students begin to solve multiplication problems and develop the earliest foundations of multiplicative reasoning by directly modelling multiplicative situations (Carpenter, et al., 1999). Early on, a student might make a physical model of the groupings, model the objects within the groups using manipulatives or a drawing, and then count the total number of objects. Students who have a slightly stronger understanding of unitizing may be able to use an existing visual of a linear growing pattern and extend that visual in order to determine information about other terms in the pattern. Students who have an even stronger understanding of unitizing may use a shorthand visual or model to help them unitize accurately and either count by ones, skip-count, or use a combination of the two, to determine information about other pattern terms (see Figure 9).

As students become more confident with direct modelling and counting strategies, they generally progress toward more sophisticated counting strategies that make use of skip-counting (Carpenter et al., 1999). The earliest skip-counting strategies that students use when working with linear functions involve creating a shorthand model of groupings in order to track and help them skip-count. Beginning skip-counting strategies are slightly more sophisticated and a student who can use beginning skip-counting strategies can skip-count independently but only by some numbers (e.g. they can skip-count by 2 or 5 but not by 3 or 7) or only for part of a sequence (e.g. to determine 4 x 4 a student might skip-count for part of the sequence then count by ones: *4, 8, 12 (13, 14, 15…) 16*) (Carpenter et al., 1999). Then as students practice skip-counting strategies, they may move on to proficient skip-counting strategies where they can skip-count for long sequences and by any number, or they may be able to start at any number and skip-count on from there (see Figure 9).

Students may then move towards thinking about multiplication in terms of repeated addition. Students will often begin to think about repeated addition by employing doubling strategies (e.g. to solve 5 x 4 a student might double 5 to find that two 5s are 10 and then double that again to find that four 5s are 20) (Carpenter et al., 1999). Doubling strategies can be efficient in some contexts however it is important for students to move beyond doubling and toward “a rich, dense structuring of multiplicative relations” (Fosnot & Jacob, 2010, p. 55). As students move on from doubling strategies they may employ more efficient repeated addition strategies that may involve regrouping the parts being repeatedly added (e.g. to solve 10 x 3 a student might think about the problem like 10 + 10 + 10 and then group the addends to make the problem easier (10 + 10) + 10 = 20 + 10) (see Figure 9).

4.5.2.2 Strategies in the beginning of multiplicative reasoning. After students have developed an understanding of the early foundations of multiplicative reasoning they may move on to more sophisticated strategies founded on an understanding of the beginning of multiplicative reasoning. When students use strategies that recognize multiplicative structures, they are working with strategies that are more multiplicative in nature and less additive. Students may initially identify the multiplier and multiplicand or use known multiplication facts with only a limited number of factors or combinations of known addition facts to solve multiplication problems. Students will automatize some facts and be able to recall known multiplication facts quickly or use a combination of known facts to solve a multiplication problem (Carpenter et al., 1999). Students may also make use of the commutative property to make problems easier to solve (e.g. a student may realize that 7 x 4 is the same as 4 x 7 and that the order does not affect the amount;

they can select the order of the factors based on what is easiest for them to calculate) (Van de Walle et al., 2011).

4.5.2.3 The progression of the strategies used by the research participants. Generally, early on in the project, the six students were mainly using direct modeling and counting strategies or skip-counting strategies in order to find near and far terms of the linear functions explored. By the end of the research project, all of the students had progressed toward the use of repeated addition and a few of the student were regularly using known multiplication facts (although only with some factors) and the commutative property.

4.5.3 Turning to the Case Studies. The case studies that follow describe how six students (two Grade 2 and four Grade 3 students) progressed throughout the research project. Overall, the six students began the project with little formal knowledge of multiplication but some experience working with multiplicative contexts. Many of the students initially relied on additive strategies but by the end of the project they were beginning to construct an understanding of multiplication and some were able to use multiplicative strategies efficiently. Initially, the students primarily viewed growing patterns through a recursive lens. By the end of the project, after students had worked with the multiple representations of linear growing patterns, most students moved toward more explicit reasoning processes.

4.5.4 Case 1: Amy's progression through the five-lesson intervention. First we turn to the case study of Amy who was one of the younger students participating in the research project (Grade 2). Even by the end of the project, Amy had not yet developed an understanding of multiplication and she almost always used recursive reasoning and strategies that were additive. Amy began the research project with an understanding that patterns are made from iterated units; she primarily used beginning skip-counting strategies, but with some difficulty. She was just constructing the early foundations of multiplicative reasoning and had not yet determined how to unitize. By the end of the research project, Amy had begun to construct an understanding of the early foundations of multiplicative reasoning and she was able to use skip-counting and repeated addition strategies more effectively with some assistance from her peers.

4.5.4.1 Pre-assessment. In the pre-assessment Amy demonstrated that she knew that patterns were made from iterated units and she thought about some of the linear functions as "adding" one pattern core at each term. She was able to determine all of the next terms because she could add on one pattern core to the previous term. However, when asked to think about a near or far term of 10 or 100 respectively, she was unable to accurately determine any of the near or far terms; she could only provide a guess as to how many tiles would be needed (P31v[2] to P35v). Her *guesses,* about the near and far terms, showed that she was unable to move toward more explicit reasoning and was only able to think about adding one core at a time to each successive term in a pattern. Her *guesses* showed that she knew the near term would be bigger than the last given term and that the far term would be even bigger, but she did not know how to form a generalization about the pattern at any term.

4.5.4.2 Five-lesson intervention. During the first lesson in the five-lesson intervention, Amy showed that she was beginning to unitize and think about

[2] 'P' refers to *Primary Documents* that have been labeled with an unique identifier all beginning with 'P' in the research database. The letters following the Pnumber refer to the type of primary document: "v" indicates videos, "ws" indicates student work samples, and "on" indicates observation notes.

multiplicative growth as repeated addition, but she was not able to use repeated addition strategies independently at this time. She was also thinking about linear growing patterns as “adding” one core at each term as was evident when she explained the rule for a linear function as “adding 5, 3 times” at position 3 (P47ws). She was unable to generalize about the whole growing pattern, or write a generalized pattern rule; but, she recognized that within position 3 there were 3 groups of 5.

In their work in Lesson 3, Amy and her partner showed a more sophisticated reasoning process based on the early foundations of multiplicative reasoning. Amy and her partner made use of the visual image of the growing pattern and their first step was to circle the groups of five in the visual representation of the pattern. They used this existing visual model of groupings in order to help them begin skip-counting (see Figure 10). They first circled the group of five at position 1 and recorded “5” in the chart below, then circled the two groups of five at position 2 and said “that’s five (while circling the first group of five) and that’s ten (after circling the second group of five)” (P75v minute 0:25). They continued skip-counting by groups of five and used the visual as a model of unitizing in order to accurately determine the number of worms on day 10. After they knew that on day 10 there would be 50 worms, Amy’s partner employed a doubling whole-object strategy to find the number of worms on day 20. Amy’s partner said “there is 50 there (pointing at day 10) and 20 (day 20) is just 10 numbers ahead so it would be 100 (worms)” (P75v minute 4:30). Amy’s partner then very quickly calculated the number of worms on the 100^{th} day and she said, “on the 100^{th} day there would be 500… because it’s that number five times” (P75v minute 5:30). Amy’s partner used a general pattern rule and an efficient repeated addition strategy that involved the identification of the multiplier and multiplicand to find the far term of 100. Amy did not come up with these strategies on her own, and she likely would not have been able to use these strategies independently. She was nonetheless working with these ideas, and at times, she agreed with her partner and encouraged her partner to continue with her more sophisticated strategies. However, it was clear that Amy was more comfortable with their initial skip-counting strategy and she was more actively contributing to the partnership when they were skip-counting to find the near term of 10, compared to when her partner began using more sophisticated strategies to determine the far terms of 20 and 100. This uncertainty when working to find the far terms also indicated that Amy was likely still thinking about growing patterns as adding one core at each term. This was why she was comfortable skip-counting to find the near term of 10 but seemed confused when her partner used strategies that required more sophisticated reasoning processes to determine the far terms of 20 and 100.

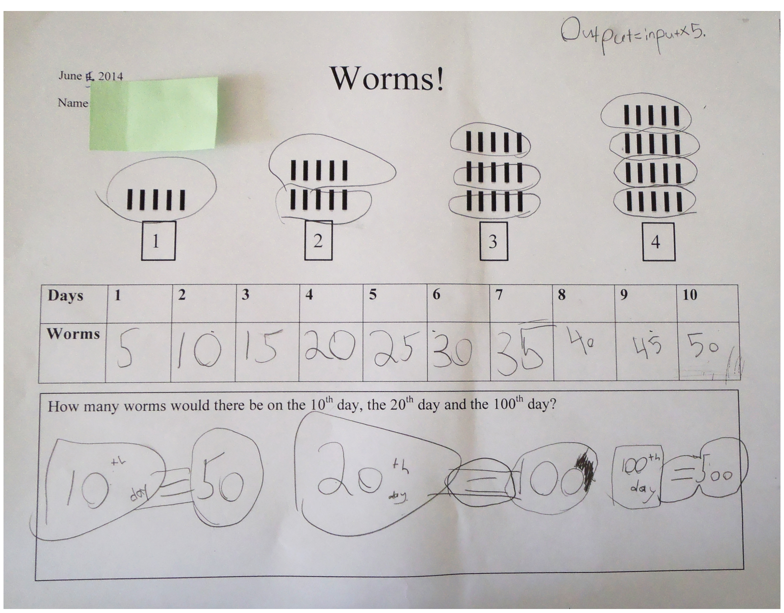


Figure 10. Amy and her partner use an existing visual model of groupings, skip-counting and repeated addition strategies

In Lesson 4 during the pattern building activity, Amy was able to build and record the next terms in her peers' patterns. She was also able to accurately record an explicit pattern rule for her peers' patterns. However, she likely did not fully understand all of the explicit reasoning involved in the symbolization process of representing a linear function as an explicit pattern rule. Although Amy could generate the pattern rule, she was unable to use that pattern rule to determine the near terms for the two patterns she studied (P193ws). This suggests she likely did not yet have a strong enough understanding of either the explicit reasoning or multiplicative reasoning (or both) necessary to generalize about any term in a linear function.

4.5.4.3 Post-assessment. In the post-assessment interviews, Amy improved slightly in her ability to determine near and far terms when compared to the pre-assessment. In the post-assessment, Amy was able to determine the near and far terms for Problem 5, likely because she was able to determine a general description of the pattern rule or pattern growth mentally and think about the pattern as *doubling* the position number. She was only able to think about a general pattern rule when she could use what she knew about doubling to conceptualize the pattern rule for Problem 5 as being "the position number doubled tells you how many tiles" rather than thinking about the rule as "Total tiles = Position number x 2".

4.5.4.4 Retention task. On the retention task Amy and her partner used a proficient skip-counting strategy beginning with use of the visual representation of the pattern to determine the total number of cucumbers on day 10. They used the visual to help them begin skip-counting by threes up to day 4 and then mentally continued to skip-count to day 10. Then, knowing that on day 10 there were 30 cucumbers, they used a combination of efficient repeated addition and skip-counting strategies to determine the total number of cucumbers on day 25 (P177ws). They decomposed day 25 into 10 days + 10 days + 5 days, and knowing that each group of 10 days was the same as 30 cucumbers, they used repeated addition to determine that 20 days would be 30 cucumbers + 30 cucumbers, and needing five more days, they skip-counted by threes to find that in five days there would be 15 cucumbers. So they came to the conclusion that 25 days was the same as 10 days + 10 days + 5 days which meant there would be 30 cucumbers + 30 cucumbers + 15 cucumbers (as can be seen at the bottom of Figure 11 below where the student circled "25th day"). They then used another efficient repeated addition strategy and identified the multiplier and multiplicand to determine the number of cucumbers on the 100th day; they stated that there would be "3 groups of 100" or 300 cucumbers on the 100th day (P177ws).

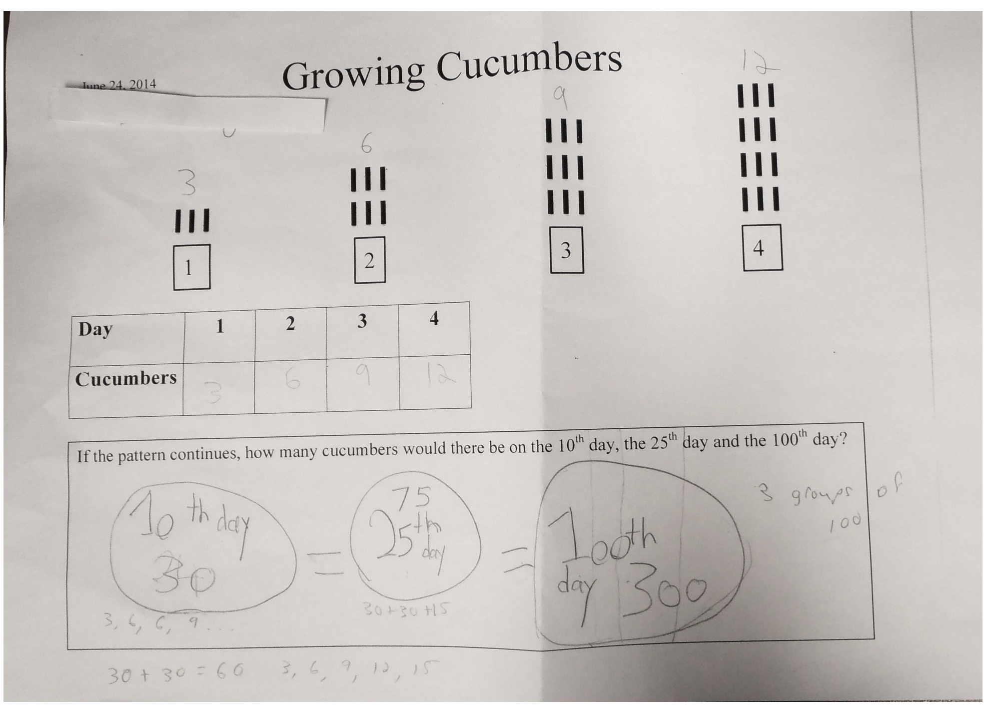


Figure 11. Amy and her partner use skip-counting and regrouping with repeated addition

4.5.5 Case 2: Nicole's progression through the five-lesson intervention. Next we turn to the case study of Nicole because, like Amy, she began the project relying mostly on recursive reasoning. Early on in the research project, Nicole thought about growing patterns as "adding" one core at a time and she depended on the visual representation of the linear functions in order to be able to visualize and then predict the structure of the pattern beyond the next term with some success. However, by the end of the project Nicole was able to generate general pattern rules that used the pattern term number consistently, and she was even able to generate and use explicit pattern rules in some instances. From the pre-assessment to the retention activity Nicole's ability to determine far terms improved and she became more proficient in her understanding of unitizing and use of skip-counting strategies. Nicole also developed some efficient repeated addition strategies and some known multiplication facts (only with the factors 1, 2, 4 and sometimes 10 and 100) by the end of the project.

4.5.5.1 Pre-assessment. In the pre-assessment Nicole thought about the linear functions as "adding" one core at a time and at times she was able to predict the visual structure of a growing pattern beyond the next term. She was able to predict the visual structure of 3 of the 5 near terms and 3 of the 5 far terms for the problems examined. It was evident that Nicole was using the visual structure of the pattern to determine what the pattern would look like and how many tiles of various colours she would need at the near and far terms with Problems 4, 5, and 6. For example, when asked to determine the number of tiles needed to build position 10 in Problem 6, Nicole used the visual structure of the last given position to generalize about the visual structure of other positions of the pattern; she said: "I would need (pointing to the column of red tiles) 10 red, (pointed to column of blue tiles) 10 blue and (pointing to the column of yellow tiles) 10 yellow" (P13v minute 1:59) (see Figure 12). Although Nicole was able to predict the visual structure of some of the patterns, she was not able to do so accurately all the time and she was unable to determine the near or far terms for Problems 3 and 7. It was clear that Nicole was relying on the visual to help her determine what other positions of the pattern would *look like* when she was working with Problem 7. With Problem 7 she was able to describe what a portion of the near term would look like but she became confused by having only a partial visual of the first four terms of the pattern (refer back to Figure 6) and then guessed that there would be about 100 tiles at position 10 and 1000 tiles at position 100 (P14v).

Figure 12. Nicole describes the visual structure of near and far terms using the existing visual of term 4 in Problem 6

4.5.5.2 Five-lesson intervention. During the first lesson Nicole showed that she was able to describe in words a general pattern rule for linear functions after examining visual representations of that function. During the function machine activity, Nicole examined a visual representation that was comprised of randomly selected positions (meaning she did not see each position of the pattern in sequential order) of a linear function and she shared with the group her finding that “every time you put in a number it’s going to double it” (P61v minute 18:50). This showed that Nicole was now able to determine a general description of a pattern rule using everyday language that also included a reference to the term number. Although Nicole showed she was beginning to think about pattern rules when working with linear functions in the first part of Lesson 1, she was not yet able to consistently generate and use a general descriptive pattern rule. Later in this first lesson, when working with a different pattern, Nicole was unable to generate a pattern rule or generalize about the nature of the pattern beyond the image that she was given. Instead, she used a direct modeling strategy and drew a picture of three groups of five in order to figure out the number of tiles at a given position; she was relying on a physical model of the groupings, which suggests she was still developing an understanding of unitizing. She was unable to communicate any information about an explicit rule or the nature of the pattern that would allow her to find the number of tiles at any given position (P44ws). This was likely because she did not yet have a strong enough understanding of unitizing to be able to think about a general pattern rule with a factor other than two; although she was comfortable doubling (or multiplying by two) she was not yet able to conceptualize multiplying by any other factors.

In Lesson 3 it was clear that Nicole was thinking about linear functions as adding one core at a time. She and her partner used a beginning skip-counting strategy to find the near term of 10 but they were unable to continue skip-counting to find the far terms of 25 and 100. Nicole and her partner used a table of values to accurately skip-count by fives up to day 10 (they may not have used the table if it had not been provided for them, more on the design of tables of values to come in the Discussion section). They then moved to a direct modeling strategy and attempted to draw and create a physical model of the 25 groups of five in order to determine the far term of 25. Nicole and her partner needed to know the previous position and count on five more from that position in order to determine the next position; they could not yet generalize about any position of the pattern with this linear function.

In Lesson 4 Nicole showed great creativity when creating her own patterns. The first pattern that Nicole created involved a constant (the bottom middle tile in each position) as well as multiplicative growth (the group of three tiles) (see Figure 13). Although her pattern was an accurate visual image of a linear function, Nicole did not fully understand this pattern. When I asked about the pattern rule for her pattern, Nicole thought that her pattern rule was Output = Input x 4. Following my questioning, Nicole realized that her visual did not have enough tiles for her desired rule at positions 2 and 3 (see Figure 13). She then created a different pattern that followed the rule Output = Input x 5. Later in the lesson, when she looked at her peers' patterns Nicole was able to determine the next terms and pattern rules but could not figure out how many tiles there would be at position 10 (P191ws). She was able to generate an explicit pattern rule but she was not able to multiply with any factor and use or apply that explicit pattern rule at this point.

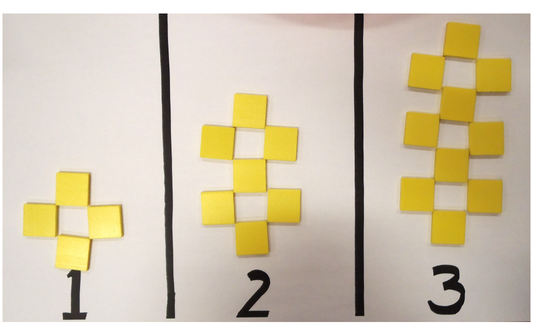


Figure 13. Nicole's first pattern

During Lesson 5, Nicole used beginning skip-counting strategies that made use of a model of unitizing. Nicole and her partner used money manipulatives to act out or physically model the situations in the problem. To figure out the total pay after 20 days with the pay rate of $2 per day and a flat fee of $10, Nicole and her partner laid a $10 bill on the table and then counted out 20 Toonies and arranged them in 4 rows of 5 (see Figure 14). This organization of the Toonies into an array helped her to be sure that there were in fact 20 Toonies on the table but she did not use the array to help her find the total amount of money. After arranging the Toonies in a four by five array, she then skip-counted by twos touching each Toonie (P106v minute 22:30). This strategy and the way that she organized the Toonies may be a pre-cursor to the development of more proficient skip-counting strategies or repeated addition strategies. In fact, when later determining the total pay after 20 days with the pay rate of $3 per day, she arranged the coins in groups with each group containing one Toonie and one Loonie demonstrating that she was beginning to unitize. She then tentatively tested a more sophisticated strategy: after counting one group of five Toonies and finding that it was $10, she put the remaining 15 Toonies into groups of five and skip-counted by $10 to find that her 20 Toonies were worth $40 and then added the $20 value of the Loonies onto the $40 value of the Toonies (P106v minute 25:40). Most of the time she skip-counted by $2 to determine the total value at each day (essentially counting by ones), but she began to successfully develop more sophisticated strategies involving the re-grouping of the coins and skip-counting by larger and more friendly numbers by the end of this lesson.

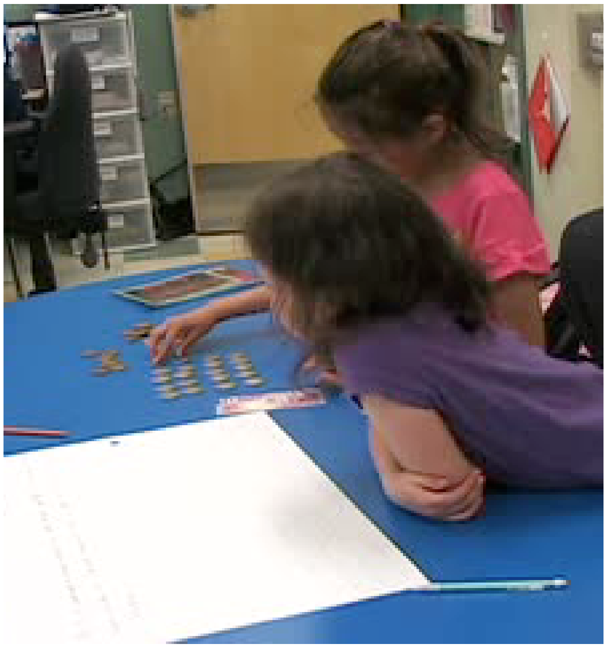


Figure 14. Nicole arranges the toonies in groups of 5 but skip-counts by 2s

4.5.5.3 Post-assessment. In the post-assessment interview, Nicole's thinking had progressed and she used repeated addition strategies and known facts to determine the near and far terms of the problems discussed. In the post-assessment, Nicole also used the visual to help her determine a general pattern rule involving the term number so that she could generalize about any term of the growing pattern. Nicole was able to determine the near and far terms for all the problems (except the far term of Problem 3) fairly quickly using a combination of repeated addition (with factors like 3) and known facts. She determined the near and far terms very quickly indicating that she was using some known multiplication facts (with factors of 1, 2, 4, and sometimes 10 or 100) and at times she described a general pattern rule using the pattern term number that she then worked with mentally to determine the near and far terms.

4.5.5.4 Retention task. In the retention task, Nicole was able to explain in words how she used repeated addition and a general pattern rule to find the number of cucumbers on any day (P155ws). Nicole used the commutative property and a number line to explain her pattern rule and she wrote "we use 10, 25 and 100 (the day numbers in the problem) 3 times each" (P155ws). Nicole was able to identify the multiplier and multiplicand but she relied on repeated addition to solve the problem. She was thinking about a general pattern rule and she was able to use the commutative property and repeated addition to solve the following multiplication problems: 3 x 10, 3 x 25 and 3 x 100. This repeated addition strategy (for example "10+10 is 20, 20+10 is 30" to solve for day 10) was much more sophisticated than Nicole's strategies from the earlier lessons that often involved some direct modelling of groups and the contents of those groups before counting the total by either skip-counting or counting by ones. It looked as if in the retention activity, Nicole and her partner began working with a strategy similar to one of her earlier strategies as they began to draw out tick marks for groups of three cucumbers. However, they likely found this strategy problematic or inefficient as they abandoned it and moved to the more sophisticated strategy previously discussed that was based on repeated addition (P174ws).

4.5.6 Case 3: Brandon's progression through the five-lesson intervention. Next we turn to the case of Brandon who had a slightly stronger understanding of explicit reasoning than Amy and Nicole by the end of the project. From the pre-assessment interviews through to the retention activity, Brandon's ability to determine the core pattern growth and generalize about linear functions progressed from thinking about linear functions as "adding" one pattern core at each term, toward generating and using explicit pattern rules. The strategies that Brandon used to determine pattern terms progressed from direct modeling strategies that used an existing visual and beginning skip-counting strategies toward efficient repeated addition strategies and the use of some known multiplication facts (only with a limited number of factors) by the end of the project.

4.5.6.1 Pre-assessment. In the pre-assessment interviews, similar to Amy and Nicole, Brandon described the growth in the various patterns as "adding" one pattern core or a certain number of tiles at each successive term. Like Nicole, Brandon was also able to predict the visual structure of the patterns beyond the next term with some accuracy for some of the patterns. With the first problems that he looked at (Problem 3 and 4) Brandon used a beginning skip-counting strategy to determine the near terms but was unable to find the far terms (P17v and P18v). He then developed a strategy that was similar to Nicole's; he used the visual structure of the pattern to generalize about the pattern beyond the next position. He used his understanding of the visual structure of any position of the pattern along with a doubling strategy (with Problems 5 and 7 to solve problems involving multiplication by two and four respectively) and possibly some repeated addition (with Problem 6 to solve problems involving multiplication by three) to find the near and far terms (P19v to P21v).

4.5.6.2 Five-lesson intervention. During the first lesson in the sequence of the five-lesson intervention, Brandon initially used a counting by ones strategy that relied on an existing visual of a pattern term but then moved on to a skip-counting strategy that was similar to his work in the pre-assessment. Brandon showed that he understood patterns were made from iterated units and he conceptualized linear functions as adding one core at each successive pattern term. When working with the Function Machine Activity (see Figure 7), to determine the number of tiles in an image of position 10 of a new linear function (Figure 15 shows position 4 of the same pattern), Brandon used the existing visual and counted the tiles by ones. At this point in the activity, the students could only see one position of the pattern and in order to determine the total number of tiles used in that position Brandon counted the individual tiles rather than considering groups of tiles within the image and skip-counting by fives (P61v minute 37). Had Brandon been asked to examine the first three terms of the pattern and then find a near or far term he may have used strategies similar to those he employed in the pre-assessment where he predicted the visual structure of the pattern and used skip-counting or repeated addition to solve for specific terms. Some of Brandon's peers who were more comfortable with unitizing and skip-counting or repeated addition saw that the tiles were arranged in groups of five and used the arrangement to determine the total number of tiles more efficiently. After one of his peers shared this idea, Brandon realized that every two groups or lines in Figure 15 were ten tiles; he employed whole-object reasoning and was able to very quickly use a beginning skip-counting strategy to count by tens and determine the total number of tiles. While Brandon's initial strategy to count the individual tiles was less sophisticated, he revised his strategy based on suggestions from his peers and was able to use a beginning skip-counting strategy to determine the number of tiles. This showed that Brandon was developing an understanding of unitizing and one of the key ideas involved in multiplication; however, he was not yet completely confident in his use of strategies that relied on that key idea. He was able to unitize in some cases but he fell back on less sophisticated counting strategies before revising his thinking and developing more efficient strategies that used groupings and skip-counting. It was clear that the activities in Lesson 1 pushed Brandon to consider some of what he knew about multiplicative growth. For example, when looking at the pattern term of a linear functions he stated “it’s like 10 x 2” (P61v minute 22:10). Brandon was able to use multiplication notation and vocabulary when discussing a possible pattern rule for a specific pattern term but did not yet have a strong enough understanding of these conventions (and the convention to use a variable or even a word to symbolize *any* pattern term number in an equation) in order to adequately describe a general pattern rule as was also evident in his Math Journal entry for Lesson 1. When asked to describe a possible pattern rule in his Math Journal, Brandon wrote that the pattern rule is “the number 5 because if you put in a 2, two 5s would come out” (P45ws). He was able to predict the visual structure of the pattern beyond the next term but did not yet have the multiplication vocabulary to be able to articulate a general or explicit pattern rule (P45ws).

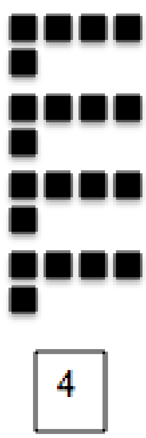


Figure 15. Position 4 of a linear function used in Lesson 1

During Lesson 3, Brandon was moving toward being able to generate an explicit pattern rule but was not quite there yet, and he used various skip-counting strategies in order to determine the near terms of the given pattern. With the *Worms in the Garden Problem* (see Figure 7 for lesson overview), Brandon used the last given term of 20 worms in the garden on day 4, and he counted by fives to figure out the total number of worms in the garden on day 10 (P69ws). He was also able to create his own pattern rule and build a visual representation of that pattern when playing the *Guess my Rule Game* with this math partner (see Figure 7 of lesson overview). When he built position 10 of his pattern it was clear that he was developing a stronger understanding of unitizing as he clearly counted and pointed to the number of groups and knew that there were three tiles within each of those groups (see Figure 16) (P70v minute 2:08). During this activity Brandon also explained to his partner the structure that can be used to write an explicit rule for a linear function; he said "if you are doing the equals (the pattern rule) you do Output = Input x___" (P70v minute 02:57). Although Brandon knew how to generate an explicit pattern rule, he was not yet able to efficiently multiply and still needed to use a skip-counting strategy to determine how many groups of the pattern core there would be at a given pattern term (the "input" in his pattern rule). After helping his partner record his pattern rule using the aforementioned format, he and his partner used a beginning skip-counting strategy and counted by threes to figure out how many tiles they used to make the tenth term that they knew consisted of 10 groups of three tiles. When skip-counting by threes, the boys only got stuck once they reached 24 and needed to count by ones to get to 27 and then skip-counted by another group of three to get to 30 and accurately determine the total number of tiles at position 10 of Brandon's pattern (P70v minute 03:47).

Figure 16. Brandon counts the groups of three in his pattern

Interestingly, also during the *Guess my Rule Game* in Lesson 3 when building a pattern based on his own pattern rule, Brandon varied the visual structure of his pattern. First he created position 10 of his pattern with groups of three tiles stacked in 10 piles. Then when creating position 6 he laid out the tiles in a different arrangement; the tiles were still in distinct groups of three tiles but they were lined up on their side instead of stacked in a flat pile (see Figure 17). Brandon thought that it would be okay to do this because he still had the correct number of tiles in position 6 according to the pattern rule and he still had the tiles arranged in groups of three with the position number indicating the number of groups of three (i.e. his pattern rule was Output = Input x 3 and at position 6 he had 18 tiles and at position 10 he had 30 tiles which was what the rule indicated he needed) (P70v minute 6:50). At this point in the lesson sequence, Brandon had an understanding of unitizing and knew that the number of groups and the number of tiles within each group was important and needed to remain consistent. However, it was not clear whether Brandon's inconsistency in the structure of the visual of his linear function indicated a lack of understanding of the way in which his pattern grew physically or if this indicated that he no longer needed the visual as a support when thinking about multiplicative growth. This also showed that he was able to produce a very creative visual representation of a linear function that was different from any pattern he had previously seen.

Figure 17. Brandon built two terms of his pattern using different visual structures

In Lesson 4, Brandon only made one small error when finding the near term for his peer's growing pattern; he used a beginning skip-counting strategy to determine the near term for a peer's pattern with the pattern rule of Output = Input x 5 and his answer was off by one group of 5 (P192ws). Although Brandon was able to accurately generate an explicit pattern rule for both of the patterns he studied, he was still not strong enough to multiply and instead used a skip-counting strategy to determine the Output with a given Input (position 10).

During Lesson 5, Brandon worked with Corey and both boys exhibited complex thought processes when confronted with a difficult problem based on a narrative context (more on Corey to follow). Brandon and Corey used graph paper to create a shorthand model of unitizing and to organize groups (they were keeping track of the multiplier and multiplicand in the pattern) that they used to ensure accuracy when later using a proficient skip-counting strategy to find the answer to various aspects of the problem (see Figure 18) (P107ws). The boys organized each group of two or three by placing one "2" or “3” in a square on the graph paper, they then skip counted to find how much you would have if you had 10 groups of three for example. Then in order to more efficiently skip-count they used whole-object reasoning and skip-counted by double the amount, reducing the number of skips. In the upper centre of the page it is clear that the boys circled the threes into groups of two threes; they then skip-counted 10 groups of six instead of skip-counting 20 groups of three as the original question had asked. On the bottom left it is clear that they employed the same strategy again; instead of skip-counting 20 groups of two and then adding on the constant of 10 in this pattern, they skip-counted 10 groups of four and added on the constant of 10 (P107ws and P108v minute 12:06 onward). When initially trying to skip-count by fours the boys recorded 20 groups of four and when the classroom teacher asked why they were counting by fours, the boys noticed their error while they were explaining their reasoning and adjusted their solution so that they had 10 groups of four instead of 20 groups of four. The teacher and I questioned Brandon and his partner while they were working on this problem and it seemed as though both boys understood the strategies they were using and had good ideas as to why these strategies would work in this context. However, this strategy was Corey’s idea and Brandon was able to help his partner implement all aspects of the strategy but he may not have accurately used a doubling and halving strategy to increase the efficiency of skip-counting on his own. Brandon was nonetheless working with these complex ideas, contributing to this partnership and actively working with Corey to find the far terms.

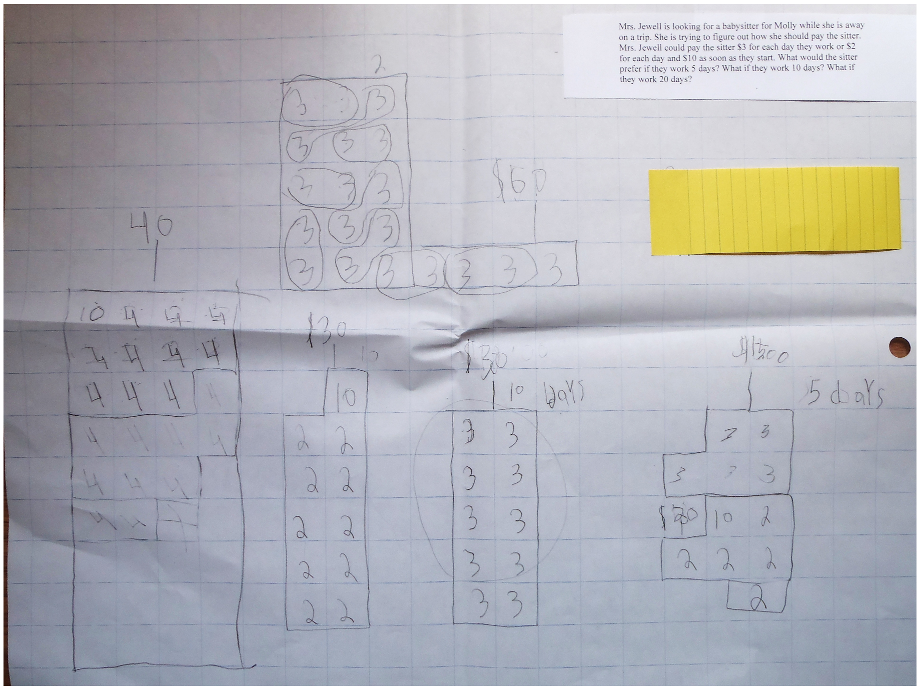


Figure 18. Brandon and Corey used the squares on the graph paper to create a shorthand model of unitizing

4.5.6.3 Post-assessment. In the post-assessment interviews it was clear that Brandon employed more sophisticated strategies when compared to his pre-assessment interview. In the post-assessment interview Brandon exhibited strategies that used skip-counting and efficient repeated addition or known facts. In the pre-assessment interview it took Brandon a while to figure out the near and far terms for many of the patterns because he was trying to skip-count on from the last given term. Surprisingly, in the post-assessment he was able to determine the near and far terms almost instantly in many cases, suggesting that he was generating and using an explicit pattern rule along with some known facts and possibly some mental repeated addition (P131v to P135v).

4.5.6.4 Retention task. Brandon's work from the retention task suggested that he was able to use a shorthand model of unitizing and proficient skip-counting strategies with accuracy. On the retention task, Brandon used the existing visual and extended it in order to create a visual model of the groups of cucumbers on each day. However, he then abandoned this diagram and moved to a more efficient shorthand model of unitizing and proficient skip-counting strategy. He was able to build on what he knew about the 10^{th} day in order to figure out the number of cucumbers on the 25^{th} day using a proficient skip-counting strategy. Brandon said that if he knew there would be 30 cucumbers on day 10 that would help him figure out how many cucumbers there would be on day 25 because 25 is 15 more than 10 so he needed 15 more groups of three on day 25 (Figure 19). Knowing there were 30 cucumbers on day 10 he was then able to start at 30 cucumbers on day 10 and create his own table of values which he used as a shorthand model of unitizing to accurately skip-count 15 more groups of three in order to determine that on day 25 there would be 75 cucumbers (see the red box in Figure 19). He was also able to determine the far term of 100 and most likely he mentally considered an explicit pattern rule then used the commutative property and repeated addition to think about 3 x 100 as being 100 + 100 + 100 to find that on the 100^{th} day there would be 300 cucumbers.

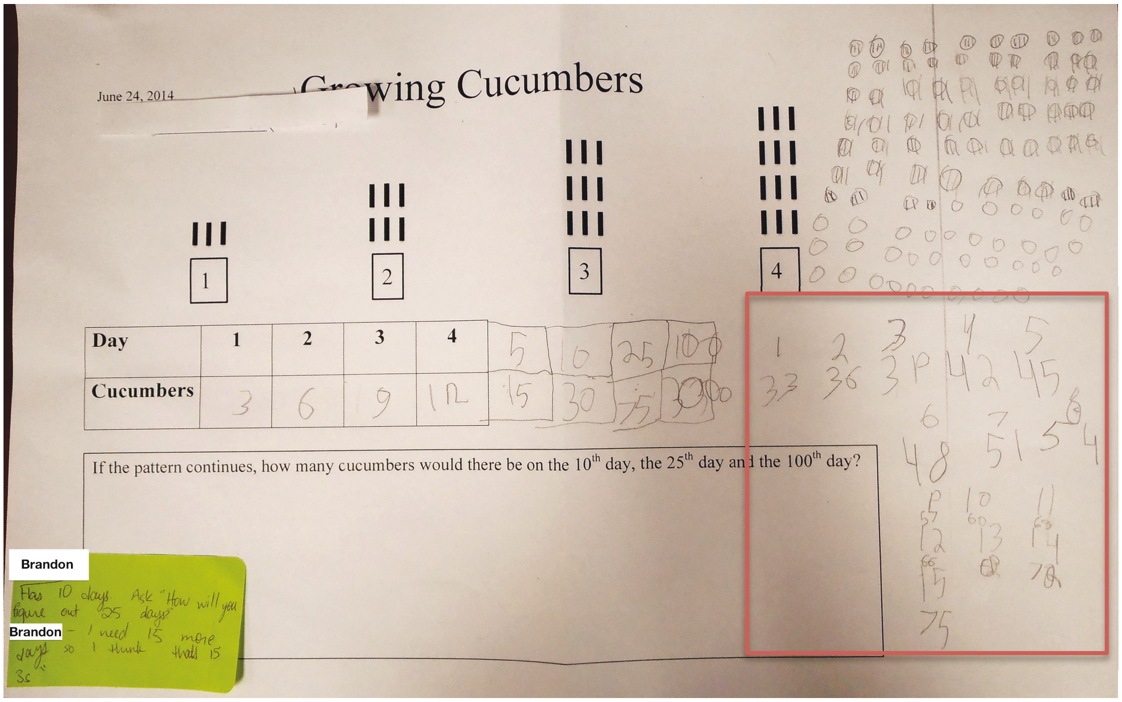


Figure 19. Brandon uses a table to start at 30 and skip-count 15 more groups of three

4.5.7 Case 4: Eric's progression through the five-lesson intervention. Now we turn to the case of Eric who, like Brandon, made notable gains in his understanding of multiplication by the end of the project. Eric was one of the younger students participating in the research project (Grade 2) and he focused on the visuals early on. He was able to use the visual representations of linear growing patterns to construct an understanding of multiplication. Initially Eric used the visuals in order to predict the visual structure of the pattern beyond the next term and he used his understanding of the visual as a general pattern rule including the term number. He then used repeated addition along with his pattern rule in order to generalize about any term of a growing pattern. His focus on the visual and visualizing the growing patterns also led to his discovery of the commutative property later in the project. By the end of the project he was comfortable generating explicit rules and general rules with pattern term numbers that he could efficiently use along with repeated addition or known multiplication facts (with factors of 2, 4, 10 and 100) to find near and far pattern terms.

4.5.7.1 Pre-assessment. In the pre-assessment it was clear that Eric was working with and using the visuals to predict the visual structure of the pattern beyond the next term. He was able to determine all near and far terms with some difficulty but, accurately nonetheless. He used his understanding of how the pattern grew visually to create and apply a general pattern rule that used the term number; he would think about how many groups of various coloured tiles there would be at a specified position. For example, when examining Problem 7 where the students were only given a partial visual of the pattern Eric initially struggled (see Figure 6). It was difficult for Eric to picture how many groups of 100 there would be at position 100 without being able to see how many groups of one there were at position 1 and how many groups of two there were at position 2 and so on. To help solve the problem, Eric decided to create an extra column of red tiles to represent the new column of green tiles and he built a full visual of position 1, 2, 3 and 4 based on my verbal description of the new pattern (see Figure 20). After having built the first four positions of the pattern he was able to determine a general rule that used the term number and he identified the multiplier and multiplicand when explaining that in the 100^{th} position "there would be 4 hundreds so there would be 400" (P28v minute 3:05). Eric initially approached some of the patterns with a focus on "adding" one core at a time and he skip-counted to find some of the near terms. However, he then switched to strategies that used explicit reasoning when thinking about the near and far terms; he used his understanding of the visual structure of the pattern to think about how many groups of red, blue, or yellow tiles there would be as well as how many individual tiles there would be of each colour (or in each group) at any given term. He was able to determine the near and far terms by using repeated addition and a general rule involving the term number that was based on the visual structure of the pattern.

Figure 20. Eric creates a full visual for Problem 7

4.5.7.2 Five-lesson intervention. During Lesson 1 Eric showed that he was thinking about the nature of multiplicative growth in the patterns explored, as well as the ways in which various aspects of different pattern representations are connected. During the group discussion, Eric asked "will the machine work in reverse? If I put in an output (the image of a specific position) could I get out an input (the position number)?" (P61v minute 8:56). Eric's question showed that he was thinking about the relationship between the visual image and the position number; he was considering how to determine what the position would *look like* from a given position number or what the position number would be from looking at a visual of a given position. Eric's idea to use the function machine in reverse is also strongly tied to the concepts used to rearrange traditional algebraic equations in order to solve for different variables. He was working with an idea similar to the following example: with an equation like $y = 4x$, if I know x then I can solve for *y,* or if I know *y* then I can solve for *x*. However he did not yet have a strong understanding of the vocabulary used to describe multiplicative growth. When describing a pattern rule, Eric wrote "it will go up by 5s and down" which is not an explicit rule but a description of the pattern based on "adding" one core at each successive position (P46ws). However, he likely was also able to predict the visual structure of the pattern beyond the next term but was unable to communicate this on paper due to his low writing ability (with a scribe he may have been able to record something closer to a general pattern rule based on the visual structure of the pattern).

During the group discussion introducing the problem for Lesson 3, Eric demonstrated that he was able to unitize as well as use repeated addition and the commutative property to determine the near and far terms of a linear function with great efficiency. When looking at a visual representation of the linear function used to introduce Lesson 3 (see Figure 21), many of the students said that on day 3 there would be three groups of two bugs. However, to increase efficiency, Eric impressively suggested that "if you turn it over… then I see there are two separate groups with three on one side and three on the other side… there are six bugs because I know 3 + 3 = 6" (P85v minute 2:30). To help his peers understand Eric's suggestion, I then drew on chart paper what he had explained to the class and Eric clarified his understanding of the commutative property. Eric said that position 3 could look like three groups of two or if you turned it over then he could think about it like two groups of three which was faster for him to calculate (see Figure 22). He also explained to his peers that it was the same amount whether you thought about it like two groups of three or three groups of two because you were not changing the amount, the only thing that changed was the orientation. His analysis of the visual along with his flexibility in making groups and re-arranging those groupings allowed him to 'create' a new problem of 2 x 3 that he could easily think about in terms of repeated addition (3 + 3) and use a known addition fact to solve. In his work from later in Lesson 3, Eric used an explicit rule and likely some mental repeated addition to determine the far terms of 20 and 100 (P72ws). He also used skip-counting to determine the number of bugs for some of the closer days. Throughout this lesson, Eric showed a very high comfort level with many different strategies and he was able to quickly assess some of the relationships between numbers and operations so that he could efficiently solve all the problems.

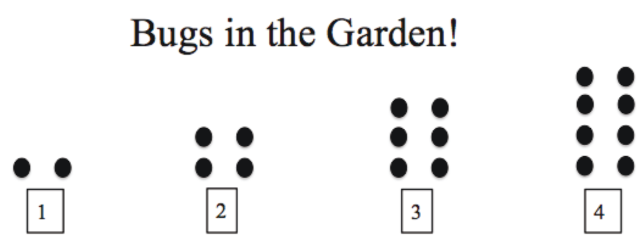


Figure 21. Bugs in the garden problem visual

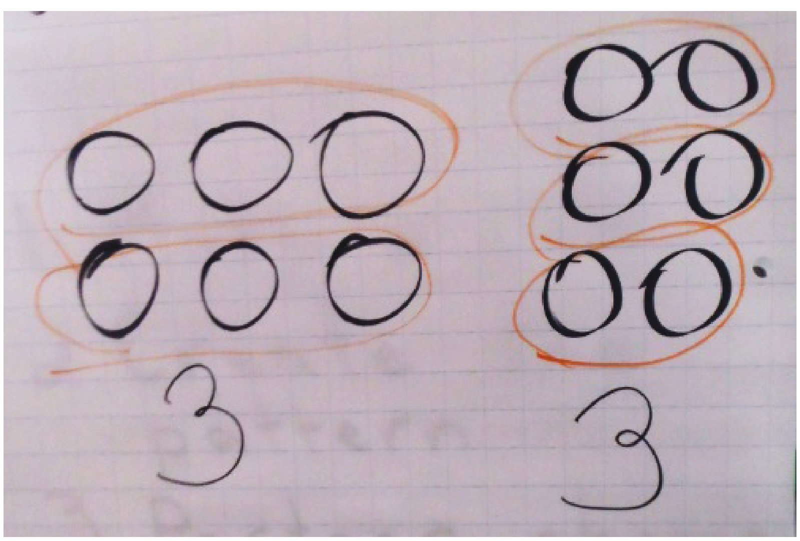


Figure 22. Eric discovers the commutative property

In Lesson 4 during the introduction to the pattern building activity, Eric again brought up the commutative property. After looking at Problem 1 and Problem 2 (see Figure 23), Eric asked to come up to the SMART Board to show his peers how position 3 of Problem 2 was "the same" as position 2 of Problem 1 (P103v minute 11:30). With Problem 2 up on the SMART Board, Eric illustrated the commutative property by moving the tiles in position 3 to prove that the tiles in position 3 of Problem 2 could be rearranged to look like position 2 of Problem 1 (see Figure 24). His manipulation of the image was essentially a proof for the commutative property as he illustrated that 3 x 2 = 2 x 3. During this lesson Eric also demonstrated his very strong understanding of the connections between various representations of linear functions and he explained to the class how he used the visual to identify the multiplier and multiplicand along with a pattern rule that involved the term number to almost instantly find the near term of 10 (P103v minute 7:15). He explained that with Problem 1, "for position 1 there are 1 threes, for position number 2 there are 2 threes, for position number 3 there are 3 threes, for position number 4 there are 4 threes so at position number 10 there would be 10 threes" (P103v minute 7:15). In this way, he visualized the pattern structure and used it as a pattern rule to determine the total number of tiles at any given term number.

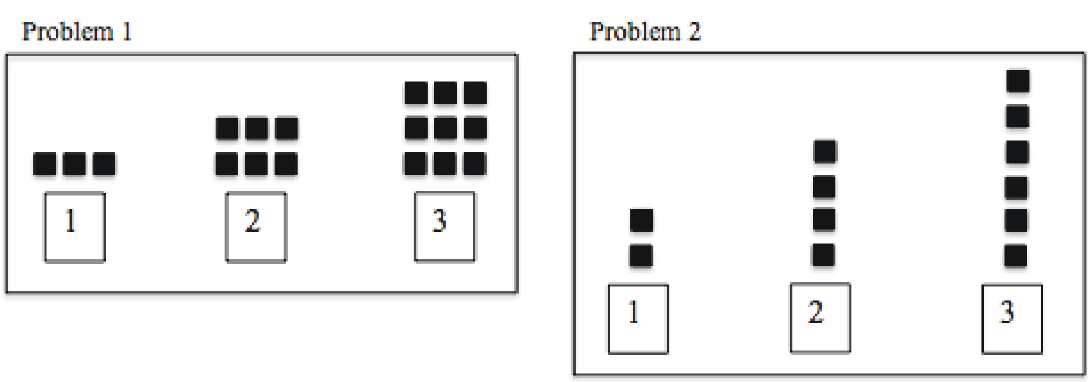


Figure 23. Lesson 4, Problems 1 and 2

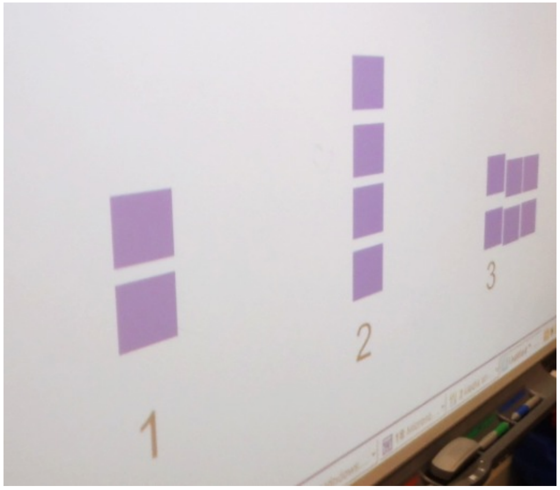


Figure 24. Eric re-arranged position 3 of Problem 2, illustrating and proving the commutative property

In the final lesson of the five-lesson intervention, Eric and his partner tried to use a skip-counting strategy in order to compare the two linear functions but they were unable to do so accurately. Their mistakes were mostly due to the pair's very unorganized record of their findings and were likely magnified by Eric's low writing ability (had Eric had access to a scribe when working on this problem, he may have been able to solve the problem with greater efficiency and accuracy). It is also possible that Eric may have still needed a visual representation of the linear functions in order to be able to think about the growing patterns in Lesson 5 in a more general way; his earlier work relied heavily on using the visual and he may have struggled with Lesson 5 because no visual was provided.

4.5.7.3 Post-assessment. During the post assessment Eric was again able to accurately determine all the near and far terms for all the problems explored. In the post-assessment Eric calculated the near and far terms very quickly; he determined the answers accurately and almost instantly where as in the pre-assessment he needed some time to think about how he was going to find the near and far terms. His increased speed in the post-assessment suggested that he was determining an explicit pattern rule, then using known facts (with factors of 2, 4, 10 and 100) and possibly some very efficient mental repeated addition.

4.5.7.4 Retention task. In his work sample from the retention task, Eric and his partner used an explicit rule and either known facts or they may have mentally used efficient repeated addition to determine each far term (P176ws). He was also able to explain in words a general pattern rule that used the term number and he provided a few examples to prove that his rule worked for the pattern in question (P157ws).

4.5.8 Case 5: Corey and Alison's progression through the five-lesson intervention. The final case study focuses on Corey and Alison who both began the project with a stronger understanding of multiplication than the other students. Alison and Corey consistently demonstrated that they were able to select from a range of strategies based on what they believed to be the most efficient way to solve various problems. Throughout the entire project they moved between strategies that used proficient skip-counting, doubling, efficient repeated addition and known facts depending on what they believed to be most accurate and efficient for a specific problem. They often worked with strategies that made use of their strong mental computational abilities and Corey did not rely on the visual representations of the linear functions in many instances, perhaps because he was focusing on other aspects of the relationship between the two sets of data involved in the linear functions. Even in their pre-assessment interviews, Alison and Corey were already able to generate and use a general pattern rule that used the term numbers to help them efficiently determine the near and far terms. By the end of the project they were able to generate and apply pattern rules; they considered the multiplicative nature of an explicit pattern rule for a linear function and used a known multiplication fact of a combination of strategies to determine the required multiplication fact.

4.5.8.1 Pre-assessment. In the pre-assessment Corey was able to accurately and very quickly determine the near and far terms for all of the problems except for the far term of Problem 3. When he saw Problem 3, the most difficult problem, he said "it's a growing pattern" and later added that he had seen this type of pattern before "with numbers" (P38v minute 1:04). It was clear that this was how he approached all the patterns that were studied in the interview; he did not use the visuals except to determine the *number* of tiles at each position. Corey made one error when determining the near term of Problem 3: he used a proficient skip-counting strategy to count by twos starting at three (position 1) to find position 10. Knowing position 10 he then tried to use whole-object reasoning to find position 100 by repeatedly adding groups of 21 (the number of tiles at position 10). Corey was focused on the number of tiles at each position and he was able to efficiently generalize about the linear functions but he was not sure whether the visual arrangement of the tiles mattered as was evident when he built the next term of Problem 4 (see Figure 25). For all of the other problems, Corey was able to very efficiently and quickly find the near and far terms for the patterns by determining a general pattern rule that used the pattern term numbers and then using known multiplication facts or repeated addition to apply his pattern rule to any specified term.

Figure 25. Corey wonders if the visual arrangement of the tiles *matters* when building position 4 of Problem 4

In the pre-assessment interviews Alison relied on the visual more than Corey did and she was able to predict the visual structure of the patterns beyond the next term. At times she was also likely mentally working with a general pattern rule that used the pattern term number. She used some repeated addition and known multiplication facts to quickly and accurately determine the far terms for all patterns, even the most difficult ones (P3v to P7v). She only made one error when determining a near term and her error indicated that she was working with a doubling strategy. When asked to determine the near term (position 10) in Problem 6, Alison looked at position 4 (the last position built) and saw that there were 12 tiles. She then said that at position 10 there would be 24 tiles because she made a small error in her doubling strategy; if she had been looking at position 5 and there were 12 tiles, then doubling both the position number to get to position 10 and the number of tiles to get to 24 tiles would have given her the correct number of tiles for position 10. Although she made an error, in that she did not double position 5 to find position 10 and instead doubled position 4 which would have actually given her information about position 8 not position 10, this strategy showed that Alison was using whole-object reasoning in a logical way, she understood how the pattern was growing multiplicatively and she was able to use a doubling strategy (P6v minute 1:24).

4.5.8.2 Five-lesson intervention. During Lesson 1 Corey used a number line to describe a general pattern rule that also used the pattern term numbers. He related the multiplicative growth of the linear function to the jumps on a number line and he knew that the position number indicated how many jumps of five to take on the number line (P48ws) (see Figure 26 where he is identifying the multiplier and multiplicand when tracking both the number of jumps and the size of each jump, along with a running total). Corey's solution also showed that he was using an efficient repeated addition strategy.

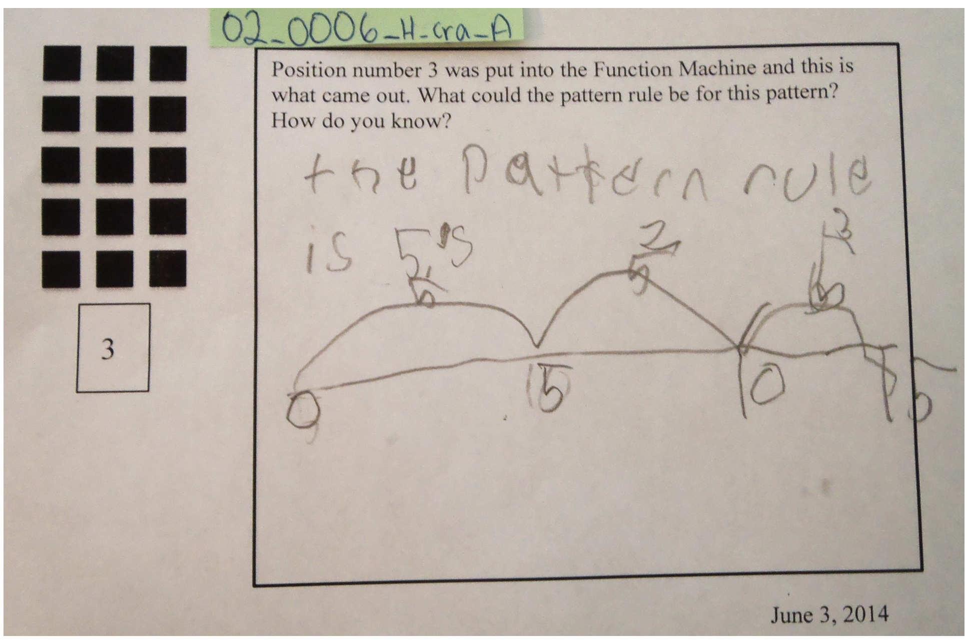


Figure 26. Corey used a number line to explain his pattern rule

During Lesson 1 Alison was able to generate an explicit rule and she showed that she had an understanding of the language and notation often used with multiplication (P43ws). When asked to examine an image of one term of a pattern and describe a possible pattern rule for that pattern, she wrote “the pattern rule could be: any number x 5” and she used the visual to circle five groups within an image of a pattern term (P43ws). Surprisingly, even in this first lesson, Alison already exhibited a strong understanding of how to determine and clearly communicate an explicit pattern rule involving multiplicative growth in linear functions.

In Lesson 3, Corey and his partner used an explicit rule and a known fact to determine the number of worms on day 10 and then used a doubling strategy in order to determine the number of worms on day 20 (see Figure 27). Corey and his partner explained that they used what they knew (that there were 50 worms on day 10) to use whole-object reasoning and a doubling strategy to determine the number or worms on day 20. They explained that they knew how to double, therefore knowing day 10 they doubled the day number and doubled the number of worms to find day 20. They then likely used the commutative property along with a known fact or some mental repeated addition in order to use their explicit rule to determine the number of worms on the 100^{th} day (P76ws).

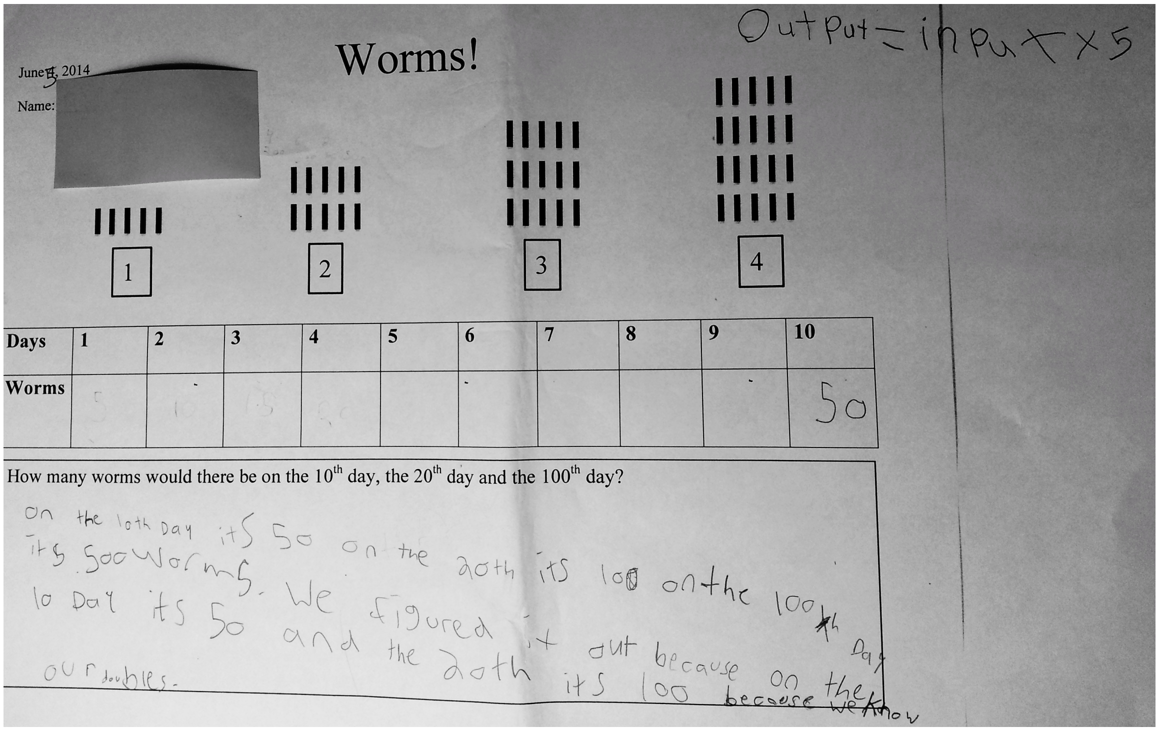


Figure 27. Corey and his partner's doubling strategy

During Lesson 4, Corey described to the whole group how he used a skip-counting by twos strategy to determine position 10 of one of the functions. He was also able to accurately determine an explicit pattern rule as well as describe the next and near terms for his peers' patterns (P194ws).

During Lesson 4, Alison accurately created at least two visual representations of a linear function from her own pattern rule (P87ws and P88ws). She also accurately identified the explicit pattern rule, next terms, and near terms (likely using an explicit rule and a known multiplication fact) for two different patterns that her peers' had created (P190ws).

As previously discussed, Corey worked with Brandon during Lesson 5 and his work for this lesson can be found in section 4.5.5.2. Corey used proficient skip-counting strategies and a shorthand model of unitizing to find the near and far terms of the linear functions in Lesson 5.

In Lesson 5, like Corey, Alison clearly used a short hand model of unitizing along with some skip-counting to solve the problem. However her model more closely resembled a visual representation of specific terms within the growing patterns and she used repeated addition in order to work with the problem more efficiently. In the top left corner Alison and her partner identified the two linear functions as “A” (you are paid $2 each day plus a $10 lump sum payment to start) and “B” (you are paid $3 each day), then below that they created a visual representation first for B (see Figure 28 solid line box) and then for A (see Figure 28 dotted line box) (P104ws). For both linear functions the girls drew a circle to represent each day and then for function B placed three tick marks within each circle and for function A placed two tick marks within each circle and drew the constant of $10 separately. The way Alison represented the functions with a circle for each day and then the daily pay represented within each circle shows an understanding of unitizing. She then used some skip-counting and an efficient repeated addition strategy to find the amount of pay you would receive for working 20 days using the pay rate of $3 per day (function B); she likely skip counted earlier on to find that four groups of three is 12, she then boxed around groups of four days showing those are each $12 and then used another box to show an extra $6 which she then added together to figure out the total pay for half of the 20 days. She then likely used whole-object reasoning to figure out the total pay for 20 days having already found the pay for 10 days. Alison made one addition error when determining the pay after 10 days using the pay rate of $2 per day and a lump sum payment of $10 (function A), however the way she created and used a visual representation of the functions involved in the problem suggests that she would be able to generalize about any term within each of the linear functions examined.

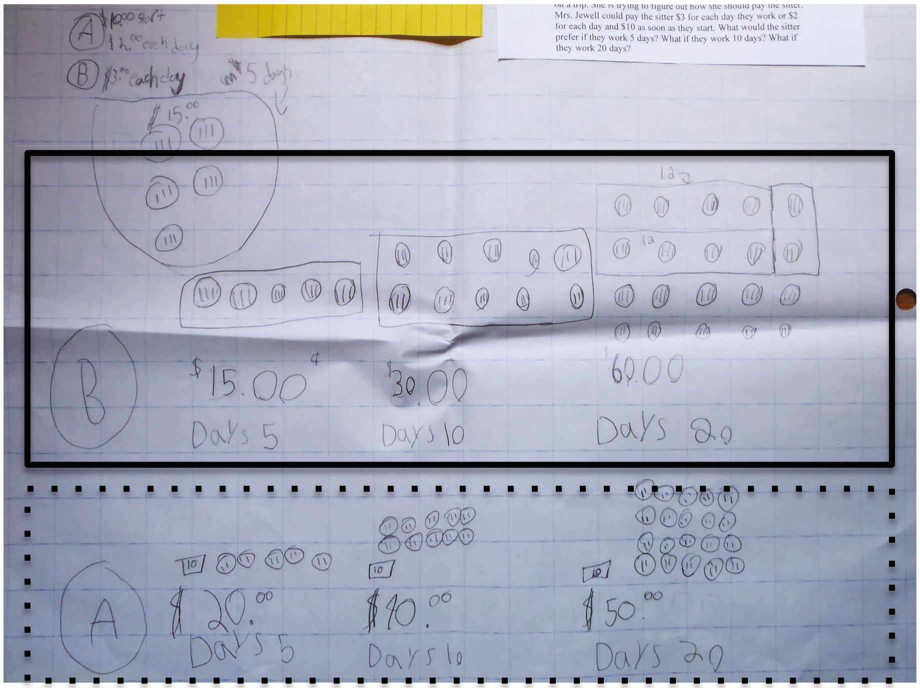


Figure 28. Alison creates a visual representation of specific terms of the linear functions to find the specified pattern terms

4.5.8.3 Post-assessment. In the post-assessment, much like in the pre-assessment, Corey was able to determine the near and far terms for the problems with great efficiency by generating and using an explicit rule along with known facts. When finding the near term for Problem 3, Corey tried to use whole-object reasoning and a doubling strategy, which resulted in an error because this problem involved a constant. However he did not repeat this error when finding the far term of the same problem; he was able to revise his strategy to accurately find the far term.

In the post-assessment interviews Alison again only made one error and it was on the same near term. However, in the post-assessment the error that she made was of a different nature: when looking at the first few terms of the pattern Output = Input x 3 she said that at position 10 (the near term) there would be 40 tiles (the correct answer would have been 30 tiles) (P122v). She answered the question very quickly indicating that she was multiplying, using known facts and using an explicit pattern rule, but she made an error when multiplying 10 x 3 likely because she was trying to use a doubling or repeated addition strategy and made an error (i.e. 10 + 10 = 20, 20 + 20 = 40 rather than 10 + 10 = 20, 20 + 10 = 30).

4.5.8.4 Retention task. Corey and his partner's work from the retention task showed that they began by using a skip-counting strategy to extend the given table of values and they recorded the day number and number of cucumbers up to day 12. Then having found the number of cucumbers in 10 days, they tried to use this information to help them find the number of cucumbers after 25 and 100 days. They tried using repeated addition to add groups of 30 cucumbers (or groups of 10 days) to find the number of cucumbers on day 25 and 100 but this strategy was problematic for them so they revised their strategy. They then began thinking in terms of an explicit rule and identified the multiplier and multiplicand so they could use repeated addition to find a derived fact: Corey said that for 25 days "it's like tripling: 25, 25, 25 so 75" (P178ws). This repeated addition strategy made use of a general pattern rule involving the term number and also employed the commutative property. Instead of thinking about the problem as being twenty-five groups of three (as was indicated in the problem context), Corey was able to use the commutative property and generate a general pattern rule that involved tripling the day number or in this case finding three groups of twenty-five. Although he did not show any work indicating how he found the far term of 100, Corey likely used the same "tripling" rule to mentally determine position 100.

Alison's work from the retention activity made use of an explicit rule and repeated addition. Alison and her partner were able to very quickly and efficiently determine the near and far terms of this linear function. Initially, to find the near term of 10, which was only six terms away from the last given term, the students skip-counted using a table as can be seen in the bottom left half of their solution (see Figure 29) (P173ws). Then knowing the pattern rule, which they generated from the visual and recorded on the far right of their paper, the students used repeated addition to determine the far terms of 25 and 100. The bottom right half of their solution shows how they added 25 three times and then added 100 three times in order to solve 25 x 3 and 100 x 3 respectively. Later in her math journal, Alison also described a general pattern rule that used the term number of this function as being "I multiply the day number by 3" (P152ws). She was able to independently and very clearly describe the way in which you could use an explicit rule to determine the number of cucumbers on any day but needed to use repeated addition to *multiply* by three.

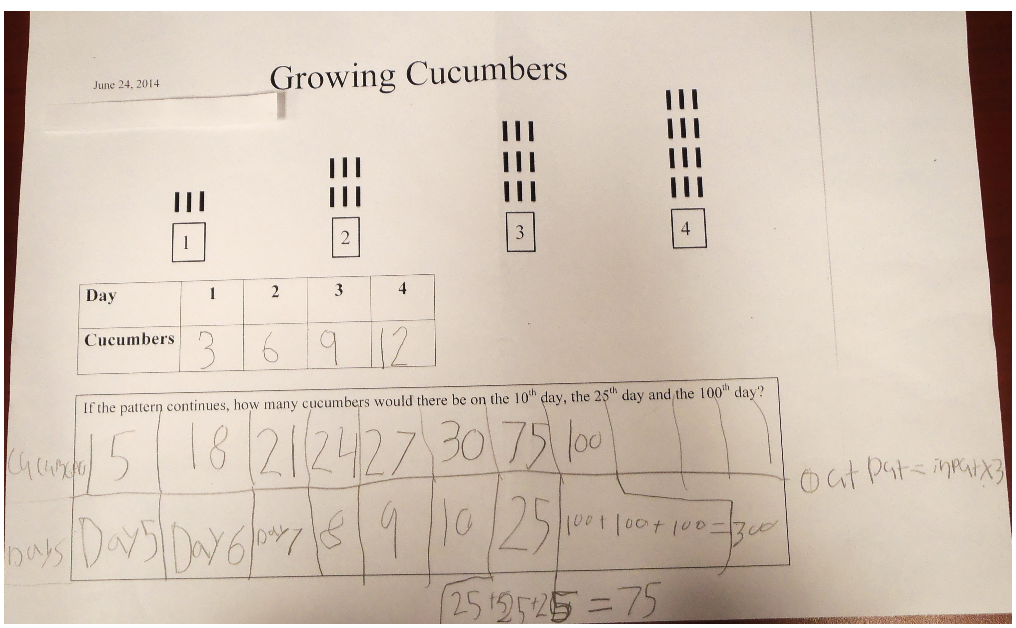


Figure 29. Alison's skip-counting moves to repeated addition to find far terms

Chapter 5: Discussion

The general purpose of this study was to explore the following research question: do primary students have the capacity to intentionally use, or develop, explicit reasoning skills when examining linear functions using designed visual representations (in most cases with these visuals being designed by the teacher and provided to the students)? The findings indicate that, within the small sample of students used in this study, most Grade 2 and 3 students are able to develop varying levels of explicit reasoning – even with little to no previous knowledge of multiplication. More specifically, I asked: how does working with designed visual representations of linear functions affect the algebraic thinking of young students? Can students move beyond additive or recursive thinking when working with linear functions? The results indicate that working with designed visuals of linear functions, even over a very short time (the lesson intervention had a duration of only one week), can encourage most students to develop explicit reasoning skills and refine their abilities to generalize about any term of a given linear function.

This study investigated the research questions within the context of *simple* linear functions due to the young age of the study's participants (Grades 2 and 3). The linear functions that were explored throughout the study all involved multiplicative growth with *easy* factors such as 2, 3, 4, and 5. Very few linear functions that involved a constant were used throughout the study and most of the near and far terms that students were asked to determine were also *easy* numbers to work with (e.g. 10, 20, 25, 100). Simple linear functions were used in the study to ensure that the young students, who had little prior knowledge of multiplication, would be able to work with the problems, and so that I could focus on the children's development of multiplicative reasoning processes. Had the students been confronted with problems involving more complex linear functions (e.g. linear functions with more challenging multipliers like 9 or 13, linear function with constants, or if students were asked to find far terms like 63 or 79) the results of the study may have been very different.

5.1 Developing Explicit Reasoning Skills Through Working with Linear Functions

I found, along with Beatty and Bruce (2012), Chapin and Johnson (2000), and Moss and London McNab (2011), that the use of visuals combined with various other representations of linear functions (tables, pattern rules, verbal or written descriptions of pattern rules, narrative contexts) can encourage students to focus on the relationships between the two sets of data involved in linear functions. Similar to Carraher, et al. (2008) and Beatty and Bruce (2012), I found that the concurrent use of multiple representations of linear functions is a very powerful support for students working with linear functions. Moreover, students can develop strategies that employ explicit reasoning to varying levels of complexity. Figure 8, found in the Results Chapter (page 55), depicts a progression of young students' abilities to generalize about any term of a linear function (i.e. their level of explicit reasoning); the characteristics of students at each level of explicit reasoning will be discussed in the sections that follow. The strategies that students commonly use as they transition from recursive to explicit reasoning will also be discussed (Figure 9 on page 56). The sophistication of a student's algebraic reasoning (see Figure 8) does not always align with the sophistication of their strategies (see Figure 9). For instance, a student may be able to use high explicit reasoning processes (they may understand the relationship between the two data sets and be able to identify an explicit pattern rule), but they may be relying on less sophisticated strategies and have to use a skip-counting strategy to find a specified term.

5.1.1 Indicators of recursive reasoning. Most students who exhibit recursive reasoning rely on a visual representation of a linear function in order to identify the core of the pattern, however, they are often unable to generalize about any term in the pattern. These students are able to consistently isolate the pattern core in a visual of a linear function and identify the next terms as well as some closer near terms. They are able to determine next and some near terms because they are able to add on the required amount of pattern cores. They can usually only determine near terms if they have the opportunity to directly model (either with manipulatives or some kind of drawing) the required number of groups of pattern cores to add; generally, they cannot determine near terms mentally. Many of these students rely heavily on the use of direct models of groups of the pattern core and then they either count by ones or use beginning skip-counting strategies to determine near and far terms. These students are unable to successfully determine far terms (e.g. usually not able to determine far terms of 20, 25 or 100 independently) likely because they are not able to accurately skip-count or directly model and count by ones very far ahead of the last given term. They are limited in their understanding of linear functions (they conceptualize linear functions as being made of iterated units or adding one core at each subsequent term) and consequently only have direct modeling and counting or beginning skip-counting strategies to draw upon.

5.1.2 Indicators of moderate levels of explicit reasoning. Students with moderate levels of explicit reasoning can successfully generalize about *some aspects* of any term of a linear function and they often can determine next, near and far terms with some consistency. These students may draw upon whole-object reasoning as they begin to focus on the pattern's growth. Then, as students develop a more thorough understanding of the relationships between the two sets of data involved in a linear function, some may develop the ability to use the visual of a linear function as a means to facilitate their generalizations about that pattern; this is similar to the findings of Beatty, et al. (2013) and Moss and London McNab (2011). These students may also use the visual structure of one or more given terms of the linear function in order to communicate a generalization about any pattern term, about a specified unknown pattern term or about the pattern's growth in general. For these students, as Beatty and Bruce (2012), Beatty, et al. (2013) and Moss and London McNab suggest, the inclusion of a term number in activities involving linear functions is instrumental to their understandings of linear functions and their development of explicit reasoning. Students with moderate explicit reasoning are often working with, or beginning to develop, repeated addition strategies and they use what they know about the visual structure of the pattern almost as a formula to govern or carry-out their repeated addition strategy (they use the visual structure to identify how many are in each group and how many groups need to be added). These students are also developing or have an early understanding of unitizing and they use their understanding of addition and unitizing in order to solve problems involving multiplicative growth. Although this study did not include any data to support this speculation, if given a pattern rule and asked to determine information about specific terms of that pattern, students at the moderate level of explicit reasoning would likely need to first build or draw a visual of a linear function.

5.1.3 Indicators of high levels of explicit reasoning. Students with high levels of explicit reasoning are able to communicate a generalization about any term in a linear function and they can use their generalization in order to determine the characteristics of any given term. These students are able to use the information about a linear function provided in a representation of that function in order to generate: a general pattern rule that uses every day language and a reference to the term number, or an explicit pattern rule that is structured like an equation and uses symbolization processes to represent unknown values. Most of these students have an understanding of multiplicative growth and are able to use repeated addition or multiplication in order to determine and apply an explicit rule. However, some students may have strong enough explicit reasoning skills to generalize about any term of a pattern, and yet they may still be figuring out the mechanics of multiplication; they may be able to determine a general or explicit rule yet not be able to actually carry out the required calculation at this time.

5.2 Explicit Reasoning Sparks a Need to Invent Multiplication

The final research question was: how do the various representations of linear growing patterns help or encourage students to invent multiplication? The findings suggest that working with visual representations (along with other representations) of linear functions, and activities that encourage explicit reasoning lead some students to invent repeated addition and, or, multiplication. Most students are able to fairly quickly develop moderate or high levels of explicit reasoning as previously discussed. The students' use of explicit reasoning and their abilities to generalize about linear growing patterns can lead some of them realize that addition alone is not sufficient when working with problems that ask them to find near and far terms. Therefore, these students can develop a way to more efficiently work with groups and many of them can use the visual of a linear function as a tool to help them unitize and repeatedly add groups of the same size. Other students who have stronger understandings of unitizing can also refine their understandings of multiplication through their work with the visual representations of linear functions. Many of the visual images of linear functions used throughout the research project were structured as an array, with one pattern core (a row or column of the array), added at each subsequent pattern term. This structure of the visual may have encouraged students to develop an understanding of some of the structures of multiplication. Young-Loveridge (2005) suggests that students often do not encounter multiplication as both repeated addition and arrays; most instruction in multiplication is limited to multiplication being described as repeated addition in the earlier grades. The activities used in this project represented many linear functions using an array structure and the students were never told that multiplication was like adding a certain number of equal size groups. Interestingly, students can use the visuals of linear functions (especially those structured as arrays) to unitize, and repeated addition strategies seem to follow naturally.

Similar to Beatty and Bruce (2012), Beatty, et al. (2013), and Moss and London McNab (2011), I also found that the purposeful use of supporting visuals that included pattern term numbers and coloured tiles can help make aspects of linear functions accessible to young students. The use of visuals of linear functions provide students with a concrete model of unitizing that they can use to generalize about a linear function. In the visual, students can see the impact of adding on a pattern core, or one group, at each successive pattern term. This helps some students conceptualize the differences between the impacts that additive and multiplicative growth have on the total; students can *see* that increasing by one (one pattern term) has a different impact on the total than increasing by one in an additive context. The visuals of linear functions, especially those that are arranged as arrays, help set the foundation for this important idea that will later lead to an understanding of area and proportional reasoning. Furthermore, as Clark and Kamii (1996) suggest, multiplication is complex due to the many different inclusion relationships that occur concurrently. I believe the visual representations of linear functions, especially those designed using an array structure, make some of the inclusion relationships involved in multiplication more transparent to young students. When examining a visual representation of a simple linear function (see Figure 30), students can simultaneously see: within each group of five there are five 1s involved in inclusion relationships (1 is included in 2, 2 is included in 3, 3 is include in 4 and 4 is included in 5), and within each pattern term there are groups of five that could also be involved in inclusion relationships (1 group of five is included in 2 groups of five, 2 groups of five are included in 3 groups of five, 3 groups of five are included in 4 groups of five).

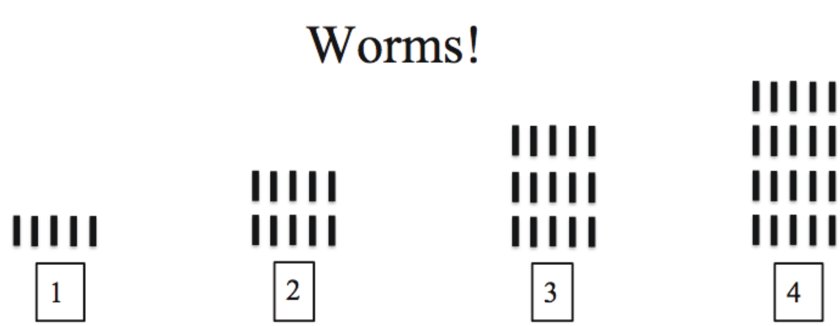


Figure 30. Worms in the garden problem visual uses an array structure

5.2.1 Students discover and prove the commutative property. Like Carpenter et al. (2003), I also found that the examination and proof of conjectures, in this case mainly the commutative property of multiplication (a x b = b x a), encouraged students to refine their understandings of important concepts. The study showed that designed visuals of linear functions can encourage students to discover the commutative property, and the visuals can even be used to prove that the commutative property of multiplication holds true for any set of factors. Similar to the suggestions of Beatty and Bruce (2012), Blanton and Kaput (2011), and Moss and London-McNab (2011), I found activities that asked students to do more than simply determine the next term in a linear function led to powerful discoveries. When teachers use designed visuals of linear functions and ask students to determine next, near and far terms (not just next terms as is common in many textbooks and traditional classrooms) students will likely discover and use the commutative property. For example, if students are asked to look at the worms in the garden problem and they are asked to determine the next term, the near term of 10 and the far term of 100, it is almost impossible for young students to find the far term of 100 without using the commutative property (see Day 4 in Figure 7 and see Figure 30). When the students see the visual (Figure 30 which uses an array structure) of the first four terms of the pattern, it is logical and efficient to think about a general pattern rule as being the day number tells you how many groups of five you have (this could be recorded as: worms = day number x 5) and this pattern rule would be easy to extend to find the next term and even a near term. To find the near term of 10, students may model or record groups of five until they have a total of 10 groups of five. However, when students then have to find the far term of 100 it is very difficult for some of them to calculate 100 groups of five. At times they will begin modeling the 100 groups of five as they did when finding the near term and then realize that it is too difficult to keep track of the running total and the number of groups of five. Therefore, these students will develop a new strategy in order to make it possible for them to work with this problem and many students quickly figure out that they can use the commutative property to make their problem easier and calculate five groups of 100 instead. Although many students make use of the commutative property in order to make a problem possible for them to calculate, as previously discussed with the worms problem, all of these students may not necessarily understand that the commutative property will hold true for any set of factors.

5.3 Conclusions

Algebra is something that Canadian students often do not encounter until their final years of elementary school. This abrupt (and often procedure dominated) introduction to algebra has made it a field that many students struggle to connect to their lives and existing understandings of mathematics (Kaput, 2008; Stephens et al., 2013). This study has demonstrated that all six of the grade 2 and 3 students who participated in the study could successfully develop and apply varying degrees of early algebraic concepts through explorations of multiple representations of linear functions. Designed visuals of linear functions, along with tables of values, pattern rules and narrative contexts provide students with multiple ways to identify and analyze multiplicative relationships between two sets of data.

This study was conducted in order to determine primary students' capacities for explicit reasoning when working with linear functions. It also examined the ways in which designed visuals affected young students' reasoning processes when working with linear functions. Some students used the designed visuals as a support in order to begin generalizing about the growth of a linear function and to think about the characteristics and structure of any term within a given linear function. The designed visuals, combined with other representations of linear functions, helped most students to generalize in some way about the relationship between the two sets of data involved in a linear function. Some of the Grade 2 and 3 students also saw the need to *invent* multiplication through their work with representations of linear functions as this new operation was necessary in order to be able to efficiently and accurately generalize about any term of a linear function. Other students (those who began the project with a stronger understanding of multiplication) did not need to rely on the designed visuals in order to generalize about a linear function. However students relying on recursive reasoning or moderate levels of explicit reasoning seemed to rely more on the visuals in order to reason with a linear function. By the end of the project, the students were also able to use various symbolization processes and many students were confidently able to use variables or other symbols to represent values in rudimentary algebraic equations.

The findings of the study showed that Grade 2 and 3 students are able to develop varying levels of explicit reasoning. The use of visuals combined with various other representations of linear functions (tables, pattern rules, verbal or written descriptions of pattern rules, narrative contexts) encourages students to focus on multiple aspects of the relationships between two sets of data involved in linear functions. Each student can focus on different aspects of the relationship between the data sets and the visuals, and given the way in which the lessons were designed, each student was able to work with a wide selection of strategies (i.e. there were many ways to come to the *right* answer). Many of the students who participated in this study were working with some kind of pattern rule and multiplication or repeated addition by the end of the project and they were not relying solely on recursive thinking.

5.4 Considerations for Future Research

Given the small sample size, it would be useful to repeat the study with a larger number of students in order to generalize the findings. Additionally, the analysis of lessons that involve designed visuals of linear functions (many of which involve an array structure) could lead to a more comprehensive understanding of the possible ways in which this style of activity could be used as an *introduction* to multiplication in the early primary years. Therefore, a longitudinal study with a larger sample size comparing the experiences of young primary students who were introduced to multiplication through activities involving designed visuals of linear functions and those who were introduced to multiplication in a different way could indicate whether or not these activities lead young students to develop a comprehensive understanding of the structures of multiplication.

Future research should also ensure that when using a table of values as a representation of linear functions the students are not provided with an empty table to fill out from the next to near term. This style of table of values can actually encourage recursive thinking, influence the students' strategies and almost force them into using skip-counting strategies. It is important that lessons and all activities encourage students to explore the relationships between the two sets of data that are a part of the linear function in question (Beatty & Bruce, 2012; Beatty, et al., 2013). However, the improper design of a table of values can prevent students from focusing on the relationships between data sets. In some of the activities used in this research project, the students were given a visual of the linear function and a blank table of values with spaces for term 1 to term 10. The later activities were revised to only provide spaces for the given terms (i.e. only for the terms provided in the visual, term 1 to term 4) because when students were given spaces for the first ten terms they felt that they were required to fill in all of the provided boxes in the table and almost exclusively used skip-counting strategies to find the near term.

An analysis of additional algebraic structures that arise out of activities that involve the study of linear functions with young students could also be useful. Exploring the impact of the study of linear functions in the early elementary years on students' symbolization processes, their abilities to generate and use equations that involve some form of variable, as well as their understandings of mathematical conventions when involved in algebraic equations could be used to inform the development of early algebra curricula.

Furthermore, a longitudinal study in the future could explore the impact that patterning activities involving multiple representations of linear functions in the primary years may have on student success when confronted with the more formal study of linear functions in the intermediate and senior years. More specifically, it could be useful to study the generation and structure of pattern rules as students progress from the primary to the intermediate grades. As Beatty and Bruce (2012) suggest, it is beneficial to structure pattern rules in any activities with linear functions so that they will easily translate to the structure of the equation for the slope of a line (a different, graphic, representation of linear functions). The students in this study were able to generate their own structure for pattern rules based on the *Function Machine Activity* (Output = Input x ___); it would be interesting to explore the experiences of students as they attempt to translate these types of pattern rules into to the traditional notation for the slope of a line: $y = \mathrm{m}\,x + \mathrm{b}$.

Additionally, video recorded student interviews (like those done in the pre and post-assessment) are a very rich source of data and should be used in similar research projects in the future. However, it would be valuable to always ask students “How did you come to that answer?” or “How did you solve that problem?” especially after they have determined a near or far term mentally.

Finally, future research could extend this study and additionally examine young primary students’ abilities to reason explicitly with more challenging linear functions. This research project involved simple linear functions, almost exclusively those with small multipliers of two through five, very few constants, and students were only asked to find “friendly” near or far terms such as 10, 20 or 100. It would be interesting to determine whether or not young primary students can successfully work with more complex linear functions (e.g. functions with challenging multipliers such as 7, and some that involve constants) or determine more challenging far terms (e.g. 17, 34, 93). An extension of this study that included similar activities with more complex functions may have different findings and could shed light on additional ways in which linear functions could be used to foster students’ understandings of multiplication and various algebraic concepts.

References

Battista, M. (1994). Teacher beliefs and the reform movement in mathematics education. *Phi Delta Kappan*, *75*(6), 462-470.

Baxter, P., & Jack, S. (2008). Qualitative case study methodology: Study design and implementation for novice researchers. *The Qualitative Report*, *13*(4), 544-559.

Beatty, R. (2010). Supporting algebraic thinking: Prioritizing visual representations. *OAME/AOEM Gazette*, 28-34.

Beatty, R., & Bruce, C. D. (2012). *From patterns to algebra: Lessons for exploring linear relationships*. Toronto: Nelson Education.

Beatty, R., Day-Mauro, M., & Morris, K. (2013). *Young students' explorations of growing patterns: Developing early functional thinking and awareness of structure*. Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Chicago, IL.

Blanton, M. L., & Kaput, J. J. (2003). Developing elementary teachers' "algebra eyes and ears". *Teaching Children Mathematics*, *10*(2), 70-77.

Blanton, M. L., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5-23). Berlin: Springer.

Carpenter, T. P., Fennema, E., Loef Franke, M., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth: Heinemann.

Carpenter, T. P., Loef Franke, M., & Levi, L. (2003). *Thinking mathematically: integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

Carraher, D. W., Schliemann, A. D., & Brizuela, B. M. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, *37*(2), 87-115.

Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235-272). New York: Taylor & Francis Group.

Chapin, S., & Johnson, A. (2000). *Math matters: Understanding the math you teach*. Sausalito, CA: Math Solutions Publications.

Chapin, S., O'Connor, C., & Anderson, N. (2009). The tools of classroom talk. In *Classroom Discussions: Using Math Talk to Help Students Learn* (pp. 11-18). Saisalito: Math Solutions.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education*, *27*(1), 41-51.

Coulombe, W. N., & Berenson, S. B. (2001). Representations of patterns and functions: Tools for learning. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics: 2001 yearbook* (pp. 166-172). Reston, VA: National Council of Teachers of Mathematics.

Creswell, J. W. (2014). *Research design: Qualitative, quantitative, and mixed methods approaches* (Fourth ed.). Thousand Oaks, CA: Sage.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, *6*(4), 232-236.

Ferrini-Mundy, J., Lappan, G., & Phillips, E. (1997). Experiences with patterning. *Teaching Children Mathematics*, *3*(6), 282-289.

Fosnot, C. T. (2007). *Investigating number sense, addition, and subtraction: Grades K-3*.

Portsmouth, NH: Heinemann.

Fosnot, C. T., & Jacob, B. (2010). *Young mathematicians at work: Constructing algebra*. Portsmouth: Heinemann.

Geist, E. (2000). Lessons from the TIMSS. *Teaching Children Mathematics*, *7*(3), 180-185.

Gravemeijer, K. (2002). Preamble: From models to modelling. In K. Gravemeijer, R. Lehrer, B. van Oers, & V. L. (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 7-22). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Hiebert, J., Carpenter, T. P., Fennema, E., Fusion, K. C., Human, P., Murray, H., . . . Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, *25*(4), 12-21.

Hufferd-Ackles, K., Fuson, K. C., & Gamoran Sherin, M. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*, *35*(2), 81-116.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5-17). New York: Taylor & Francis Group.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19-55). New York: Taylor & Francis Group.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317-326.

Lamon, S. J. (1996). The development of unitizing: It's role in children's partitioning strategies. *Journal for Research in Mathematics Education*, *27*(2), 170-193.

Lannin, J. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning*, *7*(3), 231-258.

Moss, J., & Beatty, R. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *Computer-Supported Collaborative Learning*, *1*, 441-465.

Moss, J., & London McNab, S. (2011). An approach to geometric and numeric patterning that fosters second grade students' reasoning and generalizing about functions and co-variation. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 277-301). Berlin: Springer.

Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics*, *33*(2), 203-233.

Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 303-322). Berlin: Springer.

Russell, S. J., Schifter, D., & Bastable, V. (2011). Developing algebraic thinking in the context of arithmetic. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 43-69). Berlin: Springer.

Schifter, D., Monk, S., Russell, S. J., & Bastable, V. (2008). Early algebra: What does understanding the laws of arithmetic mean in the elementary grades? In J. J. Kaput,

D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 413-447). New York: Taylor & Francis Group.

Schifter, D., Russell, S., & Bastable, V. (2009). Early algebra to reach a range of learners. *Teaching Children Mathematics*, *16*(4), 230-235.

Smith, M. S., & Stein, M. K. (2011). *5 Practices for orchestrating productive mathematics discussions*. Reston: The National Council of Teachers of Mathematics Inc.

Stephens, A., Knuth, E., Blanton, M., Isler, I., Gardiner, A., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *Journal of Mathematical Behavior*, *32*, 173-182.

Van de Walle, J., Folk, S., Karp, K., & Bay-Williams, J. (2011). *Elementary and middleschool mathematics: Teaching developmentally* (3 ed.). Toronto: Pearson.

Van de Walle, J., Karp, K., Bay-Williams, J., McGarvey, L. M., Folk, S., & Wray, J. (2015). *Elementary and middleschool mathematics: Teaching developmentally* (4 ed.). Toronto: Pearson.

van Oers, B. (2002). Informal representations and their improvements. In K. Gravemeijer, R. Lehrer, B. van Oers, & V. L. (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 25-28). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Yin, R. K. (2009). *Case study research: Design and methods* (4 ed.). Thousand Oaks, CA: Sage.

Young-Loveridge, J. (2005). Fostering multiplicative thinking using array-based materials. *Australian Mathematics Teacher*, *61*(3), 34-40.

Appendices

Appendix A: Parent Letter

(to be printed on updated letterhead)

May 2014

Dear Parent or Guardian of Potential Participant,

My name is Kate Gillies and I am working on my Master of Education degree at Lakehead University. My goal for my thesis is to investigate an area of mathematics instruction that is often difficult for students, with the hope of improving instructional techniques. The focus of my research is algebra instruction and the ways in which teachers can use patterning activities to help students build an understanding of algebraic concepts. The title of my research project is “Beyond recursive patterning: Visual representations to promote algebraic thinking with primary students”.

I will be observing mathematics lessons in your child’s classroom during the unit on patterning and algebra. The unit will be taught for 2 weeks in June 2014. With parental permission, students will participate in interviews to determine their understanding of patterning and algebra before and at the end of the unit. Samples of students’ work will also be collected. During some of the lessons, [teacher’s] teaching methods and the responses of students will be videotaped. Also, with permission, some groups of students will be videotaped so that I will be able to listen more carefully to how they have solved particular problems. Their conversations may also be transcribed and quoted anonymously in my final thesis project. [Teacher], my supervisor Dr. Lawson, or I may also make use of some of the edited classroom footage and work samples for the professional development and training of other teachers. Upon completion of the project, you will be welcome to obtain a summary of the research findings by contacting me at the phone number or email address given below, or by providing your mailing or email address on the consent form.

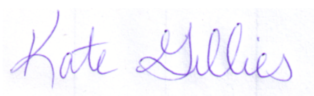
Your child will not be identified, except with an alias, in any written publication, including my thesis, possible journal articles or conference presentation. If video data is used for professional development, your child will be identified by first name only. The raw data that is collected through out the course of the research project will be securely stored at Lakehead University for a minimum of five years and then destroyed. Participation in this study is voluntary and you may withdraw the use of your child’s data at any time. This study has been approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team please contact Sue Wright at the Research Ethics Board at 807-343-8283 or research@lakeheadu.ca. The research has also been approved by the [school board] and the Principal of [school].

Please note that this research does not affect classroom instruction time, with the exception of some of the interview time. The lessons are being carried out by [teacher] and myself in the same manner and length of time as they would be without the research project. This research will not take away from the normal learning environment in the classroom and there is no apparent risk to your child. The research is simply being conducted to make note of the effects of using patterning activities and visual representations of patterns to promote algebraic understanding. If you choose not to have your child participate, he or she will still be engaged in the math lessons. The only difference is that his or her data will not be used. If you give permission for your child to participate, your child will also be asked whether or not he or she is willing to take part in the research project.

You are welcome to contact me at 355-1311 or kegillie@lakeheadu.ca if you have any questions concerning this research project. I would be very pleased to speak with you.

If you agree to allow your child to participate in the study, please sign the attached letter of consent and return it to [teacher]. Please keep this letter incase you would like to contact any of us.

Sincerely,



Kate Gillies

Kate Gillies
Master of Education Student
Lakehead University
807-355-1311
kegillie@lakeheadu.ca

Alex Lawson PhD
Thesis Supervisor
Lakehead University
807-343-8720

[Principal]

Sue Wright
Lakehead University Research Ethics Board
807-343-8283
research@lakeheadu.ca

Appendix B: Parent Consent Form
(to be printed on updated letterhead)

I DO give permission for my son/daughter,________________________________,
(name of student/please print)
to participate in the study with Kate Gillies as described in the attached letter.

I understand that:

1. My child will be videotaped in the classroom environment as a part of the research
2. My child's participation is entirely voluntary and I can withdraw permission at any time, for any reason.
3. There is no apparent danger of physical or psychological harm.
4. In accordance with Lakehead University's policy, raw data will remain confidential and securely stored at Lakehead University for a minimum of five years and then it will be destroyed.
5. All participants will remain anonymous in any publication resulting from the research project.
6. The video clips of the classroom or student work may be included in Professional Development for teachers conducted by Kate Gillies, [teacher], or Dr. Lawson. If my child appears in the video clips he or she will be identified by first name only.

☐ I initial this box to give permission for my child to appear in video clips which may be used for Professional Development purposes, as outlined above in 6.

7. I can receive a summary of the project, upon request, following the completion of the project, by contacting the researcher or providing my address or email address below.

Please keep the introductory letter on file should you have any further questions.
If you agree to let your child take part in the study, please complete this page and have your child return it to [teacher].

____
Name of Student (please print)

_______________________________________ ______________
Signature of Parent or Guardian Date

____
Address or email address if you would like a summary of the findings

Appendix C: Script for Student Consent

The children involved in the research project are in Grade 2 or 3. Due to their young age, I will not ask them to sign anything and they will not receive a letter or consent form. However, the teacher will follow the script below to ensure the students are verbally informed of the project. The students will also be given the opportunity to decide to participate or not participate as indicated in the script.

The following will be read aloud to the class by the classroom teacher prior to the start of the research project:

"Over the next two weeks, Ms. Gillies and I are going to try to learn more about how you think about patterns. During our math classes, Ms. Gillies will ask you questions about your thinking and she might take notes when you are talking. She will also collect some of your work and videotape you as you work with your partner or during our group lessons. Ms. Gillies and I want to help other teachers learn about what we do in our class for math so we might show other teachers some of the videos of you working or samples of your work. If at any time you do not want to be on camera or would rather not have your work shared, please tell me or Ms. Gillies and we will make sure that does not happen."

Appendix D: Teacher Letter

(to be printed on updated letterhead)

May 2014

Dear [teacher],

Thank you for considering participation in this study. My goal for my Master of Education thesis is to investigate an area in mathematics where students often struggle and to find ways to improve instruction in this area throughout the early elementary years. My research focuses on the ways in which primary teachers can help their students to build a strong foundation for later studies of algebra. This study was designed to explore the ways in which working with many representations of patterns impacts students' abilities to reason algebraically. Presently, there is very little information on the ways in which various representations of patterns, such as building patterns with colour tiles, writing the pattern rule, tables of values and graphs, can be used together to encourage students to reason algebraically about a specific pattern. The title of my research project is "Beyond recursive patterning: Visual representations to promote algebraic thinking with primary students".

In order to gather information needed for the study, I will be observing mathematics lessons in your classroom during the unit on patterning and algebra. The students will participate in a pre-intervention interviews, post-intervention interviews and a retention test to determine what they have learned in the unit. You will have access to the results for assessment purposes. Some samples of the students' work will also be collected. During some of the lessons, your teaching methods may be videotaped. Also, with permission, some groups of students will be videotaped so that I will be able to listen carefully to how they have solved the problems. Conversations may be transcribed and quoted anonymously in my final project in order to illustrate the ways in which students interpreted the multiple representations of patterns. You, my supervisor Alex Lawson, or myself may also make use of some of the edited classroom footage and student work samples for professional development for teachers.

As a part of the project you will need to: distribute and collect the cover letters and permission forms from parents or guardians and students, collect student work and allow time for the interviews and retention test. I will ensure that you have any of the resources needed for the lessons. I hope that you will participate for the duration of the study. However, you may withdraw at any time, for any reason, without penalty, as your participation is entirely voluntary. I do no anticipate any negative consequences as a result of participation in this study.

You and your students will not be identified in any written publication, except with an alias, including my thesis, possible journal articles or conference presentations. If video data is used for professional development, your students will be identified by first name only, but if children use your surname it may be revealed. The raw data that is collected will be securely stored at Lakehead University for five years after completion of the project and then it will be destroyed. A report of the research will be available upon request. I can be reached at 355-1311 or kegillie@lakeheadu.ca.

This study has been approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team please contact Sue Wright at the Research Ethics Board at 807-343-8283 or research@lakeheadu.ca.

If you agree to participate in the study, please sign the attached letter of consent and return it to me.

Sincerely,

Kate Gillies

Kate Gillies
Master of Education Student
Lakehead University
807-355-1311
kegillie@lakeheadu.ca

Alex Lawson PhD
Thesis Supervisor
Lakehead University
807-343-8720

[Principal]

Sue Wright
Lakehead University Research Ethics Board
807-343-8283
research@lakeheadu.ca

Appendix E: Teacher Consent Form
(to be printed on updated letterhead)

I,________________________________, do agree to participate in the study with Kate
(name of teacher/please print)
Gillies as described in the attached letter.

I understand that:

1. I will be videotaped in the classroom as a part of the project.
2. My participation is entirely voluntary and I can withdraw permission at any time, for any reason without penalty.
3. There is no apparent danger of physical or psychological harm.
4. In accordance with Lakehead University's policy, the raw data will remain confidential and securely stored at Lakehead University for a minimum of five years and then it will be destroyed.
5. I will remain anonymous in any publication resulting from the research project.
6. The Video clips of the classroom or student work may be included in Professional Development for teachers conducted by Kate Gillies, Dr. Lawson or myself. If I appear in the video clips I may be identified by surname.

☐ I initial this box to give permission for me to appear in video clips which may be used for Professional Development purposes as outlined in above in number 6.

If you agree to participate in my study, please complete this form and return it to me.

____
Name of Third Party Witness (please print)

____ __________________
Signature of Third Party Witness Date

____
Name of Teacher (please print)

____ __________________
Signature of Teacher Date

Appendix F: Principal Letter
(to be printed on updated letterhead)

May 2014

Dear [Principal],

Thank you for considering participation in this study. My goal for my Master of Education thesis is to investigate an area in mathematics where students often struggle and to find ways to improve instruction in this area throughout the early elementary years. My research focuses on the ways in which primary teachers can help their students to build a strong foundation for later studies of algebra. This study was designed to explore the ways in which working with many representations of patterns impacts students' abilities to reason algebraically. Presently, there is very little information on the ways in which various representations of patterns, such as building patterns with colour tiles, writing the pattern rule, tables of values and graphs, can be used together to encourage students to reason algebraically about a specific pattern. The title of my research project is "Beyond recursive patterning: Visual representations to promote algebraic thinking with primary students".

In order to gather the information needed for the study, I will be observing and co-teaching mathematics lessons in [teacher's] classroom during the unit on patterning and algebra. The students will participate in interviews at the beginning and end of the unit as well as a retention test two to three weeks after the unit has been completed to determine what they have learned. [teacher] will have access to the data that I collect for assessment purposes. Some samples of student work will be collected to assist [teacher] with assessment, and with permission this work may also be included in my study. During some of the lessons, [teacher's] teaching methods will be videotaped. Also, with permission, some groups of students will be videotaped so that I will be able to listen carefully to how they have solved the problems. Conversations may be transcribed and quoted anonymously in my final project in order to illustrate the students' understandings of patterns and algebra. [Teacher], my supervisor Alex Lawson or myself may also use some of the edited classroom footage and student work samples for professional development for teachers.

This research does not affect classroom instruction time, with the exception of some of the interviews with the students. [teacher] and myself are carrying out the lessons in the same manner as when I volunteer in [teacher's] classroom. This research project will not take away from the normal learning environment in the classroom and there is no apparent risk to any of the participants involved. If parents choose not to have a child participate, the child will still be engaged in all of the math lessons. The only difference is that his or her data will not be used. If parents give permission for a child to participate, the child will also be asked whether or not they would like to be a part of the research.

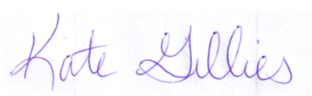
I hope that [teacher] and her students will participate for the duration of the study. However, you may withdraw your permission at any time, for any reason, without penalty, as participation is entirely voluntary. I do not anticipate any negative consequences as a result of participation in this study.

The school board, [school], [teacher] and the students will not be identified in any written publication, including my thesis, possible journal articles or conference presentations. If video data is used for professional development purposes, the students will be identified by first name only and [teacher] may be identified by surname if the students use it in the video clip. The raw data will be securely stored at Lakehead University for a minimum of five years after completion of the project and then it will be destroyed. A report of the research will be available upon request. I can be reached at 355-1311 or kegillie@lakeheadu.ca.

This study has been approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team please contact Sue Wright at the Research Ethics Board at 807-343-8283 or research@lakeheadu.ca.

If you give permission for participation in the study, please sign the attached letter of consent and return it to me.

Sincerely,



Kate Gillies

Kate Gillies
Master of Education Student
Lakehead University
807-355-1311
kegillie@lakeheadu.ca

Alex Lawson PhD
Thesis Supervisor
Lakehead University
807-343-8720

Sue Wright
Lakehead University Research Ethics Board
807-343-8283
research@lakeheadu.ca

Appendix G: Principal Consent Form
(to be printed on updated letterhead)

I,________________________________, do agree to participation in the study with Kate
(name of principal/please print)
Gillies as described in the attached letter.

I understand that:

1. [teacher] and her students will be videotaped in the classroom as a part of the project.
2. Their participation is entirely voluntary and I can withdraw permission at any time, for any reason without penalty.
3. There is no apparent danger of physical or psychological harm.
4. In accordance with Lakehead University's policy, the raw data will remain confidential and securely stored at Lakehead University for a minimum of five years and then it will be destroyed.
5. The [school board], [school], [teacher] and the students will remain anonymous in any publication resulting from the research project.
6. The video clips of the classroom or student work may be included in Professional Development for teachers conducted by Kate Gillies, Dr. Lawson or [teacher]. If [teacher] appears in the video clips she may be identified by surname. If students appear in the video clips, they will be identified by first name only.

☐ I initial this box to give permission for [teacher] and her students to appear in video clips which may be used for Professional Development purposes, as outlined above in 6.

If you approve of participation in my study, please complete this form and return it to myself or [teacher].

____
Name of Principal (please print)

______________________________________ ____________________
Signature of Principal Date

Appendix H: Interview Guide

(Adapted from R. Beatty, personal communication, March 3, 2014)

Problem 1

Show student 3x3 grid.

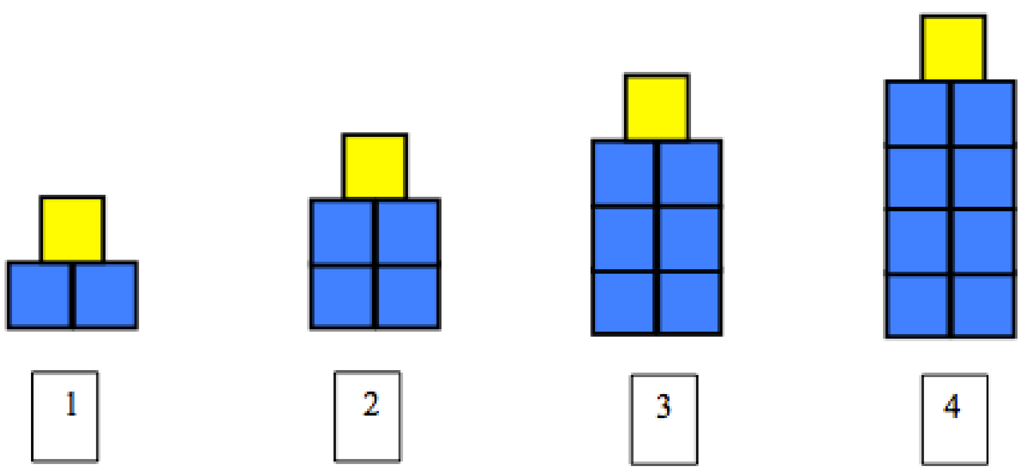
1. Ask them to copy the grid using pencil and paper.
2. Ask them to copy the grid using square tiles.

Problem 2

Ask the student "What is a pattern?"
If the student seems unsure of how to answer the question, point out the manipulatives, paper, and markers available to see if they can communicate an answer. "Is there anything here you'd like to use?"

Problem 3

"Now I'm going to show you a different kind of pattern." (After pattern is built, ask "have you seen this type of pattern before?")



Build the pattern (using tiles and position cards) and think-aloud as you build it.

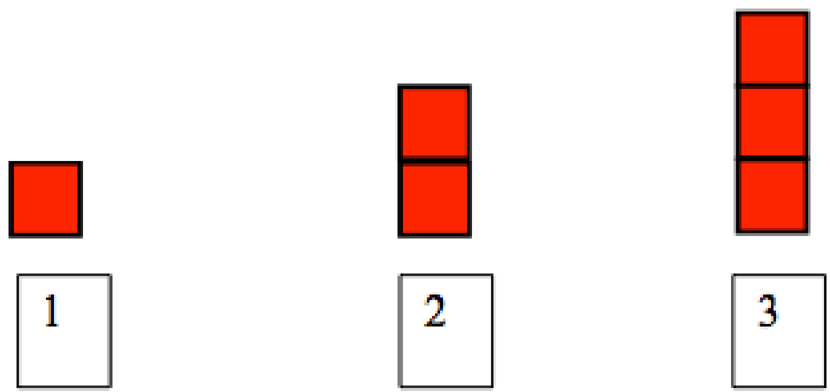
1. At the first position, position 1, I'm going to put 2 blue tiles…1, 2 (count the tiles) , and one yellow tile.
2. At the second position, position 2, I'm going to put 1, 2, 3, 4 blue tiles and 1 yellow tile.
3. At the third position, position 3, I'm going to put 1,2,3,4,5,6 blue tiles and 1 yellow tile.
4. At the fourth position, position 4, I'm going to put 1,2,3,4,5,6,7,8 blue tiles and 1 yellow tile.

Questions:

1. What's happening in this pattern? What do you see?
2. What would come next, at the fifth position (position 5)? Can you build it using the tiles?
3. How many tiles would there be in the 10^{th} position?
4. How many tiles would there be in the 100^{th} position?

Problem 4

Build positions 1, 2 and 3 of this pattern:

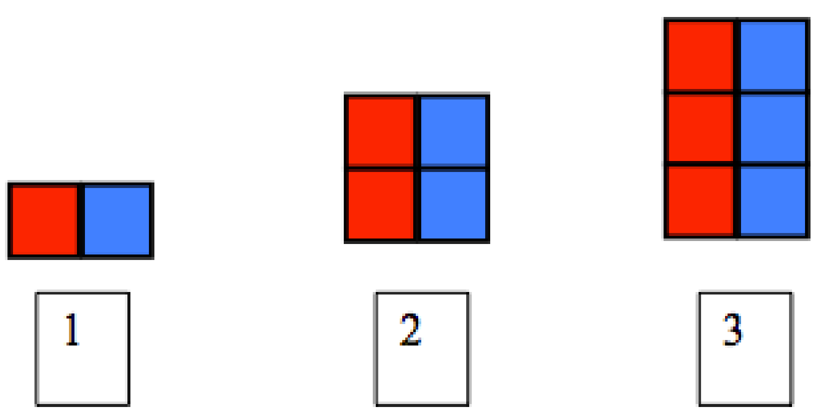


Questions:

1. What's happening in this pattern? What do you see?
2. What would come next, at the fourth position (position 4)? Can you build it using the tiles?
3. How many tiles would there be in the 10^{th} position?
4. How many tiles would there be in the 100^{th} position?

Problem 5

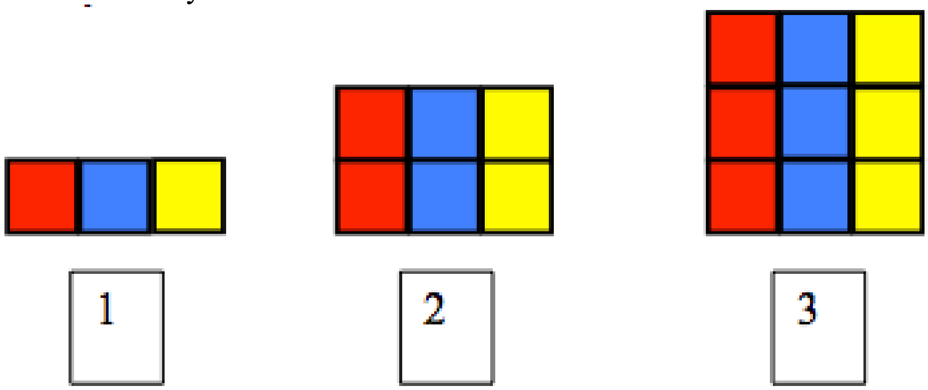
Next, add a column of blue tiles:



Questions:

1. Now what's happening in this pattern? What do you see?
2. What would come next, at the fourth position (position 4)? Can you build it using the tiles?
3. How many tiles would there be in the 10th position?
4. How many tiles would there be in the 100th position?

Problem 6
Add a column of yellow tiles:

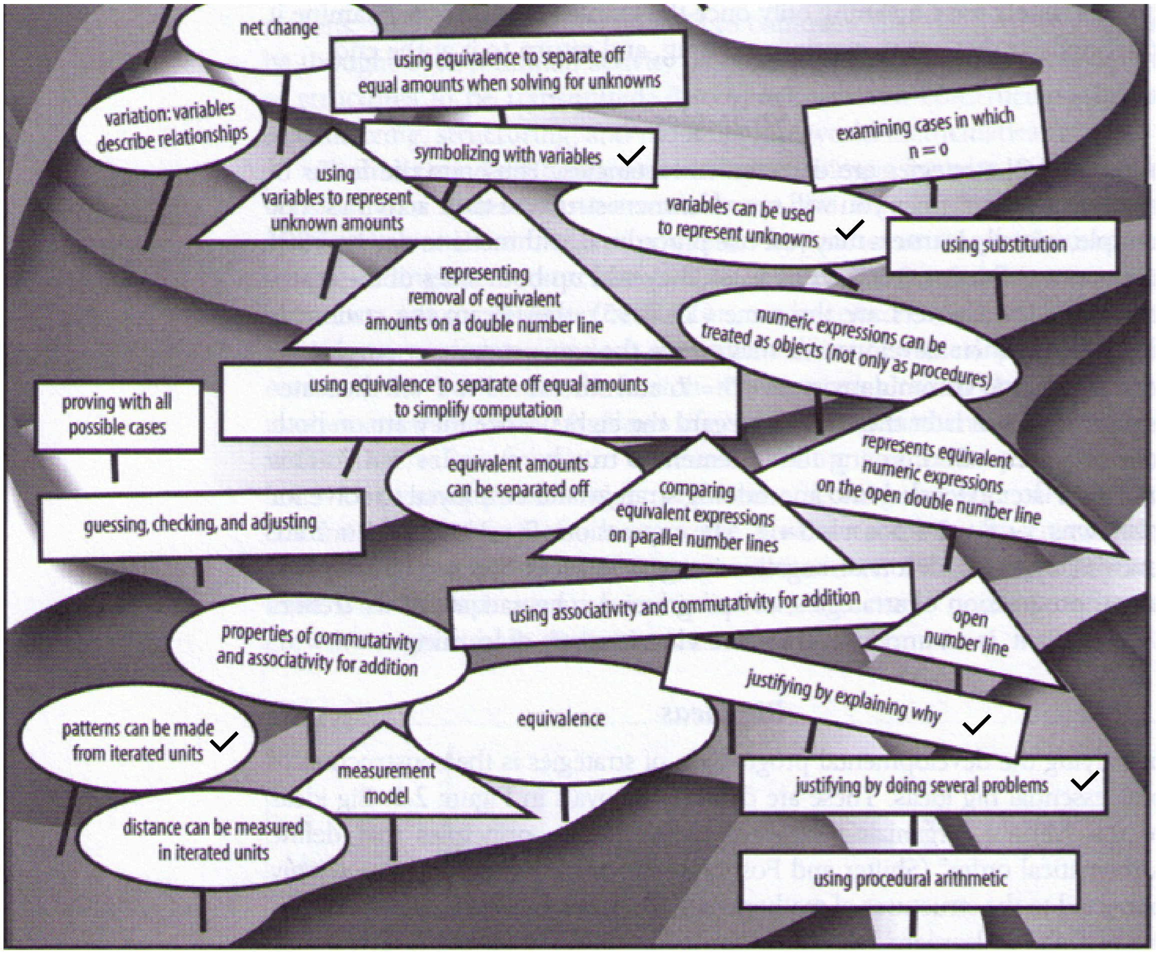


Questions:
1. What's happening in this pattern? What do you see?
2. What would come next, at the fourth position (position 4)? Can you build it using the tiles?
3. How many tiles would there be in the 10th position?
4. How many tiles would there be in the 100th position?

Problem 7

What if we had four colours (what if we added green tiles)? How many tiles would be at position 10?
How many tiles would be at position 100?

Appendix I: Modified Landscape of Learning



(Adapted from Fosnot and Jacob, 2010, p. 30)[3]

[3] Checkmarks indicate strategies and big ideas that were used as codes in this study.

Appendix J: Definition of Codes

Codes used	Code name	Definition as it pertains to this project	Source
✓	A close	answer is not completely incorrect but not completely correct either	
✓	A correct	answer is correct	
✓	A incorrect	answer is incorrect	
✓	A with teacher assistance	student is able to come to correct answer with teacher prompting or questioning	
✓	T identification of far terms	identifying or describing pattern terms that do not directly follow the last given term and are very different from the last term	(Beatty & Bruce, 2012)
✓	T identification of near terms	identifying or describing pattern terms that do not directly follow the last given term and are still close to the last term	(Beatty & Bruce, 2012)
✓	T identification of next terms	identifying or describing the pattern term that follows the last given term	(Beatty & Bruce, 2012)
✓	Stgy beginning unitizing	student is beginning to think about how much is in a group and the number of groups but needs a diagram or some support to do so	
✓	Stgy unitizing	student is able to think about how much is in a group and the number of groups	(Fosnot & Jacob, 2010)
✓	Stgy commutative property	student uses the commutative property (a x b = b x a)	(Carpenter et al., 2003)
✓	Stgy multiplication as repeated addition	student uses repeated addition to solve a multiplication problem	(Carpenter, Fennema, Loef Franke, Levi, & Empson, 1999; Young-Loveridge, 2005)
✓	Stgy counting by ones	student solves the problem by counting by ones, may have drawn a picture with each object	(Van de Walle et al., 2015)

✓	Stgy doubling reasoning	student uses doubling or doubling multiple times to solve the problem.	(Carpenter et al., 1999)
✓	Stgy beginning multiplicative reasoning	student uses multiplication to solve the problem	
✓	Stgy proportional reasoning	student reasons with known value to find unknown value based on equal proportions	(Van de Walle et al., 2015)
✓	Stgy skip counting	student uses beginning or proficient skip-counting strategies	(Carpenter et al., 1999)
✓	Stgy whole-object reasoning	student uses proportional reasoning in ways that do not apply to the situation or may not be logical in the context of that specific function (e.g. if I know the output for 4, double it to find the output for 8 but this will not be logical if there is a constant in the function)	(Beatty & Bruce, 2012; Moss & London McNab, 2011)
✓	EorR "adding"	student describes pattern growth as "adding" one core at each successive term	(Van de Walle et al., 2015)
✓	EorR explicit reasoning	student identifies the functional relationship between multiple sets of data	(Beatty & Bruce, 2012; Moss & London McNab, 2011)
✓	EorR patterns can be made from iterated units	student recognizes that patterns involve the repetition of some unit (a pattern core	(Fosnot & Jacob, 2010)
✓	EorR reliance on recursive thinking	student focuses on only one set of data when examining a linear function	(Beatty & Bruce, 2012)
✓	EorR using the term number to explain a pattern	student recognizes a relationship between the term number and some other characteristic of the pattern	(Beatty & Bruce, 2012; Beatty et al., 2013)
✓	V visual no support	the visual representation of the pattern does not support the student's thinking	
✓	V visual supports	the visual representation of the pattern supports the student's thinking and helps to push them	

✓	V position number confusion	the visual is being used but the position number is a source of confusion for the student	
	PofG guess, check, and adjust	using trial and error to determine pattern rule	(Fosnot & Jacob, 2010)
	PofG examining cases in which n=0	considering the pattern at term 0 when describing a pattern or determining a pattern rule	(Fosnot & Jacob, 2010)
✓	PofG justifying by doing several problems	using many examples to prove a pattern rule or idea	(Fosnot & Jacob, 2010)
✓	PofG justifying by explaining why	using reasoning to prove a pattern rule or idea	(Fosnot & Jacob, 2010)
	PofG proving with all possible cases	using properties of number and/or operations to prove a pattern rule would hold for any term number	(Fosnot & Jacob, 2010)
✓	SorR describes a pattern rule with words	uses everyday language to describe a pattern rule	(Beatty & Bruce, 2012; Chapin & Johnson, 2000)
✓	SorR describes visual structure of pattern	the student describes how the pattern is growing	(Beatty & Bruce, 2012; Chapin & Johnson, 2000)
✓	SorR draws visual representation	student draws a visual of the pattern	(Beatty & Bruce, 2012; Chapin & Johnson, 2000)
✓	SorR explicit rule or symbolization using variables	student writes the explicit pattern rule and uses this to reason with the pattern	(Beatty & Bruce, 2012; Chapin & Johnson, 2000; Van de Walle et al., 2015)

✓	SorR physically builds representation	student builds the pattern with coloured tiles	(Beatty & Bruce, 2012)
✓	SorR uses table of values	student uses a table to values to symbolize and reason with a pattern	(Beatty & Bruce, 2012; Chapin & Johnson, 2000)
✓	IwP identifies growing part of pattern	student identifies the multiplicative growth within a linear function	(Beatty et al., 2013)
✓	IwP makes a connection	student makes a connection to a pattern they have previously seen or to some other pattern or idea they already know	
✓	IwP prediction spontaneous	student makes a prediction about an aspect of the pattern or problem on their own	
✓	IwP self corrects	student revises and corrects their own error or reviews their work looking for errors	
✓	O even and odd	observation about even and odd number relationships	
✓	O makes conjecture about numbers or operations	student makes a conjecture about number relationships or properties of operations	(Carpenter, Loef Franke, & Levi, 2003)
✓	O relational thinking	student uses what they know about the number system and operations to reason with a situation or to problem solve	(Carpenter et al., 2003)