ABSTRACT CONVEXITY SPACES

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Ashok K. Varma

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Thesis Advisor: John H. M. Whitfield

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ABSTRACT

The principal question discussed in this dissertation is the problem of characterizing the linear and convex functions on generalized line spaces. A linear function is shown to be a convex function. The linear and convex functions are characterized, that is, a function $f: X \rightarrow R$ is linear [convex] if and only if f_{ℓ} is linear [convex] in the usual sense on each line of a generalized line space X. We prove that if a function has at least one support at each point on its graph, then it is a convex function.

In the first chapter the basic concepts of abstract convexity spaces are introduced. The next chapter is concerned with join systems which are shown to be examples of abstract convexity spaces. On the other hand, a domain-finite, join-hull commutative abstract convexity space with regular straight segments satisfies the axioms of a join system. Consequently, such abstract convexity spaces satisfy the separation property.

In Chapter III, the linearization of abstract spaces is done using a linearization family.

The following chapter is on generalized line spaces and graphically it is shown that Pasch's and Peano's axioms do not hold in a certain generalized line space. It is also proved that the separation property may not hold, in

general, in a generalized line space.

Finally, the convex and linear functions are studied on generalized line spaces. The linearization of generalized line spaces is done by means of the properties of a linearization family.

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CHAPTER I

ABSTRACT CONVEXITY SPACE

1.1 Introduction

Convex subsets of linear spaces have been studied for a long period of time. For example, Valentine [17] is a good reference for the concepts of convexity. In this thesis, an axiomatic setting for the theory of convexity is provided by taking an arbitrary set X and distinguishing a family $\mathcal{C}$ of subsets of X which is closed under arbitrary intersection. The notion of such a pair, called an abstract convexity space, was first introduced by Kay and Womble [9]. This yields a weak type of closure or hull operator on the power set of X.

This chapter introduces some of the basic definitions and fundamental propositions of linear spaces and abstract convexity spaces on which some of the results of this paper are based. Many of the elementary and well known propositions of convex sets in linear spaces, which follow are not proved here; however, standard proofs of these propositions can be found in Roberts and Varberg [15] and Valentine [17] .

The abstract convexity spaces have been studied with many examples and then convex sets are defined axiomatically. Several other approaches were introduced. One by Prenowitz [14] which is included in this thesis, called here a join system or called a convexity space by Bryant and Webster [2]. The join system with several axioms has a meaningful theory of convexity and its axioms are stronger than those of abstract convexity space. It is shown that the join system satisfies the axioms of an abstract convexity space. However, the reverse condition is also true if we add some additional axioms to the abstract convexity space.

1.2 Linear Spaces

1.2.1 Definition

A linear space X is a set on which addition, +, is defined so that (X,+) is a commutative group; and multiplication by scalars satisfying the distributive laws

$$t(a + b) = ta + tb \text{ and } (s + t)a = sa + ta$$

where s,t are scalars, $a,b \in X$, and satisfying $(st)a = s(ta)$, and $1.a = a$.

The elements of X will be called vectors and for the purpose of this dissertation, the field of scalars will always be the reals denoted by R.

1.2.2 Definition

A subset of a linear space is convex if and only if it includes the line segment joining any two of its points. More precisely, a set A is convex provided $ta + (1-t)b$ is in A, for all scalars t satisfying $o \leq t \leq 1$, and $a,b \in A$.

1.2.3 Definition

A non-empty subset A of a linear space is called affine if $tA + (1-t)A \subset A$ for all scalars t. Thus, a set A is affine if it is a translate of a linear subspace.

1.2.4 Definition

If $t_i \in R$, $t_i \geq 0$ and $\sum_{i=1}^{n} t_i = 1$ then $a = \sum_{i=1}^{n} t_i a_i$ is called a convex combination of $a_1, a_2, \ldots, a_n$, the latter being elements of a linear space X. If the condition $t_i \geq 0$ is removed then a is called an affine combination of $a_1, a_2, \ldots, a_n$.

1.2.5 Proposition

If $\{A_i\}$, $i \in I$, is any family of convex [affine] sets, then $M = \bigcap_{i \in I} A_i$ is convex [affine].

1.2.6 Proposition

A set $A \subseteq X$ is convex [affine] if and only if every convex [affine] combinition of points of A lies in A.

1.2.7 Definition

The intersection of all convex [affine] sets in X containing a given set A is called the convex [affine] hull of A. The convex hull of A is denoted by $\mathfrak{C}(A)$.

1.2.8 Proposition

For any $A \subseteq X$, the convex [affine] hull of A consists precisely of all convex [affine] combinations of elements of A.

1.2.9 Definition

Let A be a convex subset of a linear space X. A function $f: A \to R$ is convex if

$$f(ta + (1-t)b) \leq t\, f(a) + (1-t)\, f(b)$$

for all $a, b \in A$ and t in the open interval $(0,1)$.

1.2.10 Definition

If A is a subset of a linear space X and $f: A \to R$ then the epigraph of f is the set in $X \times R$ described by

$$\text{epi}(f) = \{(a,r): a \in A,\ r \in R,\ r \geq f(a)\}.$$

1.2.11 Proposition

Let A be a convex subset of a linear space X. A function $f: A \to R$ is convex if and only if the epi(f) is a convex set in $X \times R$.

1.2.12 Definition

Let X and Y be two linear spaces. The mapping f of X into Y is called linear if

$$f(a+b) = f(a) + f(b), \quad f(t\,a) = tf(a)$$

for all $a,b \in X$ and $t \in R$.

1.2.13 Definition

Let X and Y be two linear spaces. The mapping $A: X \to Y$ is affine if for every $a \in X$, $A(a) = f(a) + t$ where f is a linear function from X into Y and t is a constant in Y.

1.2.14 Definition

Let U be a convex subset of a linear space X. A function $f: U \to R$ has a support at $a_0 \in U$ if there exists an affine function $Aa_0: X \to R$ such that $Aa_0(a_0) = f(a_0)$ and $Aa_0(a) \leq f(a)$ for every $a \in U$. The graph of a support function Aa_0 is called a supporting hyperplane for f at a_0. If U is an interval of the real line R then the affine function is defined as $Aa_0(a) = f(a_0) + m(a - a_0)$ and the supporting hyperplane is known as the line of support.

Convex functions are characterized as those which admit supporting hyperplanes at each point on its graph as indicated in the following:

1.2.15 Proposition

Let U be an open-convex subset of a linear space X. A function f: $U \to R$ is convex if and only if there is at least one supporting hyperplane for f at each $a \in U$.

In the above proposition, if U is an open interval of reals, then the supporting hyperplane is replaced by a line of support.

Next we define an abstract convexity space which is of basic importance. The remainder of this chapter will be concerned with the abstract convexity spaces and their associated properties.

1.3 Abstract Convexity Spaces

1.3.1 Definition

An abstract convexity space is a pair $(X, \mathcal{C})$ consisting of a non-empty set X and a family $\mathcal{C}$ of subsets of X, called a convexity structure for X, which (i) contains X and the empty set ϕ; and, (ii) is closed under arbitrary intersection. The members of $\mathcal{C}$ are called $\mathcal{C}$-convex sets, (or just convex sets, if $\mathcal{C}$ is understood).

1.3.2 Definition

The convex hull operator on the power set of X generated by the convexity structure $\mathcal{C}$ is defined by

$$\mathcal{C}(S) = \bigcap\{C \in \mathcal{C} : S \subset C\}$$

for each $S \subset X$. The set $\mathcal{C}(S)$ will be termed as $\mathcal{C}$-hull of S.

When $S = a_1, a_2, \ldots, a_n$ is finite, we will simply write $\mathcal{C}(a_1, a_2, \ldots, a_n)$ for its hull.

1.3.3 Proposition

The hull operator $\mathcal{C}$ has the following properties:

(i) $S \subset \mathcal{C}(S)$ for each $S \subset X$;

(ii) $S \subset T$ implies $\mathcal{C}(S) \subset \mathcal{C}(T)$;

(iii) $\mathcal{C}(\mathcal{C}(S)) = \mathcal{C}(S)$;

(iv) $S \in \mathcal{C}$ if and only if $\mathcal{C}(S) = S$.

Proof: (i) This is trivial by the definition of $\mathcal{C}(S)$.

(ii) By definition, $\mathcal{C}(T) = \bigcap \{C \in \mathcal{C}\colon T \subset C\}$ and also by (i) we have $S \subset T \subset \mathcal{C}(T)$; but $\mathcal{C}(S) \subset C$ for all $C \supseteq S$ when $C \in \mathcal{C}$. Therefore $\mathcal{C}(S) \subset \mathcal{C}(T)$.

(iii) It is sufficient to prove that $\mathcal{C}(\mathcal{C}(S)) \subset \mathcal{C}(S)$, since the reverse inclusion follows from (i). Suppose $a \in \mathcal{C}(\mathcal{C}(S)) = \bigcap \{C\colon C \in \mathcal{C}, \mathcal{C}(S) \subset C\}$. This implies $a \in C$ for all $C \supset \mathcal{C}(S)$. But by definition $\mathcal{C}(S) \subset \mathcal{C}$. So in particular $a \in \mathcal{C}(S)$.

(iv) Clearly, $S = \bigcap \{C \in \mathcal{C}\colon S \subset C\} \in \mathcal{C}$.

Conversely, suppose $S \in \mathcal{C}$. To prove $S = \mathcal{C}(S)$, it is sufficient to show $\mathcal{C}(S) \subset S$. By definition $\mathcal{C}(S) = \{C \in \mathcal{C}\colon S \subset C\}$ that is; $\mathcal{C}(S) \subset C$ for all $C \supset S$ when $C \in \mathcal{C}$. But $S \in \mathcal{C}$ and $S \subset S$, therefore $\mathcal{C}(S) \subset S$.

1.3.4 Definition

For $a, b \in X$, the set $[a,b] = \mathcal{C}(\{a,b\})$ is called a segment.

1.3.5 Definition

Segments are said to be non-discrete if for all

$a \neq b$, $[a,b] \setminus \{a,b\} \equiv (a,b) \neq \phi$.

1.3.6 Definition

Segments are called <u>decomposable</u> if for all $a \neq b$ and $c \in [a,b]$ both $[a,c] \cap [c,b] = c$ and $[a,c] \cup [c,b] = [a,b]$ hold.

1.3.7 Definition

Segments are said to be <u>extendible</u> if for each $a \neq b$, there exists $c \neq a,b$ such that $b \in [a,c]$.

1.3.8 Definition

An abstract convexity space $(X,\mathcal{C})$ is said to have <u>regular</u> segments if its segments are non-discrete, decomposable and extendible.

1.3.9 Definition

An abstract convexity space $(X,\mathcal{C})$ is said to have <u>straight</u> segments if and only if the union of two segments having more than one common point is a segment.

1.3.10 Definition

For $a \in X$ and $S \subset X$, $S \neq \phi$ the <u>$\mathcal{C}$-join</u> of a and S is the set $a_{\mathcal{C}}S = \cup\{\mathcal{C}(a,s) : s \in S\}$.

1.3.11 Proposition

For each $a \in X$ and $S \subset X$ then $a_{\mathcal{C}}\mathcal{C}(S) \subset \mathcal{C}(a \cup S)$.

<u>Proof</u>: Suppose $x \in a_{\mathcal{C}}\mathcal{C}(S)$. Then by definition of the $\mathcal{C}$-join of a and $\mathcal{C}(S)$, we have $x \in \mathcal{C}(a,s')$ for some $s' \in \mathcal{C}(S)$. By proposition 1.3.3 (ii), $\mathcal{C}(S) \subseteq \mathcal{C}(a \cup S)$, but $s' \in \mathcal{C}(S)$, which implies $s' \in \mathcal{C}(a \cup S)$. Therefore $x \in \mathcal{C}(a,s') \subset \mathcal{C}(a \cup S)$. Hence $a_{\mathcal{C}}\mathcal{C}(S) \subset \mathcal{C}(a \cup S)$.

It is to be noted that the reverse inclusion of proposition 1.3.11 is not always true . Here is an example.

1.3.12 Example

Let $X = R^2$ and $\mathfrak{C}$ be the collection of points, straight line-segments between two points, ϕ and R^2. Now take any point $a \in X$ and let S be any line-segment which does not contain a. By definition of $\mathfrak{C}$-join, we have

$$a_{\mathfrak{C}}S = \bigcup \{\mathfrak{C}(a,s) \; : \; s \in S\}$$

which are the line segments between a and S. But $\mathfrak{C}(a \cup S) = R^2$ which is the smallest $\mathfrak{C}$-convex set containing a and S. Hence $\mathfrak{C}(a \cup S)$ is not contained in $\mathfrak{C}$-join of a and S. (Strictly speaking $\mathfrak{C}(a \cup S)$ should be written $\mathfrak{C}(\{a\} \cup S)$. However, in the interest of simplicity and consistency with [9], we will use $\mathfrak{C}(a \cup S)$ instead).

1.3.13 Definition

An abstract convexity space $(X,\mathfrak{C})$ is said to be join-hull commutative if the reverse inclusion of proposition 1.3.11 is true, and in this case we have

$$\mathfrak{C}(a_{\mathfrak{C}}S) = \mathfrak{C}(a \cup S) = a_{\mathfrak{C}}\mathfrak{C}(S).$$

1.3.14 Definition

An abstract convexity space $(X,\mathfrak{C})$ is said to be finitely join-hull commutative if the definition 1.3.13 holds for finite subsets S.

1.3.15 Definition

An abstract convexity space $(X,\mathfrak{C})$ is said to have the property of domain-finiteness if $\mathfrak{C}(S) = \bigcup \{\mathfrak{C}(T) \colon T \subset S, \; |T| < \infty\}$ for each $S \subset X$. Here $|T|$ denotes the cardinality of T.).

1.3.16 Theorem

If $(X,\mathfrak{C})$ is an abstract convexity space which has the property of domain-finiteness then join-hull commutativity and finite join-hull commutativity are equivalent.

Proof: Obviously join-hull commutativity implies finitely join-hull commutativity.

Conversely, it is sufficient to show that for any $a \in X$, $S \subset X$, $\mathfrak{C}(a \cup S) \subset a_{\mathfrak{C}}\mathfrak{C}(S)$ holds since the reverse inclusion follows from proposition 1.3.11. Let $x \in \mathfrak{C}(a \cup S)$. By domain-finiteness there exists a finite set $T \subset S$ such that $x \in \mathfrak{C}(a \cup T)$, and by finite join-hull commutativity $\mathfrak{C}(a \cup T) \subset a_{\mathfrak{C}}\mathfrak{C}(T) \subset a_{\mathfrak{C}}\mathfrak{C}(S)$. Hence $x \in a_{\mathfrak{C}}\mathfrak{C}(S)$.

1.3.17 Theorem

For a join-hull commutative and domain-finite abstract convexity space, a subset A is $\mathfrak{C}$-convex if and only if $\mathfrak{C}(a,b) \subset A$ for each $a \in A$, $b \in A$.

Proof: Suppose A is $\mathfrak{C}$-convex. Then if $a \in A$ and $b \in A$, $\mathfrak{C}(a,b) \subset \mathfrak{C}(A) = A$.

Conversely, suppose for each $a \in A$, $b \in A$ then $\mathfrak{C}(a,b) \subset A$. It follows by induction and join-hull commutativity that for any finite set $T \subset A$, $\mathfrak{C}(T) \subset A$. Now by domain-finiteness it follows

$$\mathfrak{C}(A) = \bigcup\{\mathfrak{C}(T) : T \subset A,\ |T| < \infty\} \subset A.$$

Therefore $\mathfrak{C}(A) = A$.

1.3.18 Examples

(i) Suppose X is a vector space and a family $\mathfrak{C}$ consists of X, the empty set ϕ and convex subsets of X as defined by 1.2.2. Then $(X, \mathcal{C})$ is an abstract convexity space. The convex hull operator $\mathcal{C}: P(X) \to P(X)$ is $\mathcal{C}(S) = \bigcap \{C \subset X : C$ is convex, $S \subset C\}$.

(ii) Suppose X is an arbitrary set and $\mathcal{C} = \{$all subsets of $X\}$. Then $(X, \mathcal{C})$ is an abstract convexity space. The convex hull operator $\mathcal{C}: P(X) \to P(X)$ is $\mathcal{C}(S) = S$.

(iii) Suppose X is an arbitrary set and $\mathcal{C} = \{\phi, X\}$. The pair $(X, \mathcal{C})$ is an abstract convexity space. The convex hull operator $\mathcal{C}: P(X) \to P(X)$ is $\mathcal{C}(S) = \begin{cases} X & S \neq \phi \\ \phi & S = \phi \end{cases}$

(iv) Consider a topological vector space $(X, \mathcal{J})$ and $\mathcal{C}$ is a family of closed sets in X. Then $(X, \mathcal{C})$ becomes an abstract convexity space and in this case the convex hull operator $\mathcal{C}: P(X) \to P(X)$ is a topological closure operator, that is, $\mathcal{C}(S) = \bigcap \{C: C \in \mathcal{C} \quad C \supseteq S\}$.

(v) Suppose $X = R^2$ and $\mathcal{C} = \{$ points, line segments, $\phi, X\}$. The pair $(X, \mathcal{C})$ is an abstract convexity space. The convex hull of a non-empty set having three or more non-collinear points is the whole space, otherwise, it is just the line segment between two points.

(vi) Suppose $X = R^2$ and $\mathcal{C} = \{$ compact convex subsets of X, ϕ, $X\}$. The pair $(X, \mathcal{C})$ is an abstract convexity space. The convex hull of an unbounded set is the whole space and for a bounded set it is the usual closed convex hull of the bounded set.

CHAPTER II

JOIN SYSTEMS

2.1 Introduction

This chapter introduces a structure called a join system, which is an example of an abstract convexity space. Such a system was first introduced by Prenowitz [14]. An arbitrary set with an operation called a join of two points to form a segment is the basic operation. Next in importance is the consideration of an operation of extending segments, which can be stated in terms of join. These operations satisfy several axioms to form a Join System. One axiom, sometimes called Peano's axiom, which gives a formal relation between join and extension, is of basic importance.

It should be emphasized that these axioms are completely general and hold for all degenerate or "limiting" cases. For example, the associative law for join holds if all points are distinct and collinear or any two points are the same or even if all points are the same. These axioms are too weak to characterize Euclidean geometry. Much has been omitted. For example, (i) a parallel postulate, (ii) reference to congruence, (iii) the basic incidence properties are left out. Moreover, all the axioms do not imply that the points on a line are "fully ordered." Finally, note that no dimensionality restriction is included.

It is proved that a join system is an abstract convexity space and also that the converse is true if the abstract convexity space $(X,\mathcal{C})$ is domain-finite, join-hull commutative with regular straight segments.

2.2 Join Systems

2.2.1 Definition

Consider a non-empty set X and $\cdot : X \times X \rightarrow P(X)$ a function which associates with each ordered pair of elements a,b of X a subset of X called the product or join of a and b, denoted by a.b or simply ab.

2.2.2 Definition

The inverse operation is defined as

$$a/b = \{ x : a \in bx \}.$$

2.2.3 Definition

If A,B are subsets of X then the product and inverse of these sets are $AB = \bigcup \{ ab : a \in A , b \in B\}$ and $A/B = \bigcup\{a/b : a \in A , b \in B\}$ respectively.

Before beginning the join system, we explain the notational conventions which are adopted in this chapter. A finite set whose elements (not necessarily distinct) are $a_1, a_2, \ldots, a_n$ is denoted $(a_1, a_2, \ldots, a_n)$. The relation A meets B or A intersects B is defined by $A \approx B$. If $A = (a)$ and $B = (b)$ the relation reduces to the equality a = b. Also it covers "element containment" relations - for example, $b \in A$ is equivalent to $(b) \approx A$ which is simply written as

$b \not\approx A$.

2.2.4 Definition

A pair $(X,\cdot)$ is said to be a join system if it satisfies the following axioms for all $a,b,c,d \in X$:

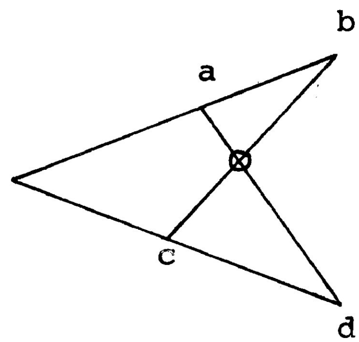
(A1) $\phi \neq ab \subset X$;

(A2) $ab = ba$;

(A3) $a(bc) = (ab)c$;

(A4) $\phi \neq a/b \subset X$;

(A5) If $a/b \approx c/d$ then $ad \not\approx bc$;



(A6) $aa = a = a/a$.

The axiom (A5) is sometimes called the transposition principle and is central since it gives a formal relation between join and extension. This is also called Peano's axiom.

2.2.5 Examples

(i) The pair $(X,\cdot)$ is a join system, if X is a real vector space and $\cdot$ is defined by

$$a.b = \{sa + tb : o < s < 1 \text{ and } s + t = 1\}$$

(ii) If $(X,<)$ is a totally ordered set such that for each $a < b$ there exists $c,d,e \in X$ with $c<a<d<b<e$, and $\cdot$ is defined by

$$a.b = \begin{cases} a & \text{if } a=b \\ (c: a<c<b & \text{if } a<b \\ (c: b<c<a & \text{if } b<a \end{cases}$$

then $(X, \cdot)$ is a join system.

(iii) Spherical Convexity. If $X = \{x \in E^n: \ \|x\| = 1$ and $x_1 > 0\}$ where $\| \ \|$ denotes the usual norm in E^n and x_1 denotes the first component of x, and $\cdot$ is defined by

$$a.b = \{ t\ (sa + (1-s)b) \subset X : o<s<1,\ o<t \}$$

then $(X, \cdot)$ is a join system. Here X is an open hemisphere and joins are minor arcs of great circles.

2.3 Formal Properties of a Join System

2.3.1 Proposition

$A \subset B$ implies $AC \subset BC$ and $CA \subset CB$.

Proof: Consider any element $x \in AC$ and deduce $x \in BC$. By definition, $x \in AC$ implies there exists $a \in A$ and $c \in C$ such that $x \in ac$. But $A \subset B$, so $a \in B$. Thus $x \in ac$, $a \in B$, $c \in C$. This implies by definition of Join of sets $x \in BC$. Similarly we can show $CA \subset CB$.

2.3.2 Corollary

$A' \subset A$, $B' \subset B$ imply $A'B' \subset AB$.

2.3.3 Corollary

$a \in A$, $b \in B$ imply $ab \subset AB$.

2.3.4 Proposition

$A \subset B$ implies $A/C \subset B/C$ and $C/A \subset C/B$.

Proof: Suppose $x \in A/C$. Then $x \in a/c$ where $a \in A$, $c \in C$. But $A \subset B$ so $a \in B$. Thus $x \in a/c$, $a \in B$, $c \in C$ implies

$x \in B/C$. We infer $A/C \subset B/C$. Similarly we can show $C/A \subset C/B$.

2.3.5 Corollary

$A' \subset A$, $B' \subset B$ imply $A'/B' \subset A/B$.

2.3.6 Corollary

$a \in A$, $b \in B$ imply $a/b \subset A/B$.

2.3.7 Proposition

$A \approx BC$ if and only if $A/B \approx C$.

Proof: Suppose $A \approx BC$. Then there exists a such that $a \in A$, $a \in BC$. The latter implies $a \in bc$ for some $b \in B$, $c \in C$. Thus, $a \in bc$ and, by definition of a/b, we have $c \in a/b$. This implies $c \in A/B$, but $c \in C$ therefore $A/B \approx C$.

Conversely, suppose $A/B \approx C$. Then there exists c such that $c \in C$, $c \in A/B$. Thus $c \in a/b$ where $a \in A$ and $b \in B$. By definition of a/b, we have $a \in bc$. Thus $a \in BC$ and, since $a \in A$, we conclude $A \approx BC$.

2.3.8 Corollary

$a \approx bc$ if and only if $a/b \approx c$.

2.3.9 Proposition

$A/B \approx C/D$ implies $AD \approx BC$.

Proof: By hypothesis there exists x such that $x \in A/B$, $x \in C/D$. Hence by definition $x \in a/b$ where $a \in A$, $b \in B$, and $x \in c/d$ where $c \in C$, $d \in D$. Thus $a/b \approx c/d$, and (A5) implies $ad \approx bc$. By Corollary 2.3.3, $ad \subset AD$ and $bc \subset BC$. Hence $AD \approx BC$.

2.4 Convex Sets

2.4.1 Definition

A set A is convex if $x,y \in A$ implies $xy \subset A$. Observe that X, the basic set, is convex and that each element a of X is a convex set since by (A6) $a = aa$.

2.4.2 Theorem

A is convex if and only if (a) $A \supset AA$ or (b) $A = AA$.

Proof: Suppose A is convex. Then $x,y \in A$ implies $xy \subset A$. By corollary 2.3.3, we have $AA \subset A$.

Conversely, if $AA \subset A$ then certainly $xy \subset A$, for $x \in A$ and $y \in A$, and A is convex.

2.4.3 Corollary

$a_1a_2a_3 \; \ldots\ldots \; a_n$ is convex.

Proof: By the generalized associative, commutative laws and (A6)

$$\begin{aligned}(a_1a_2 \ldots\ldots a_n)\,(a_1a_2\ldots\ldots a_n) &= (a_1a_1)(a_2a_2)\ldots\ldots(a_na_n)\\ &= a_1a_2a_3\ldots\ldots a_n.\end{aligned}$$

2.4.4 Corollary

If A is convex and $S,T \subset A$ then $ST \subset A$.

Proof: $S \subset A$, $T \subset A$ can be combined to yield $ST \subset AA$ or $ST \subset A$.

2.4.5 Theorem

Any intersection of convex sets is convex.

Proof: Suppose $F = \bigcap(F_i : i \in I$ and F_i is convex) and $a,b \in F$ then a,b belongs to each F_i, but F_i, $i \in I$, is convex therefore ab belongs to each F_i. Hence $ab \subset F$.

<u>2.4.6 Proposition</u>

Consider the finite set $(a_1,a_2,..,a_n)$. Let S be the union of joins of $a_1,a_2,..,a_n$ taken one or more at a time:

$$S = a_1 \cup a_2 \cup .. \cup a_n \cup a_1 a_2 \cup a_1 a_3 \cup .... \cup a_1 a_2 .... a_n.$$

Then S is the only set which satisfies the following properties:

(i) S is convex;

(ii) $(a_1,a_2,a_3,..,a_n) \subset S$;

(iii) If T is convex and $(a_1,a_2,..,a_n) \subset T$ then $S \subset T$.

<u>Proof</u>: (i) Suppose $x,y \in S$. Then x,y belong to some joins of a's. That is, let $x \in a_1 a_2 a_5$ and $y \in a_2 a_6 a_7$. Combining these we obtain $xy \subset (a_1 a_2 a_5)(a_2 a_6 a_7) = a_1 a_2 a_5 a_6 a_7$. Thus $xy \subset S$ and S is convex by definition.

(ii) This is trivial by definition of S.

(iii) Suppose T is convex and $(a_1,a_2,...,a_n) \subset T$. By Corollary 2.4.4, T contains any join of a's and so $S \subset T$.

To prove uniqueness, suppose S' satisfies (i), (ii) and (iii). Letting T = S' in (iii) we have $S \subset S'$. Similarly $S' \subset S$ so that $S = S'$.

Thus in constructing S we have converted the finite set $(a_1,a_2,..,a_n)$ into a convex set in a simplest possible way. Since by (iii), any other convex set containing $(a_1,a_2,..,a_n)$ must be larger than S, this suggests a precise formulation of the concept of convex hull containing a given (finite or infinite) set.

2.4.7 Definition

Let A be any set. Let S be the only set which satisfies the following properties (i) S is convex; (ii) $A \subset S$; (iii) If T is convex and $A \subset T$, then $S \subset T$. Then S is called convex hull of A, denoted by $\mathscr{C}(A)$.

2.4.8 Theorem

If $(X,\cdot)$ is a join system and $\mathscr{C} = \{A : A \subset X, AA = A\}$ then $(X,\mathscr{C})$ is an abstract convexity space.

Proof: This follows from theorem 2.4.5.

2.5 Axioms for a Join System

So far we have shown that a join system satisfies the axioms of an abstract convexity space. One can think of the converse, that is, when does an abstract convexity space satisfy the axioms of a join system? In order to prove this we add some properties to the abstract convexity space so that it at least implies axiom (A3), (A4), (A5), and (A6), and the following will show that any system with a join, satisfying the aforesaid axioms is a join system. The following lemma was proved by Bryant and Webster [3].

2.5.1 Lemma

If $\cdot$ is a join on X satisfying:

(i) $(ab)c \subset a(bc)$;

(ii) $a/b \neq \phi$;

(iii) If $a/b \approx c/d$ then $ad \approx bc$;

(iv) $aa = a = a/a$;

for all $a,b,c,d \in X$, then $(X,\cdot)$ is a join system.

Proof: Assume that $(X,.)$ satisfies (i) - (iv). We show that axioms (A1), (A2) and (A3) hold in it and conclude that $(X,\cdot)$ is a join system.

(A1) $ab \neq \phi$:

For $a,b \in X$ we have, by (ii), $a/b \neq \phi$. Thus $a/b \approx a/b$ and by (iii) $ab \approx ba$. Hence $\phi \neq ab \cap ba \subset ab$ as required.

(A2) $ab = ba$:

We need to show $ab \subset ba$ then (A2) will follow by symmetry. So let $c \in ab$ and by (A1) above we may choose $d \in cb$. Then $b \in d/c \cap c/a$ and so by (iii), $da \approx cc$. Thus by (i) and (iv), $c = cc \approx da \subset (cb)a \subset c(ba)$ and so $c \in c(ba)$, $c/c \approx ba$ and $c \approx ba$ as required.

(A3) $(ab)c = a(bc)$:

By (i) and (A2) above we have

$$c(ab) \subset a(bc) = (bc)a \subset b(ca) = (ca)b \subset c(ab) = (ab)c$$

and the result follows. Hence $(X,\cdot)$ is a join system.

2.5.2 Definition

In an abstract convexity space $(X,\mathfrak{C})$ the join of a and b, $a \neq b$, is $ab = \mathfrak{C}(a,b) \setminus \{a,b\}$ and extension of a and b is described by $a/b = \{x\colon a \in bx\}$.

We introduce the convention that $aa = a = a/a$ and remark that the definition 2.2.3, Axiom (A2) and formal properties of a join system from 2.3.1 to 2.3.8 will also hold in an abstract convexity space, since these properties were proved from the above definition only.

Finally, one can note that the straightness property in an abstract convexity space implies the following:

If $ab \not\approx ac$ then $b = c$, $b \approx ac$ or $c \approx ab$.

2.5.3 Definition

The line determined by a and b, if $a \neq b$, is

$\ell(a,b) = a/b \cup a \cup ab \cup b \cup b/a$.

2.5.4 Pasch's axiom

If a,b,c, are three points of X with $x \in ab$ and $y \in xc$ then there exists a point $z \in bc$ such that $y \in az$.

2.5.5 Peano's axiom

If x,b,d are three points of X with $a \in bx$ and $c \in dx$ then $ad \cap bc \neq \phi$.

2.5.6 Lemma

If $(X,\mathfrak{C})$ is an abstract convexity space with regular straight segments then for all $a,b,c,d \in X$, (i) $(ab)b = a(bb)$; (ii) $a/a \approx c/d$ implies $ac \approx ad$ and $cd \approx ad$; (iii) $a/b \approx b/c$ implies $b \approx ac$; (iv) $a/b \approx a/c$ implies $ab \approx ac$; (v) $a/b \approx c/b$ implies $ab \approx cb$.

Proof: (i) Suppose $x \in (ab)b$. We show $x \in ab = a(bb)$. By definition $x \in (ab)b$ implies $x \in yb$ for some $y \in ab$. Since X has regular segments therefore $ab = ay \cup y \cup yb$. But $x \in yb$ implies $x \in ab = a(bb)$.

Conversely, if $x \in a(bb) = ab$ then $ab = ax \cup x \cup xb$. Choose some $y \in ax \subset ab$. This implies $ab = ay \cup y \cup yb$. But, $x \in ab$ therefore, either $x \in ay$ or $x \in yb$. Suppose $x \in ay$. Then by the above result $x \in a(ax) \subset ax$, which is a contradiction. Hence $x \in yb$ and so $x \in (ab)b$.

(ii) We suppose $x \in ac$ and deduce $x \in ad$. By definition

$a \in c/d$ implies $c \in ad$, and $x \in ac$ implies $x \in a(ad) = (aa)d = ad$. Hence $ac \approx ad$. Similarly we can prove $cd \approx ad$.

(iii) Suppose $x \in a/b$ and $x \in b/c$. By definition $a \in xb$ and $b \in xc$, these imply $a \in x(xc) = (xx)c = xc$. Since X has regular segments therefore $xc = xa \cup a \cup ac$. But $b \in xc$ implies either $b \in xa$ or $b \in ac$. Clearly if $b \in xa$ then $b/x \approx a$. But $a \approx xb$ implies $b/x \approx xb$ or $b \approx x(xb) = (xx)b = xb$, which is a contradiction. Hence $b \approx ac$.

(iv) Suppose $x \approx a/b$ and $x \approx a/c$, then by definition $a \approx xb$ and $a \approx xc$; this implies $xb \approx xc$. By straightness property $b = c$ or $b \approx xc$ or $c \approx xb$. If $b = c$ then clearly $ab \approx ac$. If $b \approx xc$ then $x \approx b/c$, but $x \approx a/b$ implies $a/b \approx b/c$, by (iii) $b \approx ac$, and by (ii) $ab \approx ac$. If $c \approx xb$ then $x \approx c/b$ but $x \approx a/c$ implies $a/c \approx c/b$, by (iii) $c \approx ab$, by (ii) $ac \approx ab$.

(v) Suppose $x \approx a/b$ and $x \approx c/b$, then by definition $a \approx bx$ and $c \approx bx$. Since X has regular segments therefore $xb = xa \cup a \cup ab$, $c \approx xb$ implies $c \approx xa$ or $c \approx ab$. Suppose $c \approx xa$ then $c/a \approx x$, but $x \approx a/b$ implies $c/a \approx a/b$, by (iii) $cb \approx a$, and by (ii) $ab \approx cb$. If $c \in ab$, choose $y \in cb$, this with $c \in ab$ implies $y \in (ab)b = a(bb) = ab$. Hence $ab \approx cb$.

<u>2.5.7 Lemma</u>

If $(X, \mathcal{C})$ is a join-hull commutative abstract convexity space with regular straight segments then for all $a,b,c \in X$, $(ab)c = a(bc)$.

<u>Proof:</u> The case when $a = b$, $b = c$, $c = a$ or $a = b = c$

follows from lemma 2.5.6. If a,b,c are collinear then it can be easily proved by using the regularity and straightness properties. So we consider the case when a,b,c are not collinear. First we prove that a,b,c do not belong to a(bc). Suppose $a \in a(bc)$. This implies $a/a \approx bc$ or $a \approx bc$ which contradicts the fact that a,b,c are non-collinear. If $b \in a(bc)$ then $b \in ad$ for some $d \in bc$. By lemma 2.5.6 $bd \subset ad$ and $bd \subset bc$. Since the segment bd contains at least a countable number of points therefore by the straightness property $ad \cup bc$ is a segment, which contradicts the supposition that a,b,c are non-collinear. Similarly we can prove that c does not belong to a(bc). Next we show that if $x \in a(bc)$ then x cannot belong to ab, ac or bc. Suppose $x \in ab$ then by straightness property $ab \approx a(bc)$ implies $b \approx bc$, $b \approx a(bc)$ or $ab \approx bc$. The last case implies $a = c$, $a \approx bc$ or $c \approx ab$. One can note that in all these cases we get a contradiction. Similarly we can prove that x does not belong to ac. The proof for the case when $x \in bc$ is different and is as follows. By definition $x \in a(bc)$ implies $x \in ad$ for some $d \in bc$. Since X has regular segments therefore $bc = bd \cup d \cup dc$. But, $x \in bc$ implies $x \in bd$ or $x \in dc$. Suppose $x \in bd$, by straightness property $ad \approx bd$ implies $a = b$, $b \in ad \subset a(bc)$ or $a \in bd \subset b(bc) = bc$. In all these cases we get a contradiction. Similarly we can show x does not belong to dc and hence x does not belong to bc. We suppose

$x \in a(bc)$ and show $x \in (ab)c$. By definition and join-hull commutativity, $x \in a(bc) \subset a_{\mathfrak{C}}\mathfrak{C}(b,c) = \mathfrak{C}(a,b,c) = c_{\mathfrak{C}}\text{-}\mathfrak{C}(a,b)$. This implies $x \in \mathfrak{C}(c,d)$ for some $d \in \mathfrak{C}(a,b)$. We note that $x \neq c$. Also $x \neq d$; for if, $x = d$ then $x \in \mathfrak{C}(a,b)$ and so $x \in ab$, which is a contradiction. Therefore $x \in cd$. Similarly d cannot be equal to a or b. If so, then $x \in ca$ or $x \in cb$, which is again a contradiction. Therefore $d \in ab$. Combining $x \in cd$ and $d \in ab$ gives our result. The reverse containment follows similarly.

<u>2.5.8 Lemma</u>

If $(X,\mathfrak{C})$ is a join-hull commutative abstract convexity space with regular straight segments, then for each $a \neq b$ and $u \in \ell(a,b)$, $u \neq a$, $\ell(a,b) \subset \ell(a,u)$.

<u>Proof</u>: Suppose $u \in \ell(a,b)$ and $u \neq a$, then by definition either $u = b$, in which case the result is immediate or (I) $u \in ab$, (II) $u \in a/b$, or (III) $u \in b/a$. We suppose $x \in \ell(a,b)$ and show $x \in \ell(a,u)$. The case when $x = a$ and $x = b$ are obvious so we consider (i) $x \in ab$; (ii) $x \in a/b$; or (iii) $x \in b/a$ with each of the above three cases. Suppose (I) and (i) hold. Then $u \approx ab$ and $x \approx ab$, so $u/a \approx b$ and $x/a \approx b$. Thus $u/a \approx x/a$. It follows by lemma 2.5.6 that $ua \approx xa$. The straightness property yields $u = x$, $u \approx ax$ or $x \approx au$. In all these cases $x \in \ell(a,u)$. Suppose (I) and (ii) hold. Then $u \approx ab$ and $x \approx a/b$, so $u/a \approx b$ and $b \approx a/x$. Thus $u/a \approx a/x$. It follows by lemma 2.5.6 that $ux \approx aa = a$. Therefore $x \in \ell(a,u)$. Suppose (I) and

(iii) hold. Then $u \approx ab$ and $x \approx b/a$, so $u/a \approx b$ and $xa \approx b$ Thus $u/a \approx ax$. It follows by lemma 2.5.6 (i) that $u \approx a(ax) = ax$. Therefore $x \in \ell(a,u)$. Now we consider (II) with other three cases. Suppose (II) and (i) hold. Then $u \approx a/b$ and $x \approx ab$, so $a/u \approx b$ and $x/a \approx b$. Thus $a/u \approx x/a$. It follows by lemma 2.5.6 that $ux \approx aa = a$. Therefore $x \in \ell(a,u)$. Suppose (II) and (ii) hold. Then $u \approx a/b$ and $x \approx a/b$, so $a/u \approx b$ and $a/x \approx b$. Thus $a/u \approx a/x$. It follows by lemma 2.5.6 that $ax \approx au$. By straightness property it follows that $x \in \ell(a,u)$. Suppose (II) and (iii) hold. Then $u \approx a/b$ and $x \approx b/a$, so $a/u \approx b$ and $xa \approx b$. Thus $a/u \approx xa$ or $a \approx u(xa)$. By lemma 2.5.7, $a \approx a(ux)$ or $a/a \approx ux$. Therefore $x \in \ell(a,u)$. Similarly we can prove (III) with (i), (ii) or (iii).

<u>2.5.9 Lemma</u>

If $u,v \in \ell(a,b)$, $u \neq v$, then $\ell(a,b) = \ell(u,v)$.

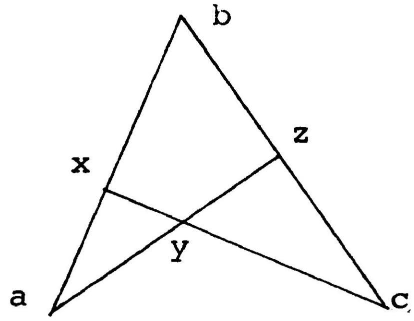
<u>Proof</u>: It now easily follows from lemma 2.5.8 that if $u \in \ell(a,b)$, $u \neq a$, then $\ell(a,b) = \ell(a,u)$, and thus if $v \in \ell(a,b) = \ell(a,u)$, $u \neq v$, then $\ell(a,b) = \ell(u,v)$; and our result follows.

<u>2.5.10 Lemma</u>

If $(X,\mathfrak{C})$ is a join-hull commutative abstract convexity space with regular straight segments then Pasch's axiom hold.

<u>Proof</u>: Suppose a,b,c are non-collinear points of X with $x \in ab$ and $y \in xc$. This implies $y \in (ab)c$, so, by lemma

2.5.7, $y \in a(bc)$. By proposition 2.3.7, $y/a \approx bc$.

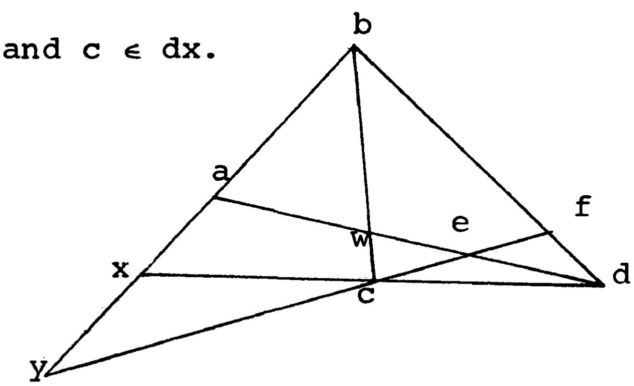


(See figure). Suppose $z \in bc$ and $z \in y/a$ then $y \in az$ and this completes our proof.

2.5.11 Lemma

If $(X,\mathfrak{C})$ is a join-hull commutative abstract convexity space with regular straight segments, then Peano's axiom hold.

Proof: Suppose x,b,d are non-collinear points with $a \in bx$ and $c \in dx$.



Since X has regular segments, therefore, for some y we have $x \in ay$. By Pasch's axiom applied to triangle ady there exists $e \in ad$ such that $c \in ey$. (See figure). By Pasch's axiom applied to triangle byd there exists $f \in bd$ such that $e \in fy$. By Pasch's axiom applied to triangle bcd there exists $w \in bc$ such that $e \in wd$. But $e \in wd$ and $e \in ad$ implies a,e,w,d lie on one line, i.e.,

$\ell(e,d)$. Now in triangle ybe, $c \in ey$ and $w \in bc$, therefore by Pasch's axiom $a \in yb$ and $w \in ae$, but $e \in ad$ implies $w \in a(ad) = (aa)d = ad$.

2.5.12 Theorem

If $(X,\mathfrak{C})$ is a domain-finite, join-hull commutative abstract convexity space with regular straight segments then it is a join system.

Proof: By lemma 2.4.1, we only have to show that axioms (A3), (A4), (A5) and (A6) of a join system hold in $(X,\mathfrak{C})$.

(A3) $a(bc) = (ab)c$ for all $a,b,c \in X$.

The proof follows from lemma 2.5.7

(A4) $a/b \neq \emptyset$ for all $a,b \in X$.

By one of the property of regular segments, i.e., extendibility of segments implies a/b is not empty.

(A5) If $a/b \approx c/d$ then $ad \approx bc$ for all $a,b,c,d \in X$.

The case when any two of them are same follows from lemma 2.5.6 and if all are distinct collinear or non-collinear follows from lemma 2.5.11.

(A6) $aa = a = a/a$ for each $a \in X$.

It follows from our definition.

2.6 Separation

In this section the separation property for convex sets is proved. The discussion of the separation of convex sets in a join system is based on the notion of a complementary pair. For a more complete treatment of separation in a join system and further references see Bryant and Webster [2] and Bair [1].

2.6.1 Definition

A complementary pair (C,D) is an un-ordered pair of non-empty convex subsets of X such that $C \cap D = \emptyset$ and $C \cup D = X$.

2.6.2 Definition

A join system $(X,\cdot)$ has the separation property if for any convex sets A and B such that $A \cap B = \emptyset$ there exists a complementary pair (C,D) such that $A \subset C$ and $B \subset D$ or $A \subset D$ and $B \subset C$.

2.6.3 Theorem

In a join system $(X,\cdot)$ the separation property holds.

Proof: Suppose A and B be disjoint , non-empty convex sets, and denote $\mathcal{F}$ a non-empty collection of all ordered pairs (A_i,B_i), where A_i and B_i are disjoint convex sets with $A \subset A_i$ and $B \subset B_i$. Define a partial order $\leq$ on $\mathcal{F}$ by writing $(A_i,B_i) \leq (A_j,B_j)$ whenever $A_i \subset A_j$ and $B_i \subset B_j$, i.e., by inclusion. Every non-empty chain in $\mathcal{F}$, say Σ, is bounded above by $C = \bigcup\{A_i : (A_iB_i) \subseteq \Sigma\}$, $D = \bigcup\{B_i : (A_i,B_i) \subseteq \Sigma\}$ So by Zorn's lemma $\mathcal{F}$ has a maximal element (C,D). We show that (C,D) is a complementary pair which separates A and B. To do this we need only to show that $C \cup D = X$. Suppose $C \cup D \neq X$, and let $x \notin C \cup D$. Then by maximality of (C,D) it follows that $\mathfrak{C}(x \cup C) = x \cup xC \cup C \approx D$ and $\mathfrak{C}(x \cup D) = x \cup xD \cup D \approx C$.

Since $x \notin C \cup D$ and $C \cap D = \phi$ we must have $xC \approx D$ and $xD \approx C$. Hence by propositions2.3.7 and 2.3.9, we have $D/C \approx C/D$ and $C = CC \approx DD = D$ which is impossible. Hence (C,D) is a complementary pair which separates A and B.

2.6.4 Corollary

If $(X,\mathfrak{C})$ is a domain-finite, join-hull commutative abstract convexity space with regular straight segments then the separation property holds in it.

CHAPTER III

LINEARIZATION OF AN ABSTRACT CONVEXITY SPACE

3.1 Introduction

Any vector space V over an ordered field together with its family of convex sets becomes the prototype for all convexity spaces, and the family of convex sets of V is called the usual convexity structure for V. A deeper question is the determination of an algebraic structure for a given abstract convexity space $(X,\mathfrak{C})$ which makes X into a vector space whose convex sets are precisely the members of $\mathfrak{C}$. This is termed the linearization problem for abstract convexity spaces.

An internal solution to this problem should use only the properties of $\mathfrak{C}$. However, we give necessary and sufficient conditions, in terms of $\mathfrak{C}$ and real-valued convexity-preserving functions on X, for the existence of a real linear structure for X such that the collection of all convex sets in the resulting linear space is precisely $\mathfrak{C}$. This characterization is an external one and was done by Mah, Naimpally and Whitfield [12]. Later in the last chapter the linearization of generalized line spaces is done by means of the results of [12]. There are internal solutions to the linearization problem. See, for example, Doignon [7].

3.2 Linearization

3.2.1 Definition

Let $(X,\mathfrak{C})$ and $(X,\mathfrak{C}')$ be abstract convexity spaces. A map $f\colon X \longrightarrow Y$ is said to be convexity-preserving provided that $f(C) \in \mathfrak{C}'$ for all $C \in \mathfrak{C}$.

3.2.2 Definition

Let R denote the set of real numbers with the usual convex sets. A convexity-preserving map from X to the reals R is called a convexity-preserving functional.

3.2.3 Definition

A family X^* of convexity-preserving functionals on X is called a linearization family for X provided that the following conditions are satisfied:

(L1) There exists a distinguished point $a_0 \in X$ such that $f(a_0) = 0$ for each $f \in X^*$, and the family X^* is point distinguishing; that is, if $f(a) = f(b)$ for each $f \in X^*$, then $a = b$.

(L2) Each $f \in X^*$ restricted to any line in X is either a bijection or a constant map.

(L3) If $f,g \in X^*$ and each separates two points a and b, then there are $s,t \in R$ such that $g(c) = s\, f(c) + t$ for each $c \in \ell(a,b)$.

3.2.4 Example

Suppose X is a vector space and if we consider the zero vector as a_0 and X^* is the set of all linear functions on X to R, then X^* is a linearization family for X.

The purpose of this section is to show that if an abstract convexity space $(X,\mathfrak{C})$ has a linearization family X^*, then X can be given a real linear structure.

The map restricted to the line $\ell(a,b)$ will be denoted by f_{ab}. We begin with a lemma which will allow us to define scalar multiplication on X.

3.2.5 Lemma

If $f,g \in X^*$ and $f(a) \neq 0$, $g(a) \neq 0$ then for each $s \in R$,

$$(f_{a_0a})^{-1}(sf(a)) = (g_{a_0a})^{-1}(sg(a)).$$

Proof: Since f and g separate a_0 and a, (L3) implies that there is a $t \in R$ such that for all $c \in \ell(a_0,a)$, $g(c) = tf(c)$. Thus, $g\left[(f_{a_0a})^{-1}(sf(a))\right] = t\, f\left[(f_{a_0a})^{-1}(sf(a))\right] = t(sf(a))$ $= s(g(a))$, and the result follows.

We are now ready to define scalar multiplication on X as follows:

3.2.6 Definition

For each $s \in R$ and $a \in X$ define

(i) $sa_0 = a_0$ and

(ii) $sa = (f_{a_0a})^{-1}(sf(a))$, for $a \neq a_0$ where $f \in X^*$ and $f(a) \neq 0$.

3.2.7 Definition

For $a, b \in X$, define (vector) addition on X as follows:

(i) $a + b = 2a$ if $a = b$, and

(ii) $a + b = 2\left[(f_{ab})^{-1}\left(\frac{f(a) + f(b)}{2}\right)\right]$ if $a \neq b$, where

f is any member of X^* that separates a and b.

To show addition is well defined, we consider $g \in X^*$ such that it separates a,b and show

$$(f_{ab})^{-1}\left(\frac{f(a) + f(b)}{2}\right) = (g_{ab})^{-1}\left(\frac{g(a) + g(b)}{2}\right).$$

By (L3), there are $s,t \in R$ such that for each $c \in \ell(a,b)$, $g(c) = sf(c) + t$. Thus,

$$\begin{aligned} g\left[(f_{ab})^{-1}\left(\frac{f(a) + f(b)}{2}\right)\right] &= sf\left[(f_{ab})^{-1}\left(\frac{f(a) + f(b)}{2}\right)\right] + t \\ &= s\left(\frac{f(a) + f(b)}{2}\right) + t \\ &= \frac{g(a) + g(b)}{2} \end{aligned}$$

and the result follows.

3.2.8 Theorem

For all $f \in X^*$, $s \in R$, $a,b \in X$, (i) $f(sa) = sf(a)$; (ii) $f(a+b) = f(a) + f(b)$.

Proof: (i) The result is obvious if $a = a_0$, so we suppose $a \neq a_0$. Now two cases arise (1) $f(a) = 0$; (2) $f(a) \neq 0$. In case (1), we have $f(\ell(a_0a)) = \{0\}$ by (L2), and since $sa \in \ell(a_0,a)$, $f(sa) = 0$. In case (2), $sa = (f_{a_0a})^{-1}(sf(a))$, and so $f(sa) = sf(a)$. (ii) The result is trivial if $a = b$, so we suppose $a \neq b$. If f_{ab} is constant, then since $(a + b)/2 \in \ell(a,b)$ (by 3.2.7(ii)), we have, by the above result,

$$f\left(\frac{a + b}{2}\right) = \frac{1}{2} f(a + b) = f(a).$$

Hence, $f(a + b) = 2\,f(a) = f(a) + f(b)$. If f_{ab} is not constant, then by (L2), f seprates a and b. By 3.2.7(ii),

$$f\left(\frac{a + b}{2}\right) = \frac{f(a) + f(b)}{2}.$$

By (i),

$$f(a + b) = 2\, f\left(\frac{a + b}{2}\right) = f(a) + f(b).$$

3.2.9 Theorem

If $(X, \mathcal{C})$ is an abstract convexity space with a linearization family X^*, then X, with addition and scalar multiplication as defined by 3.2.7 and 3.2.6, respectively, is a real linear space.

Proof: To show X is a vector space, we prove all the properties for addition and scalar multiplication which makes X a vector space.

(i) Commutativity: For all $a,b \in X$, $a + b = b + a$.

By theorem 3.2.8, we have

$$\begin{aligned} f(a + b) &= f(a) + f(b) \\ &= f(b) + f(a) \\ &= f(b + a) \text{ for all } f \in X^*. \end{aligned}$$

By (L1) it follows that:

$$a + b = b + a.$$

(ii) Associativity: For all $a,b,c \in X$, $(a + b) + c = a + (b + c)$.

Again by theorem 3.2.8 and (L1), we have

$$\begin{aligned} f((a + b) + c) &= f(a + b) + f(c) \\ &= (f(a) + f(b)) + f(c) \\ &= f(a) + (f(b) + f(c)) \\ &= f(a + (b + c)) \text{ for all } f \in X^*. \end{aligned}$$

So $(a + b) + c = a + (b + c)$.

(iii) Identity: a_0 is the unique member of X such that $a + a_0 = a$ for all $a \in X$.

This follows by theorem 3.2.8 and (L1) and since $f(a_0) = 0$ therefore

$$f(a + a_0) = f(a) + f(a_0)$$
$$= f(a).$$

Hence $a + a_0 = a$.

To prove uniqueness, consider $a + a_0' = a$ so that by (L1) and theorem 3.2.8, we have

$$f(a) + f(a_0') = f(a + a_0') = f(a) = f(a + a_0) = f(a) + f(a_0).$$

So $f(a_0') = f(a_0)$.

Hence $a_0' = a_0$.

(iv) Additive inverse: For each $a \in X$ there is a unique $b \in X$ such that $a + b = a_0$.

Let $b = (-1)a = -a$. By (L1), theorem 3.2.8

$$f(a + b) = f(a) + f(b) = f(a) + f(-a)$$
$$= f(a) - f(a) = 0 = f(a_0).$$

Hence by (L1) $a + b = a_0$.

To prove uniqueness, we suppose there exists b' such that $a + b' = a_0$. Therefore $f(a + b) = f(a_0) = f(a + b')$. So,

$$f(a) + f(b) = f(a + b) = f(a + b') = f(a) + f(b').$$

Thus $f(b) = f(b')$.

(v) Distributive Laws: For all $s,t \in R$ and $a,b \in X$, $(s + t)a = sa + ta$ and $s(a + b) = sa + sb$.

By 3.2.8, (i) we have

$$f((s + t)a) = (s + t)\, f(a) = sf(a) + tf(a)$$
$$= f(sa) + f(ta)$$
$$= f(sa + ta).$$

Hence by (L1) $(s + t)a = sa + ta$.

Similarly, $(st)a = s(ta)$ and $1.a = a$ can be easily proved.

3.3 Compatibility

Now we are ready to study the compatibility of the linear structure for X as constructed above with the family $\mathfrak{F}$. We begin with a lemma.

3.3.1 Lemma

If $a,b \in X$ and $a \neq b$, then $pa + (1-p)b \in \ell(a,b)$ for all $p \in R$.

Proof: The result is trivial if $p = 0$, and so we consider the case when $p \neq 0$. Let $x = pa + (1-p)b$ and $y = (f_{ab})^{-1}$ $\left((pf(a) + (1-p)\ f(b)\right)$, where f is any member of X* which separates a and b. We prove that $x = y$ by showing that the contrary assumption leads to a contradiction. If $x \neq y$ there exists a $g \in X^*$ which separates x and y. Then g must separate x and b, for if $g(x) = g(b)$, then $g(x) = g(b) + p\ g(a-b) = g(b)$. The latter equality implies $g(a) = g(b)$ since $p \neq 0$. This, in turn, implies $g(y) = g(a) = g(b)$ and hence $g(a) = g(b) = g(y) = g(x)$, which contradicts the fact g separates x and y. Thus g separates x and b, and consequently, must separate a and b also, for otherwise, $g(a) = g(b)$ in the following equation $g(x) = g(b) + p\ g(a - b)$ for $p \neq 0$ would imply $g(x) = g(b)$. Since g separates a and b, by (L3), there exists $s,t \in R$ such that $g(c) = sf(c) + t$ for all $c \in \ell(a,b)$. Hence

$$g(y) = s\ f\left[(f_{ab})^{-1}\left(pf(a) + (1-p)\ f(b)\right)\right] + t$$

$$= p\big(sf(a) + t\big) + (1-p)\big(sf(b) + t\big)$$

$$= p\, g(a) + (1-p)g(b)$$

$$= g(x)$$

which is absurd.

We now prove the main result of this chapter.

3.3.2 Theorem

Let $(X,\mathfrak{C})$ be an abstract convexity space which is domain-finite, (finitely) join-hull commutative, and with the property that for all $a,b,c \in X$, $\mathfrak{C}(a,b) = \mathfrak{C}(c,b)$ implies $a = c$. A necessary and sufficient condition that $\mathfrak{C}$ is the family of all convex sets generated by a real linear structure for X is that X has a linearization family X^*.

Proof: We prove only the sufficiency, the necessity being trivial (see Example 3.2.4).

By theorem 3.2.9, X^* induces a linear structure on X. Suppose $C \in \mathfrak{C}$, $a,b \in C$, $a \neq b$ and $0 \leq s \leq 1$. To show that C is convex in the real linear space X, we must prove that $sa + (1-s)b \in C$. Choose $f \in X^*$ which separates a and b. Then

$$f\big(sa + (1-s)b\big) = sf(a) + (1-s)f(b) \subset \big[f(a),f(b)\big] \subset f\,(\mathfrak{C}(a,b)),$$

since f is convexity-preserving. $\big([f(a),f(b)]$ denotes the closed interval formed by $f(a)$ and $f(b)$ in $R\big)$. Hence there is a $c \in \mathfrak{C}(a,b)$ such that $f(c) = f(sa + (1-s)b)$. Since f is bijective on $\ell(a,b)$, by lemma 3.3.1 and our definition of $\ell(a,b)$,

$$c = sa + (1-s)b \in \mathfrak{C}(a,b) \subset C.$$

Conversely, assume $\mathcal{C}$ is convex in the real linear space X. We must show $C \in \mathcal{C}$ i.e., for all $a,b \in C$, $\mathcal{C}(a,b) \subset C$. The hypothesis of the theorem implies that $\{a\} \subset C$ for each $a \in X$, and so if $a = b$ then, trivially, $\mathcal{C}(a,b) = \mathcal{C}(a) = \{a\} \subset C$. Suppose $a \neq b$ and $c \in \mathcal{C}(a,b)$. Choose an $f \in X^*$ which separates a and b. We claim that

$$f(c) \in [f(a), f(b)],$$

for if not, then $c \notin \{a,b\}$. Without loss of generality, we can assume that $f(a) \in [f(c), f(b)]$ that is, $f(a) = t\,f(c) + (1-t)\,f(b) = f(tc + (1-t)b)$ for some $t \in R$, $0 < t < 1$. Since $a \in \ell(c,b)$, then f separates b and c for if not, then $f(b) = f(c)$ in the above equation would imply $f(a) = f(b)$ which is absurd. Hence it follows that $a = tc + (1-t)b \in \mathcal{C}(c,b)$. But $c \in \mathcal{C}(a,b)$ and $a \in \mathcal{C}(c,b)$ implies $\mathcal{C}(a,b) = \mathcal{C}(c,b)$, and so $a = c$, which is absurd. Hence

$$f(\mathcal{C}(a,b)) = [f(a), f(b)], \text{ and}$$

$$f(c) = s\,f(a) + (1-s)f(b) = f\big(sa + (1-s)b\big)$$

for some s, $0 \leq s \leq 1$. Since f is bijective on $\ell(a,b)$, and $c \in \ell(a,b)$, $c = sa + (1-s)b \in C$. Thus $\mathcal{C}(a,b) \subset C$ and so $C \in \mathcal{C}$.

CHAPTER IV

PRODUCTS OF LINE SPACES

4.1 Introduction

An arbitrary set X with a family L of subsets of X satisfying three simple axioms is called a line space and was first introduced by Cantwell [4]. The axioms described here make use of Pasch's axiom, the uniqueness of line determined by two distinct points and the order structure of lines and space rather than the linear or metric structure. Thus, neither algebra nor topology play an important role in this development.

Most of our results apply to the broader class of spaces, introduced by Sandstrom and Kay [16], called generalized line spaces which are line spaces without the requirement of Pasch's axiom. Generalized line spaces are studied because they behave well with respect to products. A product of these spaces is again a generalized line space. But the non-metric character of Pasch's axiom leads us to the un-expected conclusion that Pasch's axiom in a product implies that each factor is a vector space. This result, together with most of the results are due to Sandstrom and Kay [16].

We give as an example of the product of the Moulton plane [13] and the real line, showing that the product

M x R cannot be a line space since M is non-desarguesian and therefore, is not a subspace of a real vector space. It is proved that a line space is essentially a join system, leading us to the direct conclusion that the separation property is true in line spaces. It is perhaps surprising that a generalized line space, in general, cannot satisfy the separation property. The final chapter is concerned with convex and linear functions on generalized line spaces.

4.2 Line Spaces

4.2.1 Definitions and Notation

Consider a pair (X,L) consisting of a non-empty set X whose members will be referred to as <u>points</u>, and a family L of linearly ordered subsets of X called <u>lines</u>. If $a \neq b$ lie on $\ell \in L$, that is, $a,b \in \ell$, then the <u>segment</u> joining points a and b is the set

$$[a,b] = \{c\colon a,b,c \in \ell,\ \ell \in L, \text{ and } a \leq_{\ell} c \leq_{\ell} b\},$$

where $\leq_{\ell}$ denotes the linear order defined on $\ell \in L$.

We assume every line has a given total ordering and will feel free to reverse the order when convenient. Corresponding definitions hold for the open and half-open segments denoted by (a,b) and $[a,b)$ or $(a,b]$ respectively. We introduce the convention that $[a,a] = [a,a) = (a,a] = \{a\}$. If $a \neq b$ then the unique line determined by a and b is denoted by $\ell(a,b)$.

4.2.2 Definition

If a,b,c $\in \ell$ then a,b and c are collinear.

4.2.3 Definition

Three points a,b,c constitute the triangle abc.

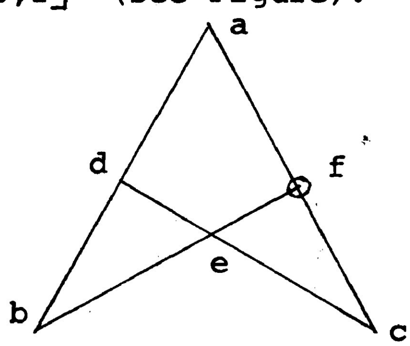
4.2.4 Definition

A line space is a pair (X,L) consisting of a non-empty set X with a family L of linearly ordered subsets of X satisfying the following axioms:

A. Each line is order-isomorphic to the reals.

B. Each distinct pair of elements of X belong to a unique line.

C. For each three points a,b,c of X with d $\in$ [a,b] and e $\in$ [d,c] there exists f $\in$ [a,c] such that e $\in$ [b,f] (See Figure).



It is easy to see that Axiom C remains true for non-collinear points a,b,c with open segments replacing segments. Indeed, in this case, the point is unique with the stated properties.

4.2.5 Example

(i) Any real linear space R or any convex subset of a real linear space R, with the property that a line

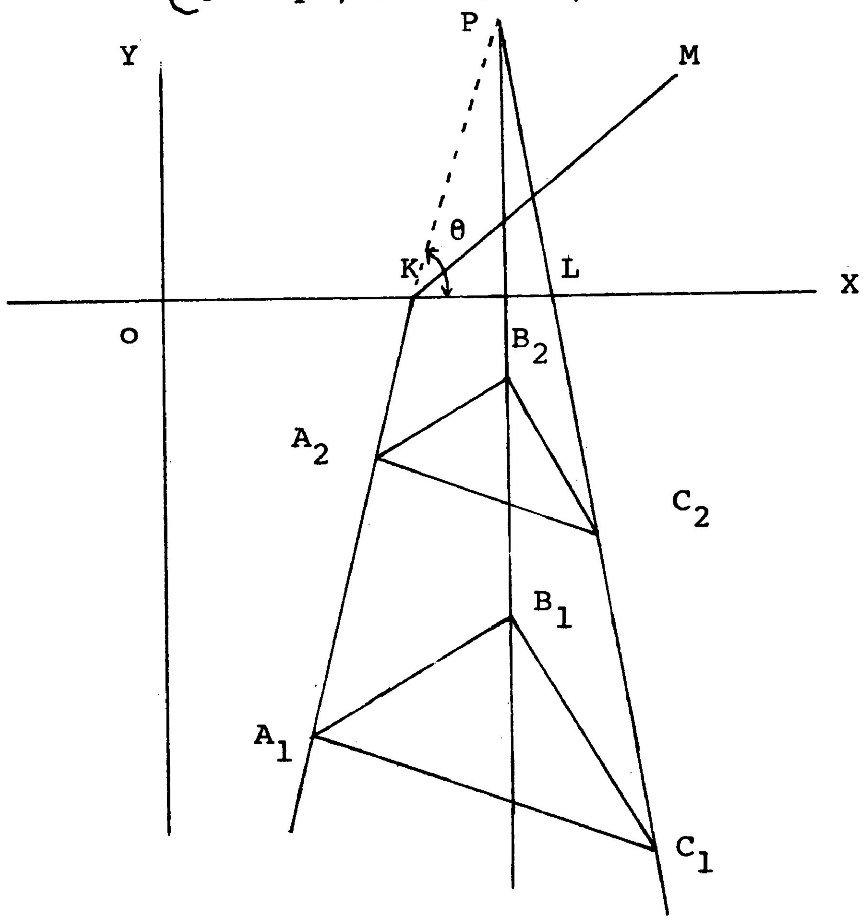
ℓ of R which meets S meets S in an open interval of ℓ, satisfies axioms A, B and C.

(ii) NON-DESARGUESIAN PLANE - The non-desarguesian plane sometimes called the Moulton plane, was introduced by Moulton [13] and is defined in terms of an ordinary euclidean plane, co-ordinatized by the field of real numbers. So, we may consider all pairs of (x,y) of real numbers to be non-desarguesian points. The euclidean straight lines, except those which have a positive slope, are non-desarguesian straight lines; the euclidean (broken) straight lines with positive slope broken at the x-axis so that the slope above is a positive constant (not unity) times the slope below are the remaining non-desarguesian straight lines. That is, the non-desarguesian straight lines are the euclidean straight lines parallel to the x- and y-axis and the euclidean (straight or broken) lines defined, in a new way, by the equation:

$$y = \delta_{y,\theta} (x - a) \tan \theta \qquad \text{(A)}$$

Here x and y are the rectangular co-ordinates of a point referred to the given axes, a is the distance from the origin to the point where the line crosses the x-axis, $\theta (o \leq \theta < \pi)$ is the angle between the positive end of the x-axis and the prolongation of the lower half of the line, and $\delta_{y,\theta}$ is a constant such that

$$\delta_{y,\theta} = \begin{cases} 1 & y < o \quad \theta = \text{any value} \\ 1 & y > o \quad \theta \geq \pi/2 \\ \tfrac{1}{2} & y > o \quad \theta < \pi/2 \end{cases}$$



Thus, in the figure, the lines A_1A_2M, B_1B_2P, C_1C_2P, OY, OX are non-desarguesian straight lines. It is easily seen that Axioms A,B and C are fulfilled in this geometry and hence, it is a line space which is not a subspace of a real linear space.

For example, if we take $x' = (x_1,y_1)$, $y' = (x_2,y_2)$ in M (Moulton plane) then we have four possible cases: (i) $x_1 = x_2$, (ii) $y_1 = y_2$, (iii) $y_1 > y_2$ and $x_1 < x_2$ or $y_1 < y_2$ and $x_1 > x_2$, (iv) $y_1 > y_2$ and $x_1 > x_2$ or $y_1 < y_2$

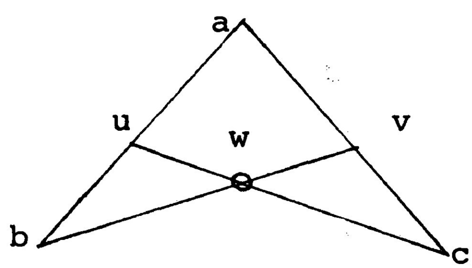
and $x_1 < x_2$. In the first three cases the lines are euclidean straight lines. In case (iv), using equation (A), we get the broken straight line at a point a on the x-axis which makes an angle θ. The particular case we are interested in is when $y_1 > o$ and $y_2 < o$ which gives us the following values for a and θ;

$$a = \frac{y_2\, x_1 - 2y_1\, x_2}{y_2 - 2y_1} \quad \text{and } \theta = \text{arc tan}\left(\frac{2y_1}{x_1 - a}\right).$$

4.3 Axiom C'

Next, we state an axiom in a line space (X,L) which is of basic importance, namely Axiom C', as follows:

Axiom C': For each three points a,b,c of X with $u \in [a,b]$ and $v \in [a,c]$ there exists $w \in [b,v] \cap [c,u]$.



Veblen in 1904 [18] proved that if the line determined by two distinct points is unique, then Axiom C implies Axiom C'.

4.3.1 Definition

In a line space (X,L), for $a \neq b$, the join of a and b, denoted by ab, is the set of points on $\ell(a,b)$ strictly between a and b, and the extension of a and b denoted by a/b is $\{x : a \in bx\}$. We introduce the convention that aa = a = a/a.

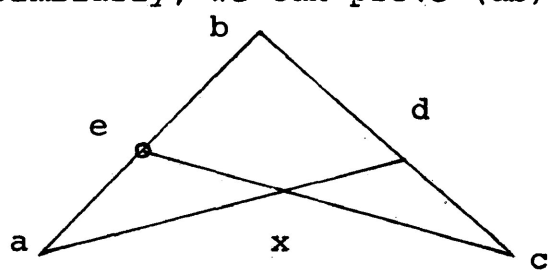
4.3.2 Theorem

A line space (X,L) satisfies the axioms of a join system.

Proof: In order to show (X,L) is a join system, by lemma 2.5.1 we need to show that axioms (A3), (A4), (A5) and (A6) of a join system hold in (X,L).

(A3) For all $a,b,c \in X$, $a(bc) = (ab)c$:

It is easy to prove that if a,b,c are collinear or any two of a,b,c are same or even all are same, then (A3) holds. So we only consider the case when a,b,c are not collinear. We suppose $x \in a(bc)$ and show $x \in (ab)c$. By definition $x \in a(bc)$ means $x \in ad$ for some $d \in bc$. By Axiom C applied to triangle abc (see figure), there exists $e \in ab$ such that $x \in ec$. Now combining these two we get $x \in (ab)c$. Similarly, we can prove $(ab)c \subset a(bc)$.



(A4) $a/b \neq \phi$ for all $a,b \in X$.

It follows immediately, since each line is isomorphic to the reals and one can find a point $x \in \ell(a,b)$ such that $a \in bx$.

(A5) If $a/b \approx c/d$ then $ad \approx bc$ for all $a,b,c,d \in X$:

Suppose a/b meet c/d at x. Then by definition $a \in bx$, $c \in dx$. Since (X,L) has the straightness property and regularity and also Pasch's axiom, i.e., Axiom C, therefore, by Lemma 2.5.11, (A5) holds.

(A6) $a/a = a = aa$:

It follows from our definition.

4.3.3 Corollary

In a line space (X,L), Axiom C implies Axiom C'.

4.4 Generalized Line Spaces

Most of our results apply to the broader class of spaces, which we shall call generalized line spaces. These spaces are studied because they behave well with respect to products.

4.4.1 Definition

A line space (X,L) without the requirement of Axiom C is known as a generalized line space.

4.4.2 Definition

In a generalized line space, a set $C \subset X$ is convex if $a,b \in C$ implies $[a,b] \subset C$. If $A \subset X$, $\mathfrak{C}(A)$ = the convex hull of $A = \bigcap\{C : C \supset A,\ C \text{ convex}\}$.

4.4.3 Definition

A convex set C is convex-open if for every $\ell \in L$, $\ell \cap C = \phi$, a point or an open interval of ℓ.

4.4.4 Definition

F is a flat if $a, b \in F$, $a \neq b$, implies $\ell(a,b) \subset F$. If $A \subset X$, $f\ell(A)$ = flat spanned by $A = \bigcap\{F \supset A : F \text{ a flat}\}$.

4.4.5 Definition

A hyperplane is a maximal proper flat of X.

4.5 Product of Generalized Line Spaces

Axiom A guarantees an isomorphism (say ϕ_ℓ) from each line ℓ to the reals R, and we can define for each such ℓ the directed distance function (relative to ϕ_ℓ) $d_\ell : \ell \to R$ by writing $d_\ell(a,b) = \phi_\ell(b) - \phi_\ell(a)$ for $a,b \in \ell$.

Since by Axiom A the line joining a and b is unique (if $a \neq b$), we can thus define a directed metric function from X x X to R as follows.

4.5.1 Definition

Let (X,L) by any generalized line space. Then the directed metric $d: X \times X \to R$ is defined by $d(a,b) = d_\ell(a,b)$ for $a,b \in \ell$.

Note that for all $a,b \in X$, and any $c \in \ell(a,b)$, we have the following obvious properties

$$d(a,b) = -d(b,a),$$

$$d(a,b) = d(a,c) + d(c,b).$$

4.5.2 Definition

Let (X_i, L_i), $i \in I$, be any collection of generalized line spaces, with d_i denoting the directed metric of X_i. The product $\prod_{i \in I} (X_i, L_i)$ is the pair (X,L) where $X = \prod_{i \in I} X_i$ and, letting a_i denote the i-th co-ordinate of $a \in X$, L is the family of all subsets of the form

$$\ell(a,b) = \{c \in X : c_i \in \ell(a_i,b_i),\ d_i(a_i,c_i)d_j(a_j,b_j) = d_j(a_j,c_j)\ d_i(a_i,b_i), \text{ for all } i,j \in I\} \text{ where } a \neq b.$$

Note the fact that if $a \neq b$ and $c \in \ell(a,b)$, and the co-ordinates of a and b are such that $a_i \neq b_i$ and $a_j \neq b_j$, then the equation defining $\ell(a,b)$ may be written as the ratio

$$\frac{d_i(a_i,c_i)}{d_i(a_i,b_i)} = \frac{d_j(a_j,c_j)}{d_j(a_j,b_j)}.$$

For, $a \neq b$, if $c \in \ell(a,b)$ and one of the co-ordinates of a and b is the same (say $a_i = b_i$) then clearly $c_i = a_i$ for that co-ordinate of c. So choosing $j \in I$ such that $a_j \neq b_j$ then $d_i(a_i,c_i) = 0$.

<u>4.5.3 Lemma</u>

For each $a \neq b$ and $c \in \ell(a,b)$ then $\ell(a,b) \subset \ell(a,c)$.

<u>Proof</u>: If $c \in \ell(a,b)$ and $i,j \in I$, then by definition

$$d_i(a_i,c_i)d_j(a_j,b_j) = d_j(a_j,c_j)d_i(a_i,b_i). \qquad (1)$$

Also, for any $z \in \ell(a,b)$, we have

$$d_i(a_i,z_i)d_j(a_j,b_j) = d_j(a_j,z_j)d_i(a_i,b_i). \qquad (2)$$

We wish to prove that $z \in \ell(a,c)$ or by definition, that $z_i \in \ell(a_i,c_i)$ and

$$d_i(a_i,z_i)d_j(a_j,c_j) = d_j(a_j,z_j)d_i(a_i,c_i). \qquad (3)$$

From the equations (1) and (2), however, one obtains

$$d_i(a_i,z_i)d_j(a_j,c_j)d_i(a_i,b_i)d_j(a_j,b_j) =$$
$$d_j(a_j,z_j)d_i(a_i,c_i)d_i(a_i,b_i)d_j(a_j,b_j). \qquad (4)$$

Now if one of the co-ordinates of a and b is equal, say $a_i = b_i$, then $c_i = a_i = z_i$ and (3) follows immediately. If not, then $d_i(a_i,b_i) \neq 0 \neq d_j(a_j,b_j)$, and we may divide the equation (4) by $d_i(a_i,b_i)d_j(a_j,b_j)$ to obtain (3).

4.5.4 Theorem

The product of any collection of generalized line spaces is a generalized line space.

Proof: Let (X_i,L_i), $i \in I$, be any collection of generalized line spaces. To show that the product $\prod_{i\in I} X_i = X$ is a generalized line space, we show that Axioms A and B hold in X. To obtain Axiom A for X, we construct a bijection $\phi_\ell : \ell \to R$ for each $\ell \in L$ as follows. Choose any two distinct points a,b on ℓ such that $a_i \neq b_i$ for some i. Then set

$$\phi_\ell(c) = \frac{d_i(a_i,c_i)}{d_i(a_i,b_i)} \qquad c \in \ell,$$

where d_i denotes the directed metric of X_i. Note that ϕ_ℓ is well defined follows from the definition of product of generalized line spaces. It follows by definition that $\phi_\ell(c) = \phi_\ell(c')$ implies $c_i = c_i'$ for some i, and in that case, for $j \neq i$ the equations

$$d_i(a_i,c_i)d_j(a_j,b_j) = d_j(a_j,c_j)d_i(a_i,b_i)$$
$$d_i(a_i,c_i)d_j(a_j,b_j) = d_j(a_j,c_j')d_i(a_i,b_i)$$

imply that $c_j = c_j'$ or $c = c'$. Hence, ϕ_ℓ is one-to-one and onto, by definition. By writing $c < c'$ if and only if $\phi_\ell(c) < \phi_\ell(c')$, one obtains that the members of L are linearly ordered sets in X. To obtain Axiom B for the product, it follows by lemma 4.5.3 that if $a,b \in \ell$ and $c \in \ell(a,b), c \neq a$, then $\ell(a,b) = \ell(a,c)$, and thus if $c,d \in \ell(a,b)$, $c \neq d$, then $\ell(a,b) = \ell(c,d)$, and Axiom B follows. This completes our proof.

4.5.5 Example

(i) If each X_i is a real vector space and L_i is the family of 1- flats in X_i, $i \in I$, then (X,L) consists of the usual product vector space and L is the correspon- ding family of 1- flats in X.

(ii) Here we consider the product of the Moulton plane (see 4.2.5(ii)) and the real line denoted by M x R. Let d_m denote the directed metric of M which is defined as follows:

$$d_m\big((x_1,y_1),(x_2,y_2)\big) = \begin{cases} x_2 - x_1 & \text{if } y_1 = y_2 \\ y_2 - y_1 & \text{if } x_1 = x_2 \\ \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} & \text{if } y_1 > y_2 \ \& \ x_1 < x_2 \\ & \text{or } y_1 < y_2 \ \& \ x_1 > x_2 \\ \pm\sqrt{(x_2 - a)^2 + y_2^2} \pm & \text{if } y_1 > y_2 \ \& \ x_1 > x_2 \\ \sqrt{(x_1 - a)^2 + y_1^2} & \text{or } y_1 < y_2 \ \& \ x_1 < x_2 \end{cases}$$

where a is a point on x- axis of the Moulton plane and the choice of the + or - sign is determined by the location of the points. For example, for any two points $a, b \in \ell$, fix the positive sign when you go towards b from a, otherwise negative. The directed metric, d_r, of real numbers is the usual directed distance between two points; (X,L) consists of the usual product, $M \times R = \{(a_m, a_r): a_m \in M, a_r \in R\}$ and L is the family of all subsets of X of the form

$$\ell(a,b) = \{c \in X \mid c_m \in \ell(a_m,b_m),\ c_r \in R,\ d_r(a_r,c_r)d_m(a_m,b_m) = d_m(a_m,c_m)d_r(a_r,b_r)\}$$

for $a \neq b$.

4.6 Axiom C in Product

Thus far in this chapter we have shown that the product of any collection of generalized line spaces is a generalized line space. Our next result of this chapter relates to the question of whether the product of line spaces is a line space; that is, does Axiom C hold in the product $\prod_{i \in I} X_i$ if each X_i is a line space? In fact, it is perhaps surprising that if Axiom C holds in a product then each factor is a vector space in the following sense.

4.6.1 Definition

A generalized line space (X,L) is said to be a vector space if and only if X has an algebraic structure over the reals that is compatible with L; that is, X is a real vector space such that the family L is precisely the set of algebraic 1- flats of X.

In order to establish the above claim, we prove a sequence of lemmas. Throughout the rest of this section, X will be considered as a generalized line space and X x R as a line space. The following results were proved by Sandstrom and Kay [16].

<u>4.6.2 Lemma</u>

If $a' = (a,r) \in X \times R$, $b' = (b,r) \in X \times R$ such that $a \neq b$, then $\ell(a',b')$ consists of the points $x' = (x,t) \in X \times R$ such that $x \in \ell(a,b)$ and

$$t = \left(1 - \frac{d(a,x)}{d(a,b)}\right) r + \frac{d(a,x)}{d(a,b)} s = \frac{d(a,x)}{d(a,b)} s + \frac{d(x,b)}{d(a,b)} r.$$

Furthermore $x' \in [a',b']$ if and only if $x \in [a,b]$ and $t \in [r,s]$.

<u>Proof</u>: Let X_1 and X_2 denote X and R respectively, and with the identity map on R as the order-isomorphism between R and the only line in X_2 (namely, X_2 itself) then $x' \in \ell(a',b')$ if and only if

$$d_1(x_1',a_1')d_2(a_2',b_2') = d_2(x_2',a_2')d_1(a_1',b_1'),$$

where d_1 and d_2 denote the directed metrics of X_1 and X_2 respectively. This equation can be written,

$$d(x,a)\ (s-r) = (r-t)\ d(a,b),$$

and solving for t yields the desired result.

For the second result, suppose $\phi_\ell : \ell(a',b') \longrightarrow R$ is an isomorphism and $x' \in [a',b']$. Then by definition

$$\phi_\ell(a') \leq \phi_\ell(x') \leq \phi_\ell(b').$$

But $\phi_\ell(x') = \dfrac{d(a,x)}{d(a,b)}$ (see 4.5.4),

Therefore, the above inequality reduces to

$$\frac{d(a,a)}{d(a,b)} \leq \frac{d(a,x)}{d(a,b)} \leq \frac{d(a,b)}{d(a,b)}$$

$$0 \leq \frac{d(a,x)}{d(a,b)} \leq 1.$$

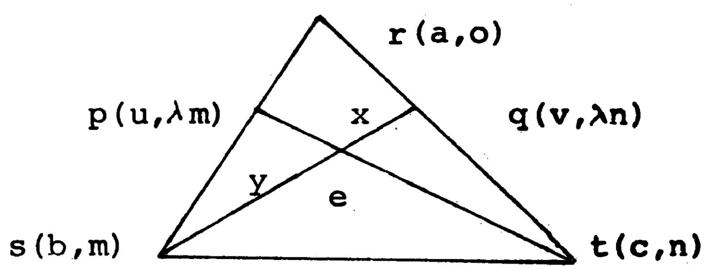
Hence by definition $x \in [a,b]$. One can easily see that t is a convex combination of r and s and hence $t \in [r,s]$. The converse is obvious.

4.6.3 Lemma

Let a,b,c be any three non-collinear points in X with $u \in [a,b]$, $v \in (a,c)$, and w is the unique point of $[c,u] \cap [b,v]$. If $d(a,u)/\, d(a,b) = d(a,v)/\, d(a,c) = \lambda$ then

$$\frac{d(u,w)}{d(u,c)} = \frac{\lambda}{1+\lambda} \quad \text{and} \quad \frac{d(u,w)}{d(c,w)} = \lambda.$$

Proof: Set $x = d(v,w)$ and $y = d(w,b)$, and consider for each real m,n the points $r = (a,0)$, $s = (b,m)$, $t = (c,n)$ in $X \times R$. Then if $p = (u,\lambda m)$ and $q = (v,\lambda n)$ it follows by lemma 4.6.2 that $p \in [r,s]$ and $q \in [r,t]$ (see figure). By Axiom C' there exists a (unique) point $e \in [p,t] \cap [q,s]$.



Again by lemma 4.6.2 $e = (w',\mu)$ for some $w' \in \ell(u,c)$ and real μ such that

$$\left(1 - \frac{d(u,w')}{d(u,c)}\right)\lambda m + \frac{d(u,w')}{d(u,c)}\, n = \mu = \frac{x'}{x'+y'}\, m + \frac{y'}{x'+y'}\, \lambda n \qquad \text{(A)}$$

where $x' = d(v,w')$ and $y' = d(w',b)$. But clearly, by lemma 4.6.2

$$w' \in [u,c] \cap [v,b]$$

and $w' = w$. Hence the above equation (A) is true without primes, and we obtain

$$\left(\lambda - \frac{d(u,w)}{d(u,c)}\lambda - \frac{x}{x + y}\right)m + \left(\frac{d(u,w)}{d(u,c)} - \frac{\lambda y}{x + y}\right)n = o.$$

The coefficents of m and n in the last equation are constant while m and n themselves are arbitrary. Hence,

$$\lambda\left(1 - \frac{d(u,w)}{d(u,c)}\right) = \frac{x}{x+y} \quad \text{and} \quad \frac{1}{\lambda}\,\frac{d(u,w)}{d(u,c)} = \frac{y}{x+y}\ .$$

Summing yields

$$\lambda + \left(\frac{1}{\lambda} - \lambda\right)\frac{d(u,w)}{d(u,c)} = 1,$$

Solving, we get

$$\frac{d(u,w)}{d(u,c)} = \frac{\lambda}{1+\lambda}.$$

To obtain the second result, applying the identity $d(u,c) = d(u,w) + d(w,c)$ in the first result we obtain

$$\frac{d(u,w) + d(w,c)}{d(u,w)} = \frac{1+\lambda}{\lambda}\ .$$

Solving, we get

$$\frac{d(u,w)}{d(w,c)} = \lambda.$$

4.6.4 Corollary

The medians of a triangle are concurrent at a point which is two thirds the distance on each median from the vertex to the midpoint of the opposite side.

Proof: Set $\lambda = \frac{1}{2}$ in the lemma 4.6.3. Then on median $[u,c]$ we have

$$\frac{d(w,c)}{d(u,c)} = 1 - \frac{d(u,w)}{d(u,c)} = 1 - \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{2}{3}\ .$$

Since w is unique on $[u,c]$ with this property it follows that all three medians pass through w and that w has the 2/3 distance property with respect to all three medians.

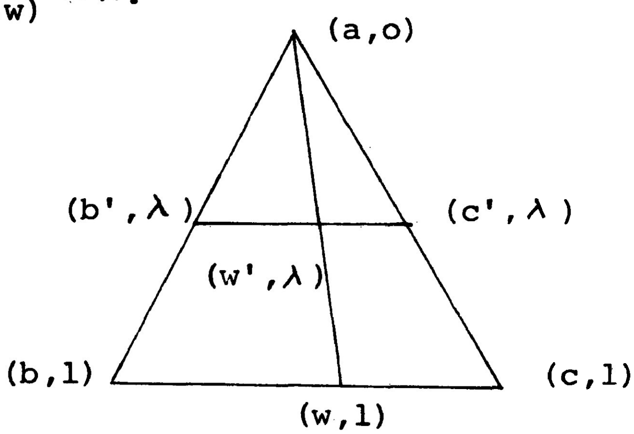
4.6.5 Lemma

Let a,b,c be any three non-collinear points in X with $w \in [b,c]$, $b' \in [a,b]$, $c' \in [a,c]$, and $w' \in [b',c'] \cap [a,w]$. If $d(a,b')/ d(a,b) = d(a,c')/ d(a,c) = \lambda$ and $d(b,w)/ d(b,c) = \lambda'$, then

$$\frac{d(a,w')}{d(a,w)} = \lambda \quad \text{and} \quad \frac{d(b',w')}{d(b',c')} = \lambda'.$$

Proof: Consider the points (a,0), (b,1) and (c,1) in X x R. Then $(b',\lambda) \in [(a,0), (b,1)]$, $(c',\lambda) \in [(a,0),(c,1)]$, $(w',\lambda) \in [(b',\lambda),(c',\lambda)]$ and $(w,1) \in [(b,1),(c,1)]$ (see figure). Axiom C implies $(w',\lambda) \in [(a,0),(w,1)]$ and hence

$$\frac{d(a,w'}{d(a,w)} = \lambda.$$



To prove the second equation consider the points (a,0),(b,1) and (c,t) in X x R, where $t = (\lambda' - 1)/\lambda'$. Then, as before $(b',\lambda) \in [(a,0),(b,1)]$, $(c',\lambda t) \in [(a,0),(c,t)]$, $(w',0) \in [(a,0),(w,0)]$, and $(w,0) \in [(b,\lambda),(c,t)]$. Hence $(w',0) \in [(b',1),(c',\lambda t)]$ and it follows that

$$\frac{d(b',w')}{d(b',c')} = \lambda'.$$

Before we prove our main theorem, we define the algebraic operations on X.

4.6.6 Definition

Choose some point $\underline{o}$ in X as origin, and define addition of $a,b \in X$ by

$$a + b = c$$

where c is the unique point such that $[\underline{o},c]$ and $[a,b]$ have the same midpoint (x is the midpoint of $[a,b]$ if $d(a,x)/d(a,b) = 1/2$.)

For scalar multiplication, if $s \in R$ then take $sa = \underline{o}$ when $a = \underline{o}$; otherwise,

$$sa = c$$

where c is the unique point on $\ell(\underline{o},a)$ such that $d(\underline{o},c) = sd(\underline{o},a)$.

4.6.7 Theorem

If X is a generalized line space and X x R is a line space then X is a vector space.

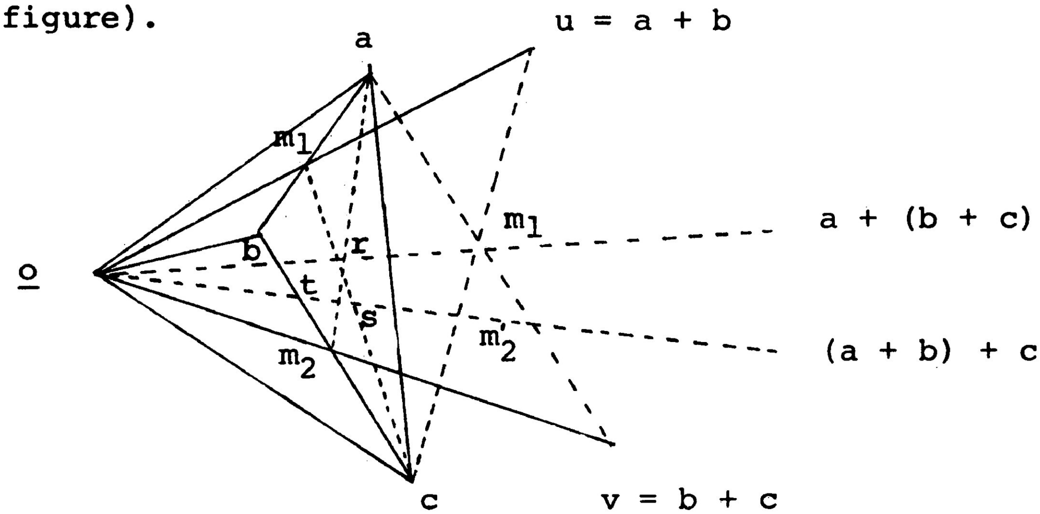
Proof: Suppose X x R is a line space and to show X is a vector space, we prove all the group properties for addition and the properties for scalar multiplication making X a vector space.

(i) Commutativity: $a + b = b + a$ for all $a,b \in X$.

Suppose $a + b = c$ and $b + a = d$ then by definition c and d lie on the lines passing through origin and the midpoints of $[a,b]$ and $[b,a]$ respectively, but the mid-point of $[a,b]$ and $[b,a]$ is unique (say x). Hence c and d lie on the same line i.e., $\ell(\underline{o},x)$. Uniqueness of x (the midpoint of $[a,b]$ and $[b,a]$ and distance property implies c and d lie on each other and the result follows.

<u>(ii) Associativity:</u> $(a + b) + c = a + (b + c)$ for all $a,b,c \in X$.

The case is trivial when $a = \underline{o}$, $b = \underline{o}$ and $c = \underline{o}$. Assume first that no three of $\underline{o}$, a, b, c are collinear; this case then implies (ii), in the case when, say, $\underline{o}$, a, b are collinear but $c \notin \ell(\underline{o}, a)$ by choosing c' so that no three of $\underline{o}$, a + b, c, c' are collinear, and the later in turn implies (iii) in the case when $\underline{o}$, a, b, c are collinear by choosing $c'' \notin \ell(\underline{o},a)$. Let $u = a + b$ and $v = b + c$, and let m_1, m_2, m_1', m_2' be the midpoints of $[a,b]$, $[b,c]$, $[a,v]$, $[u,c]$ respectively, which determine $a + (b + c)$ and $(a + b) + c$ (see figure).



The median $[a,m_2]$ of triangle $\underline{o}$ xv meets median $[\underline{o},m'_1]$ an r, while median $[c,m_1]$ of triangle $\underline{o}$cu meets median $[o,m_2']$ at s. But $[a,m_2]$, $[c,m_1]$ are medians of triangle abc, so meet at t, from which it follows that r=t=s. Hence by corollary 4.6.4

$$m_1' = \frac{3}{2} r = \frac{3}{2} s = m_2' \text{ and therefore}$$

$$a = (b + c) = 2m_1' = 2m_2' = (a + b) + c.$$

The identity element and additive inverses follow from the definition of addition.

(iv) For all $a \in X$ and $s,t \in R$, $(st)a = s(ta)$.

Since $\underline{o}$, $(st)a$, ta and hence $s(ta)$, are collinear, it suffices to prove that

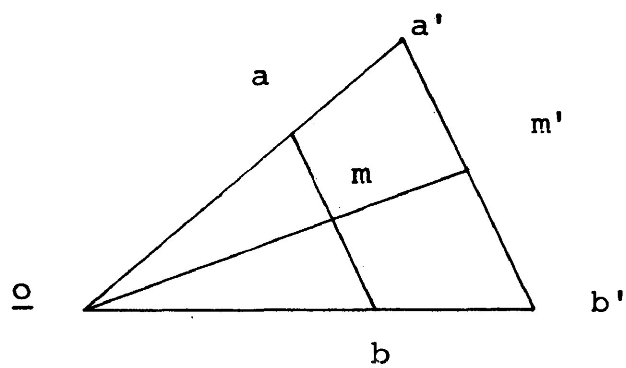
$$d(\underline{o},(st)a) = d(\underline{o},s(ta)).$$

But, by definition

$$d(\underline{o}, (st)a) = st\, d(\underline{o}, a) = sd(\underline{o},ta) = d(\underline{o},s(ta)).$$

(v) For all $a,b \in X$ and $s \in R$, $s(a + b) = sa + sb$.

We take first the case when $\underline{o}$, a and b are not collinear. With $a' = sa$, $b' = sb$, let m and m' be the midpoints of $[a,b]$ and $[a',b']$ which determine $a + b$ and $a' + b'$ (see figure). By lemma 4.6.5, since $\ell(\underline{o},m')$ meets $[a,b]$ at the midpoint of $[a,b]$ then $m' \in \ell(\underline{o},m)$, and $d(\underline{o}, m') = s\, d(\underline{o},m)$. Thus $m' = sm$.



Hence by (iii),

$$sa + sb = 2m' = 2\ sm = s(wm) = s(a + b).$$

The case when $\underline{o}$, a, and b are collinear follows from the proceeding case by choosing $c \in \ell(\underline{o},a)$ and applying (iii)

$$(sa + sb) + sc = sa + (sb + sc) = sa + s(b + c)$$

$= s(a + (b + c)) = s((a + b) + c)$

$= s(a + b) + sc.$

(vi) For all a X and $s,t \in R$, $(s + t)a = sa + ta$.

Suppose $a' = sa$ and m be the mid point of $[a',b']$. Since $\underline{o}$, sa, tb, (s+t)a, and a'+b' are collinears so it is sufficent to show that $d(\underline{o}, (s+t)a) = d(\underline{o}, sa+ta)$. By definition,

$$d(\underline{o}, (s + t)a) = (s + t)\, d(\underline{o},a') = s\, d(\underline{o},a) + t\, d(\underline{o},a)$$

$$= d(\underline{o},a') + d(\underline{o},b') + d(a',m) - d(m,b')$$

$$= 2\, d(\underline{o},m) = d(\underline{o},2m) = d(\underline{o},sa + ta).$$

The other properties of a group can be easily verified and that the question of compatibility can also be easily verified.

4.6.8 Theorem

Suppose $X = \prod_{i \in I} X_i$. For $p \in X$ and any $i,j \in I$, the space $X_i \times R$ is isomorphic to $X_i \times L_j \times (\prod_{k \neq i,j} p_k) \subset X$, where L_j is any line of X_j. Thus, if Axiom C holds for X it holds for the space $X_i \times R$ for any factor X_i.

4.6.9 Corollary

A product of line spaces is a line space if and only if each factor is a vector space.

Note: For the proof of the above theorem refer to [16].

4.6.10 Example

(i) Here is an example which not only supports theorem 4.6.7 but is also an example of a generalized line space that is not a line space. The product M x R, where M is a Moulton plane and R is the real line, is not a vector space and is not a line space, since one of the factors, M, is a non-desarguesian, so it is not a subspace of vector space. It is shown that Axiom C does not hold in this product though Axiom A and B hold in it.

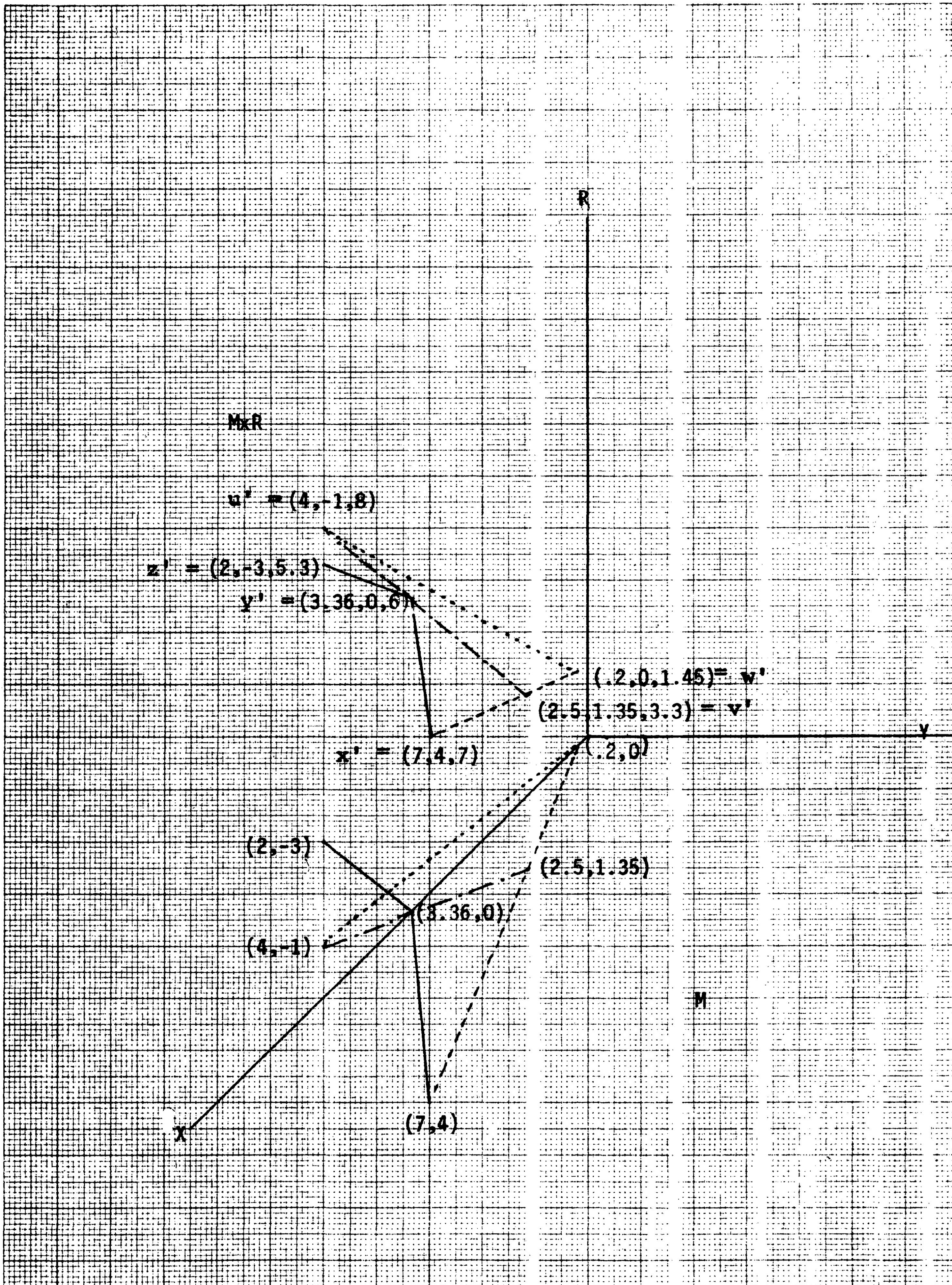
Let d_m denote the directed metric of Moulton plane and is defined as in Example 4.2.5 (ii). Let $x' = (x,r)$, $y' = (y,s) \in M \times R$ such that $x \neq y$. Then by lemma 4.6.2, $\ell(x',y')$ consists of the points $z' = (z,t) \in M \times R$ such that $z \in \ell(x,y)$ and

$$t = \frac{d_m(x,z)}{d_m(x,y)} s + \frac{d_m(z,y)}{d_m(x,y)} r \qquad \text{(A)}$$

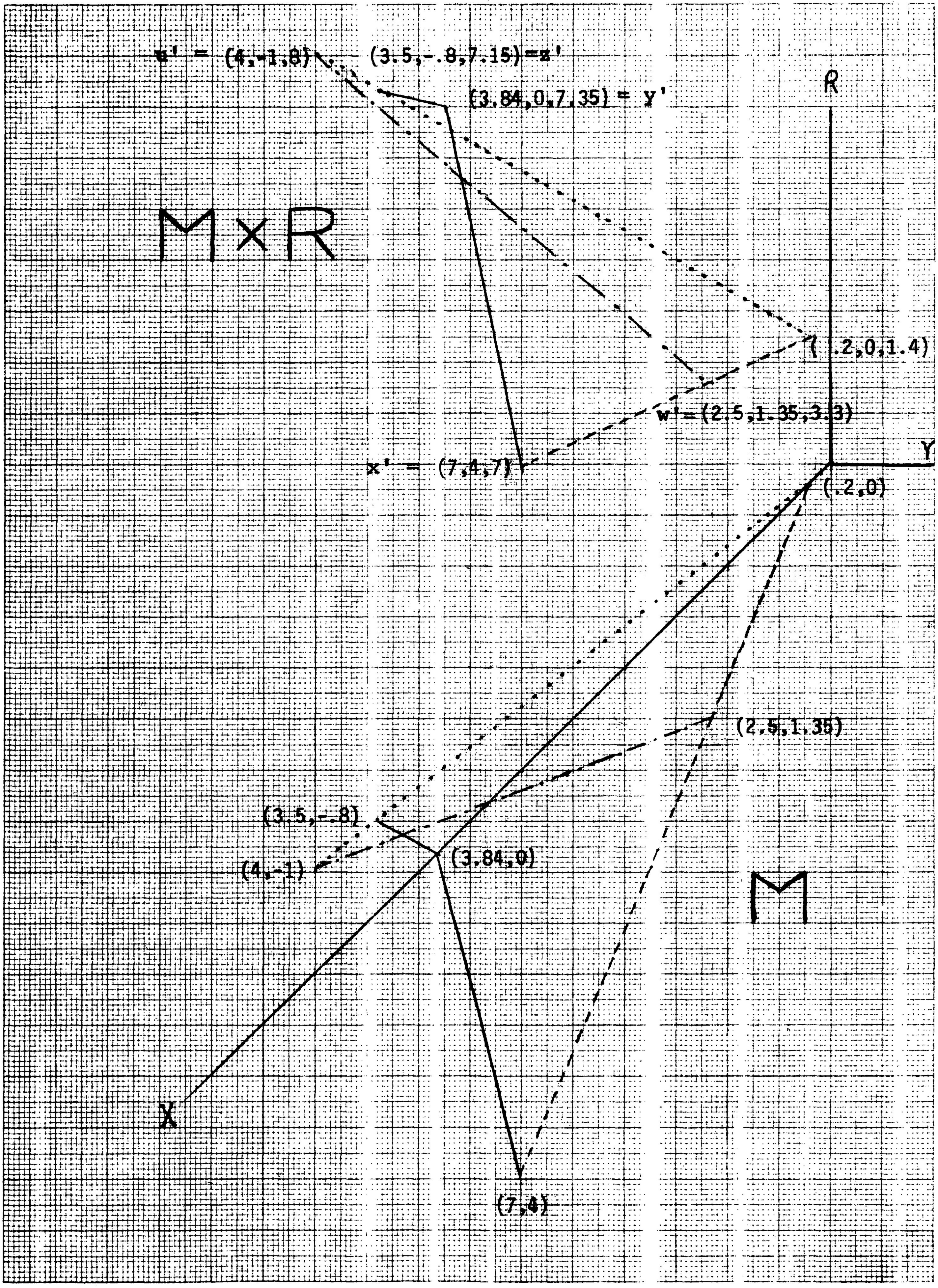
It is easy to check by using equation (A) that if the lines in M are parallel to x- axis, y-axis or of negative slope, then in M x R these are also straight lines. However, if the lines in M are of broken straight lines, then the lines in M x R behaves the same which helps us to show that Axiom C (Pasch's axiom) does not hold in M x R.

We begin with the following points:

Take $x' = ((7,4),7)$, $y' = ((3.3.6,o),6) \in M \times R$. By equation (A) we calculated $z' = ((2,-3), 5.29)$ such that $y' \in [x',z']$. Next take $u' = ((4,-1),8)$ and $y' = (3.35,o),6)$ and calculations gives $v' = ((2.5,1.35),3.3)$ so that $y' \in [u',v']$. Finally, take $v' = ((2.5,1.35),3.3)$ and $x' = ((7,4),7)$ which gives us $w' = ((.2,0,1.45)$ such that $v' \in [x',w']$. One can easily see there does not exist any point on segment $[u',w]$ so that y' belongs to the segment $[x',r']$ where $r' \in [u',w']$. Hence Axiom C does not hold in M x R. (see figure on next page).


R
MxR
u' = (4,-1,8)
z' = (2,-3,5.3)
y' = (3.36,0,6)
(.2,0,1.45) = w'
(2.5,1.35,3.3) = v'
y
x' = (7,4,7)
(.2,0)
(2,-3)
(2.5,1.35)
(3.36,0)
(4,-1)
M
(7,4)
X

METRIC



u' = (4,-1,8)
(3.5,-.8,7.15) = z'
(3.84,0,7.35) = y'
R
M×R
(.2,0,1.4)
w' = (2.5,1.35,3.3)
Y
x' = (7,4,7)
(.2,0)
(2.5,1.35)
(3.5,-.8)
(3.84,0)
(4,-1)
M
X
(7,4)

METRIC

(ii) Next we give an example which shows that Axiom C' (Peano's Axiom) does not hold in M x R. We take the following points in M x R. Take x' = ((7,4)7), y' = (3.8 ,0), 7.35). By equation (A) we calculated z'= ((3.5,-8),7.15) such that $y' \in [x',z']$. Next take u' = ((4, -1,)8) and z' = ((3.5,-.8),7.15) and calculation gives v' = ((.2,0),1.4) such that $z' \in [u',v']$. Finally, take v' = ((.2,0),1.4) and x' = (7, 4) 7) which gives us w' = (2.5,1.35),3.3) such that $w' \in [x',v']$. One can note that $[u',w'] \cap [x',z'] = \phi$. Of course, they look that they intersect in M x R but suppose $[x',y']$ and $[u',w']$ intersect. Then by lemma 4.6.2 they must also intersect in M which is not true. Hence, Axiom C' does not hold in M x R. (See figure on last page).

4.7 Separation

In this section we show that a line space has the separation property. However, it is also shown that if the separation property holds in a generalized line space then it becomes a line space.

4.7.1 Theorem

In a line space (X,L) the separation property holds.

Proof: It follows immediately from Theorem 4.3.2 that a line space is a join system and by Theorem 2.6.3 our result follows.

4.7.2 Theorem

A generalized line space (X,L) need not satisfy the separation property.

Proof: Let a,b,c,d and p be any points in X such that $a \in [p,b]$, $c \in [p,d]$ and $[a,d] \cap [b,c] = \phi$. Since $[a,d]$ and $[b,c]$ are convex sets in X, so by separation property there exists non-empty dis-joint convex set (C,D) such that $[a,d] \subset C$ and $[b,c] \subset D$ and $C \cup D = X$. Since $C \cup D = X$ implies either $p \in C$ or $p \in D$.

Suppose $p \in C$; but $[a,d] \subset C$ implies $d \in C$; therefore $[p,d] \subset C$. But $c \in [p,d] \subset C$ implies $c \in C$, contrary to hypothesis that $C \cap D = \phi$ since $c \in D$. Therefore $[a,d] \cap [b,c] \neq \phi$ which implies Axiom C' is true in X. But Example 4.6.10 shows that Axiom C' need not hold in a generalized line space. Hence this completes the proof of our thereom.

CHAPTER V

CONVEX AND LINEAR FUNCTIONS

5.1 Introduction

In this last chapter, we consider more general spaces, that is, generalized line spaces, and study convex and linear functions on these spaces. The concept of product is employed in defining these functions from one generalized line space to another. We define, a function $f : X \longrightarrow Y$ to be convex if and only if the epigraph of f is a convex set in $X \times Y$. Similarly, a map $f : X \longrightarrow Y$ is said to be linear if and only if the graph of f in $X \times Y$ is a flat. It is shown that a linear function is a convex function and the graph of a linear functional is a hyperplane. It is also proved that a function f is convex [linear] if and only if f_ℓ (the map restricted to a line) is convex [linear] in the usual sense on each line of a generalized line space.

The idea of supporting hyperplanes is also introduced. We prove that a function f defined on an open convex set U is convex if f has a support at each point of U. Finally, the linearization of generalized line spaces is done using as a linearization family the dual X^*, all linear functionals on the generalized line spaces, and the linearization theorem in Chapter III.

5.2 Convex and linear functions

Throughout this chapter, we use X and Y to denote generalized line spaces and study convex and linear functions on these spaces.

5.2.1 Definition

If $f : X \longrightarrow Y$ then the graph of f is the set in $X \times Y$ described by $\quad \text{graph } (f) = \{(a,b) : a \in X,\ b = f(a)\}$.

5.2.2 Definition

If $f : X \longrightarrow Y$ then the epigraph of f is the set in $X \times Y$ described by $\quad \text{epi}(f) = \{(a,b) : a \in X,\ b \geq f(a)\}$.

5.2.3 Definition

A function $f : X \longrightarrow Y$ is convex if and only if the epigraph of f is a convex set in $X \times Y$.

5.2.4 Definition

A map $f : X \longrightarrow Y$ is said to be linear if and only if the graph of f in $X \times Y$ is a flat.

5.2.5 Lemma

A function $f : X \longrightarrow R$ is linear if and only if it is a line-preserving map from X to R which also preserves ratios of distances on each line ℓ in X relative to the metrics d_1 and d_2 of X and R.

Proof: Suppose f is linear. By definition, graph (f) = $\{(a,f(a)): a \in X\}$ is a flat in $X \times R$. Now, if a,b,c are points of a line ℓ in X, $a \neq b$, then lemma 4.6.2 implies that for some $c' \in \ell(f(a),f(b))$, if $f(a) \neq f(b)$, of R we have $(c,c') \in \ell((a,f(a)),(b,f(b)) \subset$ graph (f). That is,

$$c' = f(c) \in \ell(f(a),f(b)), \qquad (1)$$

and again by lemma 4.6.2

$$c' = f(c) = \frac{d_1(a,c)}{d_1(a,b)} f(b) + \frac{d_1(c,b)}{d_1(a,b)} f(a).$$

Solving this, we get

$$\frac{d_1(a,c)}{d_1(a,b)} = \frac{d_2(f(a),f(c))}{d_2(f(a),f(b))} \qquad (2)$$

provided $f(a) \neq f(b)$; otherwise f is constant on ℓ since in that case one has $d(f(a),f(c)) = 0$ for all $c \in \ell$.

Conversely, it is clear that if (1) and (2) hold then graph (f) is a flat in $X \times R$.

5.2.6 Theorem

If $f : X \longrightarrow R$ is linear then f is a convex function.

Proof: Suppose $a' = (a,r)$, $b' = (b,s)$ are two distinct points of epi(f), and let $x' \in [a',b']$ where $x' = (x,t)$ and $x \in [a,b]$. We wish to show that the epigraph of f is a convex set in $X \times R$, that is, $x' \in$ epi(f). By lemma 4.6.2

$$t = \frac{d(a,x)}{d(a,b)} s + \frac{d(x,b)}{d(a,b)} r.$$

But (a,r), $(b,s) \in$ epi (f), therefore

$$t \geq \frac{d(a,x)}{d(a,b)} f(b) + \frac{d(x,b)}{d(a,b)} f(a).$$

By lemma 5.2.5 (2), we have

$$t \geq \frac{f(x) - f(a)}{f(b) - f(a)} f(b) + \frac{f(b) - f(x)}{f(b) - f(a)} f(a).$$

Simplifying the right hand side, we get

$$t \geq f(x).$$

Hence $x' \in \mathrm{epi}(f)$ and this completes our proof.

5.2.7 Theorem

A map $f : X \longrightarrow R$ is linear if and only if the graph of f in $X \times R$ is a hyperplane.

Proof: Suppose f is linear. Then by definition, graph (f) is a flat. We prove in particular that if H' is a flat in $X' = X \times R$ which properly contains graph $(f) \equiv F'$, then $H' = X'$. Suppose $x' = (x,r) \in X'$ and $h' = (h,t) \in H' \setminus F'$. We shall show that in all cases $x' \in H'$, and thus $H' = X'$ as desired. If $r = f(x)$ then $x' \in$ graph $(f) \subset H'$, so we assume $r \neq f(x)$. Further, $x \neq h$, for otherwise with $x'' = (x,f(x)) \in F' \subset H'$, then $(x,r) \in \ell((x,f(x)),(h,t))$ or $x' \in \ell(x'',h') \subset H'$ and the result follows. Also it may be assumed that

$$f(h) - t \neq r - f(x),$$

for if equality holds then one may choose $t_1 = \frac{1}{2}(f(h) + t)$ and set $h'' = (h,t_1)$, yielding $(h,t_1) \in \ell((h,t),(h,f(h))$ so that $h'' \in H' \setminus F'$ but

$$f(h) - t_1 = \frac{f(h) - t}{2} = \frac{r - f(x)}{2} \neq r - f(x).$$

Let $\ell = \ell(x,h)$. There exists $y \in \ell$ such that

$$\phi_\ell(y) = \frac{\phi_\ell(x)(f(h) - t) + \phi_\ell(h)(f(x) - r)}{f(h) - t + f(x) - r}$$

from which follows the equation

$$\frac{\phi_\ell(y) - \phi_\ell(x)}{\phi_\ell(h) - \phi_\ell(x)} f(h) + \frac{\phi_\ell(h) - \phi_\ell(y)}{\phi_\ell(h) - \phi_\ell(x)} r$$

$$= \frac{\phi_\ell(y) - \phi_\ell(x)}{\phi_\ell(h) - \phi_\ell(x)} t + \frac{\phi_\ell(h) - \phi_\ell(y)}{\phi_\ell(h) - \phi_\ell(x)} f(x).$$

That is

$$\frac{d(x,y)}{d(x,h)}\, f(h) + \frac{d(y,h)}{d(x,h)}\, r = \frac{d(x,y)}{d(x,h)}\, t + \frac{d(y,h)}{d(x,h)}\, f(x) \qquad n$$

Set $y' = (h,n)$, $x'' = (x,f(x))$ H', and $h'' = (h,f(h)) \in H'$.

By lemma 4.6.2, $(y,n) \in \mathcal{L}((x,r),(h,f(h)) \cap \mathcal{L}((x,f(x),(h,t))$.

That is, since H' is a flat,

$$y' \in \ell(x'',h') \subset H' \text{ and } x' \in \ell(h'',y') \subset H'.$$

Our next characterization of linear functions is in terms of the restriction of such function to lines. The map restricted to a line ℓ will be denoted by f_ℓ.

5.2.8 Definition

For each line ℓ in X let ϕ_ℓ denote the isomorphism guaranteed by Axiom A of a generalized line space (see definition 4.2.4), and let d be the directed metric defined as before. Define addition (relative to ϕ_ℓ) of a and b on $\ell(a,b)$ as follows:

$$a + b = \phi_\ell^{-1}(\phi_\ell(a) + \phi_\ell(b)) \quad \text{and}$$

scalar multiplication (relative to ϕ_ℓ) on ℓ is defined as

$$ra = \phi_\ell^{-1}(r\phi_\ell(a)) \quad \text{for } a \in R.$$

5.2.9 Lemma

If $a,b \in X$ and $a \neq b$, and let ϕ_ℓ be an isomorphism from $\ell(a,b)$ to the reals R. Then $x \in \ell(a,b)$ if and only if $x = ra + sb$ for r,s R such that $r + s = 1$, and furthermore, $r = \frac{d(a,x)}{d(a,b)}$ and $s = \frac{d(x,b)}{d(a,b)}$.

Proof: Suppose $x \in \ell(a,b)$. Since ϕ_ℓ is an isomorphism from $\ell(a,b)$ to the reals R, therefore for some $r, s \in R$ such that $r + s = 1$, we have

$$\phi_\ell(x) = r\,\phi_\ell(a) + s\,\phi_\ell(b)$$

or $$x = \phi_\ell^{-1}\left[r\,\phi_\ell(a) + s\,\phi_\ell(b)\right].$$

Hence $x = ra + sb$. Next we show that $r = \frac{d(a,x)}{d(a,b)}$ and $s = \frac{d(x,b)}{d(a,b)}$.

We know

$$\phi_\ell(x) = r\,\phi_\ell(a) + (1-r)\,\phi_\ell(b), \text{ and}$$

$$\phi_\ell(x) = r\,\phi_\ell(x) + (1-r)\,\phi_\ell(x).$$

Equating these, we get

$$r\,(\phi_\ell(x) - \phi_\ell(a)) = (1-r)\,(\phi_\ell(b) - \phi_\ell(x)).$$

But $d(a,x) = \phi_\ell(x) - \phi_\ell(a)$ and $d(x,b) = \phi_\ell(b) - \phi_\ell(x)$.

Therefore the above equation reduces to

$$r\,d(a,x) = (1-r)\,d(x,b)$$

which yields

$$r\,(d(a,x) + d(x,b)) = d(x,b)$$

or $$r\,d(a,b) = d(x,b).$$

Hence $r = \frac{d(x,b)}{d(a,b)}$; and $s = (1-r)$ implies $s = \frac{d(a,x)}{d(a,b)}$.

Conversely, it is clear that if $x = ra + sb$ when $r = \frac{d(x,b)}{d(a,b)}$ and $s = \frac{d(a,x)}{d(a,b)}$ then $x \in \ell(a,b)$.

5.2.10 Corollary

Suppose $a,b \in X$ and $a \neq b$. Then $x \in [a,b]$ if and only if $x = ra + (1-r)b$ for $r \in R$ such that $0 \leq r \leq 1$ and furthermore $r = \frac{d(x,b)}{d(a,b)}$.

5.2.11 Definition

Suppose $f : X \longrightarrow R$ is a function and $a,b \in X$ such that $a \neq b$. We say the map $f_\ell : \ell(a,b) \longrightarrow R$ is linear in the usual sense if for $r, s \in R$ and $r + s = 1$,

$$f_\ell\,(ra + sb) = rf_\ell(a) + sf_\ell(b)$$

is satisfied.

5.2.12 Theorem

If $f : X \longrightarrow R$ then f is linear if and only if f_ℓ is linear in the usual sense on each $\ell \in L$.

Proof: Suppose f is linear and $a, b \in \ell$ such that $a \neq b$. Let x be any point on the line (a,b) such that $x \neq a$ or $x \neq b$. Then by lemma 5.2.9, $x = ra + sb$ where $r = \frac{d(x,b)}{d(a,b)}$ and $s = \frac{d(a,x)}{d(a,b)}$. Since f is linear and $x \in \ell(a,b)$ therefore by lemma 5.2.5

$$\frac{d(a,x)}{d(a,b)} = \frac{f(x) - f(a)}{f(b) - f(a)}.$$

But $f_\ell(x) = f(x)$, $f_\ell(b) = f(b)$ and $f_\ell(a) = f(a)$, so we get

$$\frac{d(a,x)}{d(a,b)} = \frac{f_\ell(x) - f_\ell(a)}{f_\ell(b) - f_\ell(a)}.$$

Solving for $f_\ell(x)$, we get

$$\begin{aligned} f_\ell(x) &= \left(1 - \frac{d(a,x)}{d(a,b)}\right) f_\ell(a) + \frac{d(a,x)}{d(a,b)} f_\ell(b) \\ &= \frac{d(x,b)}{d(a,b)} f_\ell(a) + \frac{d(a,x)}{d(a,b)} f_\ell(b) \\ &= rf_\ell(a) + sf_\ell(b). \end{aligned}$$

Hence

$$f_\ell(ra + sb) = rf_\ell(a) + sf_\ell(b).$$

Conversely, to show f is linear, we show that condition (1) and (2) of lemma 5.2.5 hold.

(1) Let $a' = (a, f(a))$, $b' = (b, f(b)) \in \text{graph}(f)$ and $x \in \ell(a,b)$, then by lemma 5.2.9 $x = ra + sb$, where $r = \frac{d(x,b)}{d(a,b)}$ and $s =$

$\frac{d(a,x)}{a(a,b)}$. We show that for some $t \in \ell(f(a), f(b)) = R$, we have $(x,t) \in \ell((a,f(a)),f(b,f(b)) \subset$ graph (f). That is $t = f(x)$. By lemma 4.6.2,

$$t = \frac{d(a,x)}{d(a,b)} f(b) + \frac{d(x,b)}{d(a,b)} f(a). \qquad (A)$$

But $f_\ell(b) = f(b)$, $f_\ell(a) = f(a)$ and since f_ℓ is linear in the usual sense therefore

$$\begin{aligned} t &= sf_\ell(b) + rf_\ell(a) \\ &= f_\ell(sb + ra) \\ &= f_\ell(x). \end{aligned}$$

Therefore, $t = f(x)$.

(2) Substituting $t = f(x)$ into equation (A), we get

$$f(x) = \frac{d(a,x)}{d(a,b)} f(b) + \frac{d(x,b)}{d(a,b)} f(a).$$

Solving yields

$$\frac{d(a,x)}{d(a,b)} = \frac{f(x) - f(a)}{f(b) - f(a)}.$$

This completes our proof.

<u>5.2.13 Definition</u>

Suppose $f : X \longrightarrow R$ is a function and $a,b \in X$ such that $a \neq b$. The map f_ℓ: $\ell(a,b) \longrightarrow R$ is said to be <u>convex in the usual sense</u> if for all $r \in R$ such that $0 \leq r \leq 1$,

$$f_\ell(ra + (1-r)b) \leq rf_\ell(a) + (1-r)f_\ell(b)$$

is satisfied.

<u>5.2.14 Theorem</u>

If $f : X \longrightarrow R$ then f ix convex if and only if f_ℓ is convex in the usual sense on each line $\ell \in L$.

Proof: Suppose f is convex. Let ℓ be any line and $a,b \in \ell$ such that $a \neq b$, and $a' = (a,f(a))$, $b' = (b,f(b)) \in \text{epi}(f)$ then by definition $[a',b'] \in \text{epi}(f)$. Let ϕ_ℓ be an isomorphism from $\ell(a,b)$ to the reals R. Set $x' = (x,t) \in [a',b']$ where $x \in [a,b]$. Then by lemma 5.2.9, $x = ra + (1-r)b$ for $r \in R$ such that $o \leq r \leq 1$, where $r = \frac{d(x,b)}{d(a,b)}$ and $(1-r) = \frac{d(a,x)}{d(a,b)}$.

By lemma 4.6.2

$$t = \frac{d(a,x)}{d(a,b)}f(b) + \frac{d(x,b)}{d(a,b)}f(a).$$

But $f(a) = f_\ell(a)$, $f(b) = f_\ell(b)$ and $f(x) = f_\ell(x)$, therefore

$$t = rf_\ell(a) + (1-r)f_\ell(b).$$

Hence, $x' = (x, rf_\ell(a) + (1-r)f_\ell(b)) \in \text{epi}(f)$ implies

$f(x) \leq rf_\ell(a) + (1-r)f_\ell(b)$, so

$f_\ell(x) \leq rf_\ell(a) + (1-r)f_\ell(b)$

or $f_\ell(ra + (1-r)b) \leq rf_\ell(a) + (1-r)f_\ell(b)$.

Hence f_ℓ is convex in the usual sense.

Conversely, to show f is convex set in X x R. Let $a' = (a,s)$, $b' = (b,t) \in \text{epi}(f)$ and $x' = (x,u) \in [a',b']$. Then $x \in [a,b]$. By corollary 5.2.10 , $x = ra + (1-r)b$ for $r \in R$ such that $o \leq r \leq 1$ where $r = \frac{d(x,b)}{d(a,b)}$ and $(1-r) = \frac{d(a,x)}{d(a,b)}$.

Then by lemma 4.6.2 and since (a,s), $(b,t) \in \text{epi}(f)$,

therefore

$$u = \frac{d(a,x)}{d(a,b)}t + \frac{d(x,b)}{d(a,b)}s$$

or $u \geq (1-r)f(b) + rf(a)$

But $f(b) = f_\ell(b)$, $f(a) = f_\ell(a)$ and $f(x) = f_\ell(x)$

and so $u \geq (1-r)f_{\ell}(b) + rf_{\ell}(a)$.

Since f_{ℓ} is convex in the usual sense, therefore

$$u \geq f_{\ell}(ra + (1-r)b)$$
$$= f_{\ell}(x).$$

But $f_{\ell}(x) = f(x)$. Therefore $x' = (x,u) \in \mathrm{epi}(f)$ implies epi(f) is a convex set in X x R. This completes our proof.

5.3 The support of a convex function

We will now consider the results obtained by extending the concepts of support of a convex function to such functions on generalized line spaces. In order to prove our results we give the following definitions.

5.3.1 Definition

Let U be an open-convex subset of X. A function $f: U \longrightarrow R$ is said to be convex if and only if epi (f) is a convex set.

5.3.2 Definition

Let U be an open-convex subset of X. A function $f: U \longrightarrow R$ has a support at $x_0 \in U$, if there exists a linear function $A_{x_0}: X \longrightarrow R$ such that $A_{x_0}(x_0) = f(x_0)$ and $A_{x_0}(x) \leq f(x)$ for every $x \in U$. The graph of a support function A_{x_0} is called a supporting hyperplane for f at x_0.

5.3.3 Theorem

Let U be an open-convex subset of X. A function $f: U \longrightarrow R$ is convex if f has a support at each point of U.

Proof:

To show f is a convex function we prove that epi(f) is a convex set in $X \times R$. Let $a' = (a,r)$, $b' = (b,s) \in$ epi(f) and consider any point $x' = (x,t) \in [a',b']$ such that $x \in [a,b]$. We show that $x' \in$ epi(f). By lemma 4.6.2 and since $(a,r),(b,s) \in$ epi(f) therefore we have

$$t = \frac{d(a,x)}{d(a,b)} s + \frac{d(x,b)}{d(a,b)} r$$

$$\geq \frac{d(a,x)}{d(a,b)} f(b) + \frac{d(x,b)}{d(a,b)} f(a).$$

If $A_x : X \longrightarrow R$ is a linear function which supports f at x then

$$t \geq \frac{d(a,x)}{d(a,b)} A_x(b) + \frac{d(x,b)}{d(a,b)} A_x(a).$$

By lemma 5.2.5 and since A_x supports f at x therefore

$$t \geq A_x(x)$$

$$= f(x).$$

Therefore $x' = (x,t) \in$ epi(f) and this completes our proof.

5.4 Linearization of Generalized line spaces

5.4.1 Definition

The dual of a generalized line space X, denoted X^*, is the family of all linear functions from X to R.

Here we consider the results of Mah, Naimpally, and Whitfield. A convexity space X is proved to be a (compatible) vector space by means of four properties of a so- called linearization family, as shown in chapter III for abstract convexity spaces. In the case of a generalized line space it will be a vector space if and only if there is a family X_0^* of functionals from X to R satisfying:

(G0) Each $f \in X_0^*$ is a convexity-preserving map from X to R.

(G1) There exists a point $a_0 \in X$ such that $f(a_0) = 0$ for each $f \in X_0^*$, and the family X_0^*, is point distinguishing.

(G2) Each $f \in X_0^*$ restricted to any line is either a bijection or a constant map.

(G3) If $f, g \in X_0^*$ and each distinguishes two points a and b, then there exists constants $r, s \in R$ such that $g(c) = r\,f(c) + s$ for all $c \in \ell(a,b)$.

5.4.2 Theorem

A generalized line space X is a vector space if and only if its dual X* is point distinguishing.

Proof: It is obvious that if X is a vector space with the zero vector a_0 and its dual X* is the set of all linear functions on X to R, then X* is a linearization family for X and is point distinguishing.

Conversely, suppose the dual X* of a generalized line space is point distinguishing. To show X is a vector space we take

$$X_0^* = \{f \in X^* : f(a_0) = 0\}$$

for some $a_0 \in X$. We establish the properties Go - G3 for X_0^*, which by theorem 3.3.2 will show that X is a vector space.

(G0) Each $f \in X_0^*$ is a convexity preserving map from X to R:

Suppose C is any convex set in X and we show f(C) is also convex. Let f(a), f(b) be two distinct points of f(C) for some $a, b \in C$. We will show $rf(a) + (1-r)f(b) \in f(C)$

for $0 \leq r \leq 1$. Since f is linear therefore by theorem 5.2.12 $rf_{\ell}(a) + (1-r)f_{\ell}(b) = f_{\ell}(ra + (1-r)b) = f_{\ell}(x) = f(x)$. Now by Corollary 5.2.10 $x \in [a,b]$, but C is convex therefore $x \in C$. This implies $f(x) \in C$.

(G1) There exists a point $a_0 \in X$ such that $f(a_0) = 0$ for each $f \in X_0^*$ and the family X_0^* is point distinguishing:

Since X* is point distinguishing and hence (G1) follows.

(G2) Each $f \in X_0^*$ restricted to a line is either a bijection or a constant map:

Suppose $a,b \in X$ such that $a \neq b$. Then for any $c \in \ell(a,b)$ and $f \in X_0^*$, by lemma 5.2.5 we have

$$\frac{d_1(a,c)}{d_1(a,b)} = \frac{d_2(f(a),f(c))}{d_2(f(a),f(b))}.$$

If $f(a) = f(b)$ then f is constant on $\ell(a,b)$ since in that case one has $d_2(f(a),f(c)) = 0$. Now if $f(a) \neq f(b)$ then it implies f is injective. By lemma 5.2.6 if follows that f is surjective

(G3) If $f,g \in X_0^*$ and each distinguishes two points a and b, then there exists constants $r,s \in R$ such that $g(c) = r\, f(c) + s$ for all $c \in \ell(a,b)$:

Suppose $f(a) \neq f(b)$ and $g(a) \neq g(b)$ then for any $c \in \ell(a,b)$ we have by lemma 5.2.5

$$\frac{f(a) - f(c)}{f(a) - f(b)} = \frac{d(a,c)}{d(a,b)} = \frac{g(a) - g(c)}{g(a) - g(b)}$$

so that $g(c) = r\, f(c) + s$ for constants r,s. Hence the proof of our theorem is completed.

5.5 Concluding Remarks

We remark that the results of this thesis along with Doignon's [7] results can lead us to the linearization of abstract convexity spaces. This may be stated as follows.

"An abstract convexity space $(X,\mathcal{C})$ of dimension greater than 2 is an open-convex subset of an affine space if and only if it is domain-finite join-hull commutative, complete and has regular straight segments."

In Chapter V, converse of 5.3.3 is an open question and a solution appears to depend on a Hahn-Banach type theorem for generalized line spaces.

Also, the question of continuity and differentiability of linear and convex functions has not been addressed. Of course, this could require that a generalized line space be given an appropriate compatible topology.

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